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A Technique for Increasing the Accuracy of the FFT-Based Method of Numerical Inversion of Laplace Transforms

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Abstract—The FFT-based methods of numerical inversion of Laplace transforms use the trapezoidal rule to the Bromwich integral. We present in this paper a technique for reducing the truncation error in evaluating the Bromwich integral. The technique employs the differentiation property of the Laplace transform and performs the inversion on $F^{(n)}(s)$, the n^{th} order derivative of the Laplace transform of a time function $f(t)$. The improvement in the solution accuracy by incoporating the presented technique into the FFT-based numerical Laplace inversion method is demonstrated via numerical examples.

Keywords—Laplace transform inversion, Fast Fourier transform, Numerical method, Bromwich integral.

1. INTRODUCTION

Although the numerical inversion of Laplace transforms is an old problem [1–3], there is a renewed interest driven by the emerging computer technologies and the development of fast computation algorithms to obtain the improved solutions. The FFT-based methods [4–6] of numerical inversion of the Laplace transform can take full advantages of the modern computing environment. These methods are essentially based on approxmiating the Bromwich integral by the trapezoidal rule of integration and using the efficient computation algorithm of the fast Fourier transform (FFT) to evaluate inverted function values at a set of equally-spaced points. Since the Bromwich integral is an indefinite integral, the approximation procedure always involves two kind of errors. One is due to the truncation of the Bromwich integral to a definite integral, which is called the truncation error, and the other comes from trapezoidal approximation of the truncated Bromwich integral. To reduce the error in approximating the truncated Bromwich integral, Hwang *et al.* [6] have recently used a smaller step size of integration to derive a new inversion formula which can be implemented by multiple sets of FFT computations.

The purpose of this paper is to present a technique for reducing the truncation error in evaluating the Bromwich integral. The technique employs the differentiation property of the Laplace transform and performing the inversion indirectly from the n^{th}-order derivative of the Laplace transform function. The improvement in the accuracy of numerical solutions can be achieved by incorporating the presented technique with the new FFT-based numerical Laplace inversion method [6].

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Table 1. Numerical accuracy of $\mathcal{L}^{-1}\left\{1/s^2+s+1\right\}$, $\sigma=-0.05$.

n	m	$f_a(t)$ from $F(s)$	$f_a(t)$ from $F^{(2)}(s)$
1	10	$L=0.59322E-04$ $L_e=0.21945E-03$	$0.13689E-04$ $0.51069E-04$
1	11	$L=0.59322E-04$ $L_e=0.21945E-03$	$0.13689E-04$ $0.51069E-04$
1	12	$L=0.59322E-04$ $L_e=0.21945E-03$	$0.13689E-04$ $0.51069E-04$
1	13	$L=0.59321E-04$ $L_e=0.21945E-03$	$0.13689E-04$ $0.51069E-04$
2	10	$L=0.85302E-05$ $L_e=0.31608E-04$	$0.57666E-06$ $0.21518E-05$
2	11	$L=0.85302E-05$ $L_e=0.31608E-04$	$0.57666E-06$ $0.21518E-05$
2	12	$L=0.85302E-05$ $L_e=0.31608E-04$	$0.57666E-06$ $0.21518E-05$
2	13	$L=0.85302E-05$ $L_e=0.31608E-04$	$0.57666E-06$ $0.21518E-05$
3	10	$L=0.26146E-05$ $L_e=0.96918E-05$	$0.82244E-07$ $0.30691E-06$
3	11	$L=0.26146E-05$ $L_e=0.96918E-05$	$0.82244E-07$ $0.30690E-06$
3	12	$L=0.26146E-05$ $L_e=0.96918E-05$	$0.82244E-07$ $0.30690E-06$
3	13	$L=0.26146E-05$ $L_e=0.96918E-05$	$0.82244E-07$ $0.30690E-06$
4	10	$L=0.11173E-05$ $L_e=0.41424E-05$	$0.20144E-07$ $0.75171E-07$
4	11	$L=0.11173E-05$ $L_e=0.41424E-05$	$0.20144E-07$ $0.75171E-07$
4	12	$L=0.11173E-05$ $L_e=0.41423E-05$	$0.20144E-07$ $0.75171E-07$
4	13	$L=0.11173E-05$ $L_e=0.41423E-05$	$0.20144E-07$ $0.75170E-07$

2. THE MAIN RESULT

Given a Laplace transform $F(s)$, the inverse transform $f(t)$ is given by [7]:

$$f(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}F(s)\,e^{st}\,ds, \tag{1}$$

where $j=\sqrt{-1}$ and σ is a real constant greater than the abscissa of absolute convergence of $F(s)$. Replacing s by $\sigma+jw$, we have

$$f(t)=\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}F(\sigma+jw)\,e^{jwt}\,dw. \tag{2}$$

The indefinite integral in (2) is the Bromwich integral [7]. In actual computations, the Bromwich integral is evaluated over a finite frequency domain, say, $w\in[-\Omega,\Omega]$. That is, the function $f(t)$ is approximated by

$$f_a(t)=\frac{e^{\sigma t}}{2\pi}\int_{-\Omega}^{\Omega}F(\sigma+jw)\,e^{jwt}\,dw. \tag{3}$$

Table 2. Numerical accuracy of $\mathcal{L}^{-1}\left\{1/s^2+s+1\right\}$, $\sigma = 0.05$.

n	m	$f_a(t)$ from $F(s)$	$f_a(t)$ from $F^{(2)}(s)$
1	10	$L = 0.64986E-04$	$0.14899E-04$
		$L_e = 0.23969E-03$	$0.55576E-04$
1	11	$L = 0.64985E-04$	$0.14899E-04$
		$L_e = 0.23969E-03$	$0.55576E-04$
1	12	$L = 0.64985E-04$	$0.14899E-04$
		$L_e = 0.23969E-03$	$0.55576E-04$
1	13	$L = 0.64985E-04$	$0.14899E-04$
		$L_e = 0.23969E-03$	$0.55576E-04$
2	10	$L = 0.93160E-05$	$0.62560E-06$
		$L_e = 0.34437E-04$	$0.23342E-05$
2	11	$L = 0.93160E-05$	$0.62560E-06$
		$L_e = 0.34437E-04$	$0.23342E-05$
2	12	$L = 0.93160E-05$	$0.62560E-06$
		$L_e = 0.34437E-04$	$0.23342E-05$
2	13	$L = 0.93159E-05$	$0.62560E-06$
		$L_e = 0.34437E-04$	$0.23342E-05$
3	10	$L = 0.28535E-05$	$0.89161E-07$
		$L_e = 0.10553E-04$	$0.33269E-06$
3	11	$L = 0.28535E-05$	$0.89161E-07$
		$L_e = 0.10553E-04$	$0.33269E-06$
3	12	$L = 0.28535E-05$	$0.89161E-07$
		$L_e = 0.10553E-04$	$0.33269E-06$
3	13	$L = 0.28535E-05$	$0.89161E-07$
		$L_e = 0.10553E-04$	$0.33269E-06$
4	10	$L = 0.12191E-05$	$0.21832E-07$
		$L_e = 0.45097E-05$	$0.81464E-07$
4	11	$L = 0.12191E-05$	$0.21832E-07$
		$L_e = 0.45096E-05$	$0.81464E-07$
4	12	$L = 0.12191E-05$	$0.21832E-07$
		$L_e = 0.45096E-05$	$0.81464E-07$
4	13	$L = 0.12191E-05$	$0.21832E-07$
		$L_e = 0.45096E-05$	$0.81464E-07$

The approximation error due to the truncation of integration is then given by

$$e(t;\Omega) = f(t) - f_a(t) = \frac{e^{\sigma t}}{2\pi}\int_{\Omega}^{\infty}\left[F(\sigma + jw)e^{jwt} + F(\sigma - jw)e^{-jwt}\right] dw. \tag{4}$$

It is observed from (4) that the truncation error $e(\Omega; t)$ can be reduced by enlarging the maximum frequency Ω. Besides, the truncation error can be reduced if the integrand in (4) is changed such that transformed function decays faster to zero than $F(\sigma+jw)$. With these observations in mind, one may further recall that in most cases $F^{(n)}(\sigma + jw)$, the n^{th}-order derivative of $F(\sigma + jw)$, has a faster decay rate than $F(\sigma+jw)$ itself. Hence, by calculating inverted function for $F^{(n)}(s)$ from

$$g_a(t) = \frac{e^{\sigma t}}{2\pi}\int_{-\Omega}^{\Omega} F^{(n)}(\sigma + jw)e^{jwt}dw \tag{5}$$

we may then use the differentiation property of the Laplace transform

$$\mathcal{L}^{-1}\left\{F^{(n)}(s)\right\} = \left(-1\right)^n \frac{f(t)}{t^n} \tag{6}$$

to get the approximate inverted function for $F(s)$ by $f_a(t) = \left(-1\right)^n t^n g_a(t)$.

Table 3. Numerical accuracy of $\mathcal{L}^{-1}\{1/(s+1)^2\}$, $\sigma = -0.5$.

n	m	$f_a(t)$ from $F(s)$	$f_a(t)$ from $F^{(2)}(s)$
1	10	$L = 0.51650E-04$	$0.11986E-04$
		$L_e = 0.19159E-03$	$0.44724E-04$
1	11	$L = 0.51650E-04$	$0.11986E-04$
		$L_e = 0.19159E-03$	$0.44724E-04$
1	12	$L = 0.51650E-04$	$0.11986E-04$
		$L_e = 0.19159E-03$	$0.44724E-04$
1	13	$L = 0.51650E-04$	$0.11986E-04$
		$L_e = 0.19159E-03$	$0.44724E-04$
2	10	$L = 0.74521E-05$	$0.50700E-06$
		$L_e = 0.27674E-04$	$0.18921E-05$
2	11	$L = 0.74520E-05$	$0.50700E-06$
		$L_e = 0.27674E-04$	$0.18921E-05$
2	12	$L = 0.74520E-05$	$0.50700E-06$
		$L_e = 0.27673E-04$	$0.18921E-05$
2	13	$L = 0.74520E-05$	$0.50700E-06$
		$L_e = 0.27673E-04$	$0.18920E-05$
3	10	$L = 0.22858E-05$	$0.72375E-07$
		$L_e = 0.84908E-05$	$0.27010E-06$
3	11	$L = 0.22858E-05$	$0.72375E-07$
		$L_e = 0.84908E-05$	$0.27010E-06$
3	12	$L = 0.22858E-05$	$0.72375E-07$
		$L_e = 0.84908E-05$	$0.27010E-06$
3	13	$L = 0.22858E-05$	$0.72375E-07$
		$L_e = 0.84907E-05$	$0.27010E-06$
4	10	$L = 0.97712E-06$	$0.17733E-07$
		$L_e = 0.36299E-05$	$0.66178E-07$
4	11	$L = 0.97712E-06$	$0.17733E-07$
		$L_e = 0.36299E-05$	$0.66178E-07$
4	12	$L = 0.97711E-06$	$0.17733E-07$
		$L_e = 0.36299E-05$	$0.66178E-07$
4	13	$L = 0.97711E-06$	$0.17733E-07$
		$L_e = 0.36299E-05$	$0.66178E-07$

It is noted here that the definite integral (5) can be efficiently and accurately computed for a set of equally-spaced points of time by a new FFT-based algorithm recently proposed by Hwang *et al.* [6]. The algorithm is based on using the trapezoidal approximation to the integral in (5) with

$$\Delta w = \frac{\pi}{mT} \tag{7a}$$

and the upper frequency bound

$$\Omega = \left(\frac{mnN}{2} + \left[\frac{m-1}{2}\right]\right) \cdot \Delta w \triangleq M \Delta w, \tag{7b}$$

where Δw is the step length of integration, m, n and N are integers, and $[r]$ denotes the maximum integer that does not exceed r. With this approximation, the approximate function $g_a(t)$ is given by

$$g_a(t) = \frac{e^{\sigma t}}{2mT} \sum_{k=-M}^{M} F^{(n)}\left(\sigma + j\frac{k\pi}{mT}\right) \exp\left(j\frac{k\pi}{mT}\right). \tag{8}$$

By letting $t = i\Delta t$ and $\Delta t = 2T/N$, and seleting N to be a positive power of two, (8) can be

Tale 4. Numerical accuracy of $\mathcal{L}^{-1}\left\{1/s^2+1\right\}$, $\sigma=0.2$.

n	m	$f_a(t)$ from $F(s)$	$f_a(t)$ from $F^{(2)}(s)$
1	10	$L=0.68928E-04$	$0.15687E-04$
		$L_e=0.25336E-03$	$0.58512E-04$
1	11	$L=0.68930E-04$	$0.15687E-04$
		$L_e=0.25336E-03$	$0.58511E-04$
1	12	$L=0.68931E-04$	$0.15687E-04$
		$L_e=0.25336E-03$	$0.58511E-04$
1	13	$L=0.68931E-04$	$0.15687E-04$
		$L_e=0.25336E-03$	$0.58511E-04$
2	10	$L=0.98646E-05$	$0.65845E-06$
		$L_e=0.36366E-04$	$0.24566E-05$
2	11	$L=0.98647E-05$	$0.65841E-06$
		$L_e=0.36366E-04$	$0.24565E-05$
2	12	$L=0.98648E-05$	$0.65842E-06$
		$L_e=0.36366E-04$	$0.24565E-05$
2	13	$L=0.98649E-05$	$0.65842E-06$
		$L_e=0.36366E-04$	$0.24565E-05$
3	10	$L=0.30204E-05$	$0.93852E-07$
		$L_e=0.11142E-04$	$0.35017E-06$
3	11	$L=0.30205E-05$	$0.93819E-07$
		$L_e=0.11142E-04$	$0.35005E-06$
3	12	$L=0.30205E-05$	$0.93831E-07$
		$L_e=0.11142E-04$	$0.35009E-06$
3	13	$L=0.30205E-05$	$0.93830E-07$
		$L_e=0.11142E-04$	$0.35009E-06$
4	10	$L=0.12903E-05$	$0.22997E-07$
		$L_e=0.47609E-05$	$0.85803E-07$
4	11	$L=0.12903E-05$	$0.22964E-07$
		$L_e=0.47608E-05$	$0.85684E-07$
4	12	$L=0.12903E-05$	$0.22975E-07$
		$L_e=0.47608E-05$	$0.85726E-07$
4	13	$L=0.12903E-05$	$0.22974E-07$
		$L_e=0.47608E-05$	$0.85722E-07$

rewritten as

$$g_a(i\Delta t)=\frac{e^{i\Delta t}}{2mT}(-1)^n\sum_{r=-m_1}^{m_2}W^{(ir/m)}\left\{\sum_{k=0}^{N-1}F_r^{(n)}(k)W^{ik}\right\}, \tag{9}$$

where

$$m_1=\left[\frac{m-1}{2}\right],\qquad m_2=\left[\frac{m+1}{2}\right],\qquad W=\exp\left(j\frac{2\pi}{N}\right)$$

$$F_r^{(n)}(k)=\sum_{p=0}^{n_1}F^{(n)}\left(\sigma+j\frac{\pi}{T}\left(k+\frac{r}{m}+\frac{N}{2}(p-n)\right)\right),\qquad k=0,1,\ldots,N-1$$

and

$$n_1=\begin{cases}n, & k=0 \text{ and } r\neq m/2\\ n-1, & \text{otherwise.}\end{cases}$$

The values of $g_a(i\Delta t)$ for $i=0,1,\ldots,N-1$ can be obtained through m sets of N-point FFT computations or through more efficient fast Hartley transform (FHT) computations [6]. Once $g_a(i\Delta t)$ for $i=0,1,\ldots,N-1$ are obtained, the inverted function values $f(i\Delta t)$ are then computed by

$$f(i\Delta t)=(-1)^n\,(i\Delta t)\,g(i\Delta t). \tag{10}$$

Table 5. Numerical accuracy of $\mathcal{L}^{-1}\left\{e^{-4\sqrt{s}}\right\}$, $\sigma = 0.2$.

n	m	$f_a(t)$ from $F(s)$	$f_a(t)$ from $F^{(2)}(s)$
1	10	$L = 0.19300E-06$	$0.24552E-06$
		$L_e = 0.27056E-06$	$0.91368E-06$
1	11	$L = 0.19345E-06$	$0.24552E-06$
		$L_e = 0.27056E-06$	$0.91367E-06$
1	12	$L = 0.19379E-06$	$0.24552E-06$
		$L_e = 0.27056E-06$	$0.91367E-06$
1	13	$L = 0.19406E-06$	$0.24552E-06$
		$L_e = 0.27057E-06$	$0.91367E-06$
2	10	$L = 0.60220E-09$	$0.36878E-09$
		$L_e = 0.13486E-08$	$0.13764E-08$
2	11	$L = 0.60320E-09$	$0.36844E-09$
		$L_e = 0.13486E-08$	$0.13751E-08$
2	12	$L = 0.60396E-09$	$0.36842E-09$
		$L_e = 0.13486E-08$	$0.13750E-08$
2	13	$L = 0.60455E-09$	$0.36842E-09$
		$L_e = 0.13486E-08$	$0.13750E-08$
3	10	$L = 0.36852E-11$	$0.30972E-11$
		$L_e = 0.79824E-11$	$0.11534E-10$
3	11	$L = 0.36929E-11$	$0.34478E-11$
		$L_e = 0.79824E-11$	$0.12833E-10$
3	12	$L = 0.36988E-11$	$0.34629E-11$
		$L_e = 0.79824E-11$	$0.12889E-10$
3	13	$L = 0.37034E-11$	$0.34634E-11$
		$L_e = 0.79825E-11$	$0.12891E-10$
4	10	$L = 0.92879E-13$	$0.40802E-12$
		$L_e = 0.34135E-12$	$0.15076E-11$
4	11	$L = 0.92881E-13$	$0.54005E-13$
		$L_e = 0.34135E-12$	$023E-12$
4	12	$L = 0.92880E-13$	$0.38802E-13$
		$L_e = 0.34135E-12$	$0.14403E-12$
4	13	$L = 0.92884E-13$	$0.38241E-13$
		$L_e = 0.34135E-12$	$0.14195E-12$

3. NUMERICAL EXAMPLES

To illustrate the accuracy of the FFT-based numerical Laplace inversion method [6] improved by the presented differentiation technique, the following five representative Laplace transform functions are inverted:

Example 1. $F_1(s) = \dfrac{1}{s^2+s+1}$, $f_1(t) = \dfrac{2}{\sqrt{3}}\, e^{-t/2} \sin\left(\dfrac{\sqrt{3}t}{2}\right)$

Example 2. $F_2(s) = \dfrac{1}{(s+0.2)^2+1}$, $f_2(t) = e^{-0.2t}\sin(t)$

Example 3. $F_3(s) = \dfrac{1}{(s+1)^2}$, $f_3(t) = t\, e^{-t}$

Example 4. $F_4(s) = \dfrac{1}{s^2+1}$, $f_4(t) = \sin(t)$

Example 5. $F_5(s) = e^{-4\sqrt{s}}$, $f_5(t) = \dfrac{2}{\sqrt{\pi\, t^3}}\, e^{-4/t}$.

Moreover, the following two measures for the accuracy of the numerically inverted function are computed for each function:

$$L = \left(\frac{\sum_{i=1}^{N-1} [f(0.25i) - f_a(0.25i)]^2}{N-1} \right)^{1/2},$$

$$L_e = \left(\frac{\sum_{i=1}^{N-1} \left[f(0.25i) - f_a(0.25i)\right]^2 e^{-0.25i}}{\sum_{i=1}^{N-1} e^{-0.25i}} \right)^{1/2}.$$

The value of L will indicate the success of the method for large t, and L_e for relatively small t (see [1]).

For each Laplace transform $F(s)$, two versions of the numerically inverted function are computed: one is from $F(s)$ and the other is from $F^{(2)}(s)$. The computational results of L and L_e for each function are shown in Tables 1–5, along with the σ, n and m parameters used. It is clear from the values of L and L_e in these tables that the numerical accuracy of an inverted function can be increased singnificantly by the proposed technique.

4. CONCLUSION

The numerical computation of inverse Laplace transforms by calculating the indefinite Bromwich integral often involves a truncation error. We have proposed to reduce the truncation error by indirectly inverting the n^{th}-order derivative of the Lapalce transform function. It has been demonstrated by numerical examples that by incorporating the proposed technique of reducing truncation error with the new FFT-based Laplace inversion algorithm, the accuracy of the obtained numerical sloutions can be improved remarkably.

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