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Semi-parallel time-like surfaces in Lorentzian spacetime forms

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Abstract: The classification of semi-parallel surfaces in Euclidean space (by Deprez 1985) and in Riemannian space forms (by Mercuri 1991) is extended to the case of time-like surfaces in Lorentzian spacetime forms. Existence and geometry of such surfaces are investigated, especially of the exceptional and minimal ones in de Sitter spacetimes; here the minimal surfaces are the subjects of the geometrical string theory.

Keywords: Semi-parallel surfaces, time-like surfaces, minimal surfaces, strings.

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1. Introduction

1.1. Lorentzian spacetime forms. Let $N^n_s(c)$ be a space form with $ds^2 = -(\omega^1)^2 - \cdots - (\omega^s)^2 + (\omega^{s+1})^2 + \cdots + (\omega^n)^2$ of constant curvature c, Riemannian if $s = 0$ (or $s = n$) and pseudo-Riemannian if $0 < s < n$ (see [23]). In particular, $N^n_1(c)$ is called the Lorentzian spacetime form, having in mind the interpretation in the general relativity (see [8]). The standard models [23, 8] are (for $s = 0$, or $s = 1$, or $0 < s < n$, respectively)

$c = 0$ Euclidean space E^n, or Minkowski spacetime E^n_1, or pseudo-Euclidean space E^n_s,

$c > 0$ hypersphere $S^n(r) \subset E^{n+1}$, $r = (\sqrt{c})^{-1}$, or de Sitter spacetime of the 1st kind $S^n_1(r) \subset E^{n+1}_1$, or $S^n_s(r) \subset E^{n+1}_s$ (the latter two as hyperspheres of real radius $r = (\sqrt{c})^{-1}$),

$c < 0$ hyperbolic $H^n(r) \subset E^{n+1}_1$, or de Sitter spacetime of the 2nd kind (anti-de Sitter spacetime [2, 21]) $H^n_1(r) \subset E^{n+1}_2$, or $H^n_s(r) \subset E^{n+1}_{s+1}$ (as hyperspheres of imaginary radius $ir = (\sqrt{c})^{-1}$).

1.2. Semi-parallel submanifolds. Let M^m_q be a submanifold in $N^n_s(c)$ with $ds^2_M = -(\omega^1_M)^2 - \cdots - (\omega^q_M)^2 + (\omega^{q+1}_M)^2 + \cdots + (\omega^m_M)^2$. Such M^m_q is said to be *parallel* if its second fundamental form h satisfies $\bar{\nabla} h = 0$ (i.e., is parallel with respect to the van der Waerden–Bortolotti connection $\bar{\nabla} = \nabla \oplus \nabla^\perp$) and *semi-parallel* if the integrability condition $\bar{R} \circ h = 0$ of $\bar{\nabla} h = 0$ is satisfied (where $\bar{R}$ is the curvature operator of $\bar{\nabla}$).

A complete parallel submanifold M^m_q in $N^n_s(c)$ is a *symmetric orbit* in the sense that M^m_q is symmetric with respect to every its normal subspace and is an orbit of some Lie group acting in $N^n_s(c)$ by isometries (see [7, 18] for $s = 0$ and [15, 2] for $0 < s < n$).

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A semi-parallel submanifold M_q^m in $N_s^n(c)$ is a *2nd order envelope* of symmetric orbits $\widetilde{M}_q^m$ in $N_s^n(c)$ (for $s = 0$ see [11], where "semi-symmetric" is used as a synonym of "semi-parallel"; the generalization to the case $0 < s < n$ is obvious).

1.3. Semi-parallel surfaces, Riemannian case. The notion of a semi-parallel submanifold is introduced in [5], where also all semi-parallel surfaces M^2 in Euclidean spaces (i.e., for $c = 0$) are completely classified; recently this classification is extended to the case of $c \neq 0$ in [14, 1].

Theorem A (Deprez [5], Mercuri [14]). *Let M^2 be a semi-parallel surface in $N^n(c)$. There exists an open and dense part U of M^2 such that the connected components of U are of the following types*:

(i) *open parts of totally umbilical $N^2(k)$ in $N^n(c)$, $k = const \geqslant c$ (in particular, of totally geodesic $N^2(c)$, if $k = c$)*,

(ii) *surfaces with flat $\overline{\nabla}$ (i.e., locally Euclidean surfaces with flat normal connection $\nabla^\perp$)*,

(iii) *isotropic surfaces with nonflat $\nabla^\perp$ and with $\|H\|^2 = 3k - c$, where k is the Gaussian curvature and H is the mean curvature vector.*

Here a surface M^2 in $N^n(c)$ is said to be isotropic if at every fixed point $x \in M^2$ for arbitrary unit tangent vector $X \in T_x M^2$ the length of $h(X, X)$ does not depend on X.

The type (i) gives parallel surfaces, which are the parts of such symmetric orbits as spheres, horospheres etc. The other types give the 2nd order envelopes of symmetric orbits (see [11]) and can be described as follows. A surface of (ii) has diagonalizable h and thus an orthogonal net of curvature lines, hence the surface is a 2nd order envelope of products of curvature circles (or horocycles or equidistant curves, if $c < 0$) of these lines. A surface of (iii) is a 2nd order envelope of Veronese surfaces. It is shown (see [11] if $c = 0$, and [1] if $c \neq 0$) that for $n = 5$ such an envelope is a single Veronese surface or its part and thus a parallel surface. In [14, Remark 4] a conjecture is formulated that this holds also for $n > 5$, however in [12, 17] it is proved that this is not true.

A Veronese surface in E^5 belongs to a hypersphere S^4 and is minimal here, i.e., its mean curvature vector H with respect to S^4 vanishes. In [1, Proposition 3.6] it is shown that vice versa a minimal semi-parallel surface M^2 in $S^n(r)$ with nonflat normal connection $\nabla^\perp$, thus $n \geqslant 4$, turns out to be a Veronese surface or its part, hence parallel. In the proof the assertion (iii) of Theorem A is used together with a result from [3] that a minimal M^2 in $S^n(r)$ with constant Gaussian curvature $k = \frac{1}{3}r^2$ is a part of a Veronese surface. Note that the following more general result (for a special case $n = 4$) is established in [9]: any minimal M^2 with some constant k in $S^4(r)$ is a part of a Veronese surface. Recall also the well known fact that in $N^n(c)$ with $c \leqslant 0$ the only minimal M^2 with constant k are the totally geodesic surfaces, and then $k = c$.

1.4. Results of the present paper. Here the semi-parallel surfaces M_1^2 (called time-like) in $N_1^n(c)$ (called Lorentzian spacetime forms) are investigated. The main result is as follows.

Theorem B. *Let M_1^2 be a semi-parallel time-like surface in a Lorentzian spacetime form $N_1^n(c)$. There exists an open and dense part U of M_1^2 such that the connected components of U are of the following types*:

(i) *open parts of totally umbilical $N_1^2(k)$ in $N_1^n(c)$ (in particular, of totally geodesic $N_1^2(c)$)*,

(ii) *surfaces with flat $\overline{\nabla}$.*

As one can see, there is no analogue to the type (iii) of Theorem A. It is compensated by the fact that the type (ii) is here much more rich than that of Theorem A. It is caused by the fact that besides the principal case, when an adapted orthonormal frame field can be fixed on M^2_1 with flat $\nabla^\perp$ by means of h (the principal field), there exists an exceptional case when this cannot be done. (In the situation of Theorem A, type (ii), such an exceptional case is not possible; here M^2 with flat $\nabla^\perp$ always carries a unique net of curvature lines.)

Consequently, the type (ii) of Theorem B can be divided into several subcases. For every subcase also the corresponding parallel time-like surfaces are determined, the semi-parallel M^2_1 of these subcases are their 2nd order envelopes, sometimes trivially. In particular, several parallel time-like surfaces with flat $\nabla^\perp$ lie minimally in their spacetime forms. On the other hand, all minimal semi-parallel time-like surfaces in $N^n_1(c)$ are found out. The results are formulated in two propositions of the last section.

Note that before only some first steps were made in the classification of parallel and semi-parallel time-like surfaces in Lorentzian spacetime forms, concerning the parallel surfaces of low codimensions; they were made in the course of the study of parallel submanifolds with signature $(1, n-1)$ in $N^n_1(c)$ (see [13], where some interesting examples are given; also [21,2]).

In the present paper a complete classification of all such surfaces is given. Among them the results on minimal surfaces can be of some interest for the geometrical string theory, important for the theoretical particle physics [6] and cosmology [22]. (Acknowledgement is given to Professor P. Kuusk, who indicated me these two excellent survey papers on strings.)

2. Adapted frame bundle

A frame bundle $O_1(N^n_1(c))$ of the orthonormal frames $\{x, e_I\}$ in $N^n_1(c)$ (I, J etc. run $1, \dots, n$) is said to be reduced to a subbundle $O_1(M^2_1, N^n_1(c))$ of frames, adapted to a time-like surface $M^2_1 \subset N^n_1(c)$, if $e_i \in T_x M^2_1$, $e_\alpha \in T^\perp_x M^2_1$ (i, j etc. run 1, 2 and α, β etc. run $3, \dots, n$) so that $\langle e_1, e_1\rangle = -1$, $\langle e_2, e_2\rangle = 1$; of course $\langle e_1, e_2\rangle = 0$, $\langle e_i, e_\alpha\rangle = 0$, $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$. In the general formulae for the frame bundle $O_1(N^n_1(c))$,

$$dx = e_I\omega^I, \qquad de_I = -cxg_{IJ}\omega^J + e_J\omega^J_I, \qquad g_{IJ} = \langle e_I, e_J\rangle,$$
$$d\omega^I = \omega^J \wedge \omega^I_J, \qquad d\omega^J_I = -cg_{IK}\omega^K \wedge \omega^J + \omega^K_I \wedge \omega^J_K,$$
$$dg_{IJ} = g_{KJ}\omega^K_I + g_{IK}\omega^K_J$$

(where the point x is identified with its radius vector in E^n_1, if $c = 0$, or in E^{n+1}_1, if $c > 0$, or in E^{n+1}_2, if $c < 0$, in the last two cases having the origin in the centre of the standard model of $N^n_1(c)$) then $-g_{11} = g_{22} = 1$, $g_{12} = 0$, $g_{i\alpha} = 0$, $g_{\alpha\beta} = \delta_{\alpha\beta}$, thus $\omega^1_1 = 0$, $\omega^I_1 = \omega^1_I$ ($I \neq 1$), $\omega^J_I = -\omega^I_J$ ($I \neq 1$, $J \neq 1$), and for the frames of $O_1(M^2_1, N^n_1(c))$

$$\omega^\alpha = 0.$$

This, after exterior differentiation, due to Cartan's lemma, gives $\omega^\alpha_i = h^\alpha_{ij}\omega^j$, where h^α_{ij}, the components of the second fundamental form $h = e_\alpha h^\alpha_{ij}\omega^i\omega^j$, are symmetric with respect to i, j. Now a similar procedure leads to $\bar\nabla h^\alpha_{ij} = h^\alpha_{ijk}\omega^k$, where $\bar\nabla h^\alpha_{ij} = dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta$ are the components of the covariant differential $\bar\nabla h$ of h and h^α_{ijk} $(= \bar\nabla_k h^\alpha_{ij})$ are symmetric also with respect to j, k (i.e., $\overline{\nabla}_k h^\alpha_{ij} = \overline{\nabla}_j h^\alpha_{ik}$, the Peterson–Codazzi identities). The same h^α_{ijk} are the components of the third fundamental form $\overline{\nabla} h = e_\alpha h^\alpha_{ijk}\omega^i\omega^j\omega^k$ of M^2_1 in $N^n_1(c)$. The next step gives

$$\overline{\nabla} h^\alpha_{ijk} \wedge \omega^k = -h^\alpha_{kj}\Omega^k_i - h^\alpha_{ik}\Omega^k_j + h^\beta_{ij}\Omega^\alpha_\beta, \tag{2.1}$$

where

$$\Omega^j_i := d\omega^j_i - \omega^k_i \wedge \omega^j_k = -cg_{ik}\omega^k \wedge \omega^j - g_{\alpha\beta}g^{jm}h^\alpha_{ik}h^\beta_{ml}\omega^k \wedge \omega^l,$$
$$\Omega^\beta_\alpha := d\omega^\beta_\alpha - \omega^\gamma_\alpha \wedge \omega^\beta_\gamma = -g_{\alpha\gamma}g^{ij}h^\gamma_{jk}h^\beta_{il}\omega^k \wedge \omega^l$$

are the curvature 2-forms, respectively, of ∇ (with connection forms ω^j_i) and of $\nabla^\perp$ (with connection forms ω^β_α). Here the reduced coefficients at $\omega^k \wedge \omega^l$ are the components of the curvature tensors (operators) R and $R^\perp$ of ∇ and $\nabla^\perp$, respectively. Together they give the curvature operator $\overline{R} = R \oplus R^\perp$ of the van der Waerden–Bortolotti connection $\overline{\nabla} = \nabla \oplus \nabla^\perp$.

In particular, for a M^2_1 in $N^n_1(c)$,

$$\Omega^1_1 = \Omega^2_2 = 0, \quad \Omega^2_1 = \Omega^1_2 = [c - (\langle h_{11}, h_{22}\rangle - h^2_{12})]\omega^1 \wedge \omega^2, \tag{2.2}$$
$$\Omega^\beta_\alpha = -[h^\alpha_{12}(h^\beta_{11} + h^\beta_{22}) - h^\beta_{12}(h^\alpha_{11} + h^\alpha_{22})]\omega^1 \wedge \omega^2, \tag{2.3}$$

where $h_{ij} = e_\alpha h^\alpha_{ij}$.

If $\Omega^j_i = 0$ or $\Omega^\beta_\alpha = 0$ or $\Omega^j_i = \Omega^\beta_\alpha = 0$, the connections, ∇ or $\nabla^\perp$ or $\overline{\nabla}$, respectively, are said to be flat. Note that for a M^2_1 in $N^n_1(c)$ the so called normal connection $\nabla^\perp$ is flat (i.e., M^2_1 is normally flat) if and only if $h_{11} + h_{22}$ and h_{12} are collinear; it follows obviously from (2.3).

3. Reduction of adapted frame bundle

3.1. Transformation formulae. In the adapted frame bundle $O_1(M^2_1, N^n_1(c))$ the tangent part $\{e_1, e_2\}$ of the frame at a point $x \in M^2_1$ transforms according to

$$e'_1 = \varepsilon_1(e_1\cosh\varphi + e_2\sinh\varphi), \quad e'_2 = \varepsilon_2(e_1\sinh\varphi + e_2\cosh\varphi),$$

where $\varepsilon^2_1 = \varepsilon^2_2 = 1$. This leads to

$$\omega^1 = \varepsilon_1(\omega^{1'}\cosh\varphi + \omega^{2'}\sinh\varphi), \quad \omega^2 = \varepsilon_2(\omega^{1'}\sinh\varphi + \omega^{2'}\cosh\varphi)$$

and for $h = h_{ij}\omega^i\omega^j$

$$h'_{11} = h_{11}\cosh^2\varphi + 2h_{12}\sinh\varphi\cosh\varphi + h_{22}\sinh^2\varphi,$$
$$h'_{12} = \varepsilon_1\varepsilon_2[(h_{11} + h_{22})\sinh\varphi\cosh\varphi + h_{12}(\cosh^2\varphi + \sinh^2\varphi)],$$
$$h'_{22} = h_{11}\sinh^2\varphi + 2h_{12}\sinh\varphi\cosh\varphi + h_{22}\cosh^2\varphi.$$

Denoting $\frac{1}{2}(h_{11} + h_{22}) = A$, $h_{12} = B$, $\frac{1}{2}(-h_{11} + h_{22}) = H$ one has

$$A' = A\cosh 2\varphi + B\sinh 2\varphi, \quad B' = \varepsilon_1\varepsilon_2(A\sinh 2\varphi + B\cosh 2\varphi), \quad H' = H,$$

thus the vector subspace of $T^\perp_x M^2_1$, spanned by A and B, is invariant and H is an invariant vector (the mean curvature vector of M^2_1). Further

$$\langle A', B'\rangle = \varepsilon_1\varepsilon_2[\langle A, B\rangle\cosh 4\varphi + \tfrac{1}{2}(A^2 + B^2)\sinh 4\varphi].$$

Since span$\{A, B\}$ lies in a Euclidean vector space, normal to M_1^2 in $N_1^n(c)$, it holds $\langle A, B\rangle^2 \leqslant A^2 \cdot B^2$. For any two real numbers a and b from $(a^2 - b^2)^2 \geqslant 0$ it follows $4a^2b^2 \leqslant (a^2 + b^2)^2$, thus $\langle A, B\rangle^2 \leqslant \frac{1}{4}(A^2 + B^2)^2$. Here the equality is equivalent to $A = \varepsilon B$, $\varepsilon = \pm 1$. Indeed, it is trivial if $A = 0$ or $B = 0$ because then both parts of the equivalence give $A = B = 0$. If $A \neq 0$, $B \neq 0$ then the equality (i.e., instead of $\leqslant$ stands $=$) yields that for $\|A\|^2 \cdot \|B\|^{-2} = \lambda$ there holds $\cos^2\alpha = \frac{1}{4}(\lambda + \lambda^{-1} + 2)$ and this gives that $\lambda^2 - 2(2\cos^2\alpha - 1)\lambda + 1 = 0$ has a real root; thus $\cos\alpha = \pm 1$ and $\lambda = 1$. The converse is obvious.

3.2. Principal case. In general $A \neq \varepsilon B$ and there exists φ_0 such that $\langle A', B'\rangle = 0$; this φ_0 is a root of $\tanh^2 4\varphi = 4\langle A, B\rangle^2(A^2 + B^2)^{-2} < 1$. The corresponding basis $\{e_1', e_2'\}$ is determined up to redirecting its vectors. Let this transformation be already done, i.e., let in the considered principal case further $\langle A, B\rangle = 0$.

In general $A \nparallel B$ and $H \notin$ span$\{A, B\}$. (Here and further $\nparallel$ and $\|$ mean, correspondingly, "non-collinear" and "collinear".) The first three basic vectors in $T_xM_1^2$ can be taken so that $A = \frac{1}{2}(h_{11} + h_{22}) = ae_3$, $B = h_{12} = be_4$, $H = \frac{1}{2}(-h_{11} + h_{22}) = \alpha e_3 + \beta e_4 + \gamma e_5$, where $a > 0$, $b > 0$, and then

$$h_{11} = (a - \alpha)e_3 - \beta e_4 - \gamma e_5, \quad h_{12} = be_4, \quad h_{22} = (a + \alpha)e_3 + \beta e_4 + \gamma e_5. \quad (3.1)$$

The same with $\gamma = 0$ holds if $A \nparallel B$ and $H \in$ span$\{A, B\}$.

If $A \parallel B$, $A \neq \varepsilon B$ then due to $\langle A, B\rangle = 0$ either $A = \frac{1}{2}(h_{11} + h_{22}) = 0$, $B \neq 0$ or $A \neq 0$, $B = h_{12} = 0$.

In the first case, if $H \nparallel B$, then e_3 and e_4 can be taken so that $B = h_{12} = be_3$, $H = \frac{1}{2}(-h_{11} + h_{22}) = \beta e_3 + \gamma e_4$ and thus

$$h_{11} = -\beta e_3 - \gamma e_4, \quad h_{12} = be_3, \quad h_{22} = \beta e_3 + \gamma e_4, \quad b > 0; \quad (3.2)$$

this can be considered as a particular limit case of (3.1) for $a = \alpha = 0$, if the indices of e_3 and e_4 are increased by 1. The first case with $H \parallel B$ corresponds to $\gamma = 0$.

In the second case, if $H \nparallel A$ one can obtain $A = \frac{1}{2}(h_{11} + h_{22}) = ae_3$, $H = \alpha e_3 + \gamma e_4$; this is a particular limit case of (3.1) for $a > 0$, $b = \beta = 0$, if e_4 is replaced with e_5. The second case with $H \parallel A$ corresponds to $\gamma = 0$.

Hence (3.1) includes all possibilities of this principal case $A \neq \varepsilon B$ (after some renumbering, if needed).

3.3. Exceptional case. This is a case when $A = \varepsilon B$ and thus $A' = \varepsilon\varepsilon_1\varepsilon_2 B'$ for every φ. One can obtain $A' = B'$; let this be already done, i.e., let $A = B$.

If $A = B \neq 0$ and $H \nparallel B$, then e_3 and e_4 can be taken so that $H = \frac{1}{2}(-h_{11}+h_{22}) = \alpha e_3+\beta e_4$, $A = B = \frac{1}{2}(h_{11} + h_{22}) = h_{12} = be_4$, $b > 0$, thus

$$h_{11} = -\alpha e_3 + (b - \beta)e_4, \quad h_{12} = be_4, \quad h_{22} = \alpha e_3 + (b + \beta)e_4. \quad (3.3)$$

If $A = B \neq 0$ and $H \parallel B$ then here $\alpha = 0$ (and it is suitable to replace e_4 with e_3).

The subcase $A = B = 0$, $H \neq 0$ can be included into (3.3) as a limit case by $b = \beta = 0$ and the subcase $A = B = H = 0$ into the latter by $\alpha = 0$ (then $h_{ij} = 0$).

4. Proof of Theorem B

4.1. Preliminaries. In the framework of Section 2 the condition $\overline{\nabla} h = 0$ for the parallel surfaces M_1^2 in $N_1^n(c)$ is $\overline{\nabla} h_{ij}^\alpha = 0$, i.e., $h_{ijk}^\alpha = 0$. From (2.1) it follows that then

$$h_{kj}^\alpha \Omega_i^k + h_{ik}^\alpha \Omega_j^k - h_{ij}^\beta \Omega_\beta^\alpha = 0. \tag{4.1}$$

This characterizes the semi-parallel surfaces; its short operator form is $\overline{R} \cdot h = 0$.

A trivial consequence is that a M_1^2 in $N_1^n(c)$ with flat $\overline{\nabla}$ (i.e., with $\overline{R} = 0$ or, equivalently, with $\Omega_i^j = \Omega_\alpha^\beta = 0$) is semi-parallel.

Further let U be an open and dense part of M_1^2, so that on every its connected component one of the cases or subcases of the previous section holds identically. The following considerations are made on one of these components, denoted often simply by M_1^2.

4.2. Proof for principal case. In this case (3.1) yields

$$\omega_1^3 = (a - \alpha)\omega^1, \quad \omega_1^4 = -\beta\omega^1 + b\omega^2, \quad \omega_1^5 = -\gamma\omega^1, \quad \omega_1^\varrho = 0, \tag{4.2}$$

$$\omega_2^3 = (a + \alpha)\omega^2, \quad \omega_2^4 = b\omega^1 + \beta\omega^2, \quad \omega_2^5 = \gamma\omega^2, \quad \omega_2^\varrho = 0, \tag{4.3}$$

(ϱ, σ etc. run $6, \dots, n$). Hence in (2.2) and (2.3) $\Omega_1^2 = (c - a^2 + b^2 + H^2)\omega^1 \wedge \omega^2$, $H^2 = \alpha^2 + \beta^2 + \gamma^2$, $\Omega_3^4 = -\Omega_4^3 = 2ab\omega^1 \wedge \omega^2$, all other Ω_α^β are zero, and semi-parallelity conditions (4.1) reduce to

$$\begin{aligned}
&ab\beta = 0,\\
&2a(c - a^2 + 2b^2 + H^2) = 0,\\
&2b(c - 2a^2 + b^2 + a\alpha + H^2) = 0,\\
&2b(c - 2a^2 + b^2 - a\alpha + H^2) = 0.
\end{aligned}$$

The last two imply $ab\alpha = 0$. Here $ab \neq 0$ is impossible because then $\alpha = \beta = 0$, $c - a^2 + 2b^2 + \gamma^2 = c - 2a^2 + b^2 + \gamma^2 = 0$, but this leads to a contradiction $a^2 + b^2 = 0$. If $a > 0$, $b = 0$ or $a = 0$, $b > 0$ then $\Omega_1^2 = \Omega_3^4 = 0$, i.e., $\overline{\nabla}$ is flat.

4.3. Proof for exceptional case. Then (3.3) yields

$$\omega_1^3 = -\alpha\omega^1, \quad \omega_1^4 = (b - \beta)\omega^1 + b\omega^2, \quad \omega_1^\zeta = 0,$$

$$\omega_2^3 = \alpha\omega^2, \quad \omega_2^4 = b\omega^1 + (b + \beta)\omega^2, \quad \omega_2^\zeta = 0,$$

(ζ, η etc. run $5, \dots, n$); hence $\Omega_1^2 = (c + \alpha^2 + \beta^2)\omega^1 \wedge \omega^2$, $\Omega_3^4 = 0$, all other $\Omega_\alpha^\beta = 0$, thus $\nabla^\perp$ is flat. Now (4.1) reduce to $b(c + \alpha^2 + \beta^2) = 0$.

If $b > 0$, here $c + \alpha^2 + \beta^2 = 0$ and ∇ is also flat, i.e., $\overline{\nabla}$ is flat.

In the limit case $b = 0$ one can make $\beta = 0$ and $h_{11} = -\alpha e_3$, $h_{22} = \alpha e_3$, $h_{12} = 0$. By $\alpha \neq 0$ M_1^2 is totally umbilic, by $\alpha = 0$ totally geodesic.

All this proves Theorem B.

5. Existence and geometry of semi-parallel surfaces M_1^2 in $N_1^n(c)$

5.1. *Type (i).* Here $\omega_1^3 = -\alpha\omega^1$, $\omega_2^3 = \alpha\omega^2$, all other ω_i^α are zero. By exterior differentiation one obtains $d\alpha \wedge \omega^1 = d\alpha \wedge \omega^2 = 0$, thus $d\alpha = 0$. If $\alpha \neq 0$, then for $y = x + e_3/\alpha$ it follows $dy = 0$.

Let $c = 0$. The point with radius vector y in E_1^n is fixed, as well as a E_1^3, spanned by this point and vectors e_1, e_2, e_3. Thus M_1^2 is a sphere $S_1^2(r)$ with real radius $r = \alpha^{-1}$ in $E_1^3 \subset E_1^n$, or its open part. In this case M_1^2 is parallel, has nonflat ∇ with constant curvature and flat $\nabla^\perp$, and if complete, it is an orbit of a 3-parametric group of isometries of E_1^3 with a fixed point y. If $\alpha = 0$ then M_1^2 is a plane E_1^2 or its part.

Let $c \neq 0$. The point with radius vector y in E_1^{n+1} or E_2^{n+1} is fixed, as well as E_1^4 or E_2^4, spanned by this point and vectors x, e_1, e_2, e_3. This E_1^4 or E_2^4 intersects, correspondingly, $S_1^n(r)$ or $H_1^n(r)$, and similar conclusions as above can be made; the details are left to the reader.

5.2. *Type (ii), principal case.* Here the existence of semi-parallel non-parallel M_1^2 will be established and then the corresponding parallel time-like surfaces found, whose 2nd order envelope is M_1^2. For this purpose the compatibility of the Pfaff system (4.2), (4.3) is to be considered for two cases of Section 4.2. The results will be summarized below in Section 7.1, Proposition C.

Subcase (ii_1). *Existence.* Let $a = 0, b > 0$. Then it can be made $\alpha = 0$ and $c+b^2+\beta^2+\gamma^2=0$; thus such an M_1^2 can exist only for $c < 0$, i.e., in a de Sitter space-time of the 2nd kind $H_1^n(r)$ and is determined, due to (3.2), by the Pfaff system

$$\omega_1^3 = -\beta\omega^1 + b\omega^2, \quad \omega_1^4 = -\gamma\omega^1, \quad \omega_1^\zeta = 0,$$
$$\omega_2^3 = b\omega^1 + \beta\omega^2, \quad \omega_2^4 = \gamma\omega^2, \quad \omega_2^\zeta = 0$$

(ζ, η etc. run $5, \dots, n$). Exterior differentiation leads to

$$(d\beta + 2b\omega_1^2 - \gamma\omega_3^4) \wedge \omega^1 - db \wedge \omega^2 = 0,$$
$$db \wedge \omega^1 + (d\beta - 2b\omega_1^2 - \gamma\omega_3^4) \wedge \omega^2 = 0,$$
$$(d\gamma + \beta\omega_3^4) \wedge \omega^1 - b\omega_3^4 \wedge \omega^2 = 0,$$
$$b\omega_3^4 \wedge \omega^1 + (d\gamma + \beta\omega_3^4) \wedge \omega^2 = 0,$$
$$(\beta\omega_3^\zeta + \gamma\omega_4^\zeta) \wedge \omega^1 - b\omega_3^\zeta \wedge \omega^2 = 0,$$
$$b\omega_3^\zeta \wedge \omega^1 + (\beta\omega_3^\zeta + \gamma\omega_4^\zeta) \wedge \omega^2 = 0,$$

where $db = -b^{-1}(\beta d\beta + \gamma d\gamma)$. Due to Cartan's lemma

$$d\beta + 2b\omega_1^2 - \gamma\omega_3^4 = S\omega^1 + T\omega^2,$$
$$-db = T\omega^1 + U\omega^2,$$
$$-d\beta + 2b\omega_1^2 + \gamma\omega_3^4 = U\omega^1 + V\omega^2,$$

$$d\gamma + \beta\omega_3^4 = P\omega^1 + Q\omega^2, \qquad \beta\omega_3^\zeta + \gamma\omega_4^\zeta = X^\zeta\omega^1 + Y^\zeta\omega^2,$$
$$-b\omega_3^4 = Q\omega^1 - P\omega^2, \qquad -b\omega_3^\zeta = Y^\zeta\omega^1 - X^\zeta\omega^2.$$

The basis of left sides consists of $d\beta$, $d\gamma$, ω_1^2, ω_3^4, ω_3^ζ, ω_4^ζ, the rank s_1 of the polar system is $4+2(n-4) = 2(n-2)$. Among the 6 first coefficients on the right sides there exist 2 independent relations, which follow from the expression of db, thus the number of all independent coefficients is the same $s_1 = 2(n-2)$. Cartan's criterion is satisfied, this Pfaff system is compatible and determines M_1^2 in $H_1^n(r)$ with arbitrariness of $2(n-2)$ real functions of 1 argument.

Subcase (ii_1). *Geometry.* First consider the corresponding parallel time-like surface. Then the coefficients on the right hand side, as components of h_{ijk}^α, are zero. The Pfaff system for this surface consists of the previous equations and of the new ones,

$$d\beta = d\gamma = db = \omega_1^2 = \omega_3^4 = \omega_3^\zeta = \omega_4^\zeta = 0.$$

This extended system is completely integrable.

Since $d\omega^1 = 0$, $d\omega^2 = 0$, at least locally $\omega^1 = du^1$, $\omega^2 = du^2$ and for this parallel surface

$$\begin{aligned}
dx &= e_1\,du^1 + e_2\,du^2,\\
de_1 &= (cx - \gamma e_4)\,du^1 + (-\beta\,du^1 + b\,du^2)e_3,\\
de_2 &= -(cx - \gamma e_4)\,du^2 + (b\,du^1 - \beta\,du^2)e_3,\\
d(cx - \gamma e_4) &= (c + \gamma^2)\,dx,\\
de_3 &= (-\beta\,du^1 + b\,du^2)e_1 - (b\,du^1 + \beta\,du^2)e_2.
\end{aligned}$$

It is seen that this surface lies in an E_2^4 spanned by the point x and mutually orthogonal vectors $e_1, e_2, cx - \gamma e_4, e_3$, two of which are time-like, since $e_1^2 = -1$, $(cx - \gamma e_4)^2 = -(b^2 + \beta^2) < 0$, two others are space-like. The u^1- and u^2-lines are orthogonal geodesics of this surface.

The point $z \in E_2^4$ with radius vector $z = x - (c + \gamma^2)^{-1}(cx - \gamma e_4)$ is fixed for the surface, since $dz = 0$. The distance between x and z is, due to $(x - z)^2 = -(b^2 + \beta^2)^{-1}$, an imaginary constant $\tilde{r} = i(b^2+\beta^2)^{-\frac{1}{2}}$. Hence this parallel surface belongs to an $H_1^3(\tilde{r}) \subset E_2^4$. Note, that in the formulae above $cx - \gamma e_4 = \tilde{c}\tilde{x}$, where $\tilde{c} = \tilde{r}^{-2}$ and $\tilde{x} = x - z$. The asymptotic lines of the surface, with respect to the projective structure in $H_1^3(\tilde{r})$, are determined by $du^1(-\beta\,du^1 + b\,du^2) + du^2(b\,du^1 + \beta\,du^2) = 0$ or

$$\Big[\Big(b - \sqrt{b^2 + \beta^2}\Big)\,du^1 + \beta\,du^2\Big]\Big[\Big(b + \sqrt{b^2 + \beta^2}\Big)\,du^1 + \beta\,du^2\Big] = 0.$$

The tangent vectors of these lines are

$$\tilde{e}_1 = \beta e_1 - \Big(b - \sqrt{b^2 + \beta^2}\Big)e_2, \qquad \tilde{e}_2 = -\beta e_1 + \Big(b + \sqrt{b^2 + \beta^2}\Big)e_2$$

and they are linearly independent iff $\beta \neq 0$. For these vectors

$$\begin{aligned}
d\tilde{e}_1 &= \tilde{c}\tilde{x}\Big[\beta\,du^1 + \Big(b - \sqrt{b^2 + \beta^2}\Big)\,du^2\Big]\\
&\quad + e_3\sqrt{b^2 + \beta^2}\Big[\Big(b - \sqrt{b^2 + \beta^2}\Big)\,du^1 + \beta\,du^2\Big],\\
d\tilde{e}_2 &= -\tilde{c}\tilde{x}\Big[\beta\,du^1 + \Big(b + \sqrt{b^2 + \beta^2}\Big)\,du^2\Big]\\
&\quad + e_3\sqrt{b^2 + \beta^2}\Big[\Big(b + \sqrt{b^2 + \beta^2}\Big)\,du^1 + \beta\,du^2\Big].
\end{aligned}$$

On the first line $d\widetilde{x} = \beta^{-1}\widetilde{e}_1\,du^1$, $d\widetilde{e}_1 = -\beta^{-1}\widetilde{e}_1^2 \cdot \widetilde{cx}\,du^1$, on the second line $d\widetilde{x} = \beta^{-1}\widetilde{e}_2\,du^1$, $d\widetilde{e}_2 = -\beta^{-1}\widetilde{e}_2^2 \cdot \widetilde{cx}\,du^1$, where $\widetilde{e}_1^2 = 2b(b - \sqrt{b^2+\beta^2}) < 0$, $\widetilde{e}_2^2 = 2b(b + \sqrt{b^2+\beta^2}) > 0$. It is seen that these asymptotic lines are geodesics of $H_1^3(\widetilde{r})$, i.e., the straight lines of the projective structure of $H_1^3(\widetilde{r})$; hence the considered parallel surface is projectively a quadric, but from the point of view of the metric in $H_1^3(\widetilde{r})$ it is an orbit of a 2-parametric subgroup of isometries of $H_1^3(\widetilde{r})$.

The limit case, when $\beta = 0$ and the parallel surface in $H_1^3(\widetilde{r})$ is a minimal surface of $H_1^3(\widetilde{r})$, is considered further in Section 6.

Independently from β the semi-parallel M_1^2 in $H_1^n(r)$ of the subcase (ii$_1$) is a 2nd order envelope of these parallel surfaces.

Subcase (ii$_2$). Existence. Let $a > 0, b = 0$. Then it can be made $\beta = 0$ and $c - a^2 + \alpha^2 + \gamma^2 = 0$. This semi-parallel M_1^2 can exist in $N_1^n(c)$ by every value of c and is determined by the Pfaff system

$$\omega_1^3 = (a-\alpha)\omega^1, \quad \omega_1^4 = -\gamma\omega^1, \quad \omega_1^\zeta = 0,$$
$$\omega_2^3 = (a+\alpha)\omega^2, \quad \omega_2^4 = \gamma\omega^2, \quad \omega_2^\zeta = 0.$$

Similar differential prolongation as above leads to

$$d(a-\alpha) + \gamma\omega_3^4 = S\omega^1 + T\omega^2,$$
$$-2a\omega_1^2 = T\omega^1 + U\omega^2,$$
$$d(a+\alpha) - \gamma\omega_3^4 = U\omega^1 + V\omega^2,$$

$$d\gamma - (a-\alpha)\omega_3^4 = P\omega^1, \qquad d\gamma + (a+\alpha)\omega_3^4 = Q\omega^2,$$
$$(a-\alpha)\omega_3^\zeta - \gamma\omega_4^\zeta = X^\zeta\omega^1, \qquad (a+\alpha)\omega_3^\zeta + \gamma\omega_4^\zeta = Y^\zeta\omega^2$$

and the result is the same: M_1^2 in $N_1^n(c)$ exists with the arbitrariness of $2(n-2)$ real functions of one argument.

Subcase (ii$_2$). Geometry. For the corresponding parallel surface the equations $da = d\alpha = d\gamma = \omega_1^2 = \omega_3^4 = \omega_3^\zeta = \omega_4^\zeta = 0$ are to be added. The extended system is completely integrable and yields

$$dx = e_1\,du^1 + e_2\,du^2,$$
$$de_1 = [ae_3 + (cx - H)]\,du^1, \qquad d[ae_3 + (cx - H)] = 2a(a-\alpha)e_1\,du^1,$$
$$de_2 = [ae_3 - (cx - H)]\,du^2, \qquad d[ae_3 - (cx - H)] = -2a(a+\alpha)e_2\,du^2.$$

The vectors in square brackets are orthogonal due to $a^2 - (c + H^2) = 0$, thus in general this parallel surface lies in a E_1^4, $E_{1,1}^4$ or E_2^4, spanned by x and the mutually orthogonal vectors e_1, e_2, e_3, $cx - \gamma e_4$, and in the case of E_1^4 or E_2^4 is a product of two plane lines of constant curvature. Here $(cx - \gamma e_4)^2 = c + \gamma^2 = \alpha^2 - a^2$ can be positive, zero or negative and $d(cx - \gamma e_4) = (c + \gamma^2)\,dx$.

If $c + \gamma^2 \neq 0$, the point z with radius vector $z = x - (c+\gamma^2)^{-1}(cx - \gamma e_4)$ in E_1^4 or E_2^4 is a fixed point, since $dz = 0$. The distance between x and z, is due to $(x - z)^2 = (c+\gamma^2)^{-1}$, a constant $\widetilde{r}$, real for E_1^4 and imaginary for E_2^4. Hence this parallel surface belongs to an $S_1^3(\widetilde{r}) \subset E_1^4$ or an $H_1^3(\widetilde{r}) \subset E_2^4$.

If $c + \gamma^2 = 0$ then $\alpha^2 = a^2$ and $d(cx - \gamma e_4) = 0$. Let, for example $\alpha = a$; then $de_1 = (cx - \gamma e_4)\,du^1$, $de_2 = [-(cx - \gamma e_4) + 2ae_3]\,du^2$. The lines $u^2 = \text{const}$ on the parallel surface have $dx/du^1 = e_1$, $d^2x/(du^1)^2 = k_1 = \text{const}$ $(= cx - \gamma e_4)$, thus $x = \frac{1}{2}k_1(u^1)^2 + k_2u^1 + k_3$ and they are congruent parabolas which differ only by translations. The lines $u^1 = \text{const}$ are due to $d[-(cx - \gamma e_4) + 2ae_3] = -4a^2 e_2\, du^2$ the congruent circles of the curvature $2a$ on the parallel planes E^2, spanned by x, e_2 and $-(cx - \gamma e_4) + 2ae_3$. Hence the parallel surface is a translation surface of a parabola on $E^2_{1,1}$ and a circle on E^2 with totally orthogonal $E^2_{1,1}$ and E^2 in $E^4_{1,1}$.

The semi-parallel M^2_1 of this subcase (ii$_2$) is a 2nd order envelope of these parallel product- or translation surfaces. Its h is diagonalizable, thus it has a set of curvature lines, enveloped by the generating plane lines of these parallel surfaces.

5.3. *Type (ii), exceptional case.* This is the case of Section 4.3 for $b \neq 0$, $c + \alpha^2 + \beta^2 = 0$. The corresponding M^2_1 can exist only if $c \leqslant 0$, i.e., $S^n_1(r)$ is excluded.

The Pfaff system in Section 4.3 gives, after exterior differentiation and some rearrangement,

$$
\begin{aligned}
&(d\alpha - \beta\omega^4_3) \wedge (\omega^1 + \omega^2) = 0,\\
&(d\beta + \alpha\omega^4_3) \wedge (\omega^1 + \omega^2) = 0,\\
&(d\alpha - \beta\omega^4_3) \wedge (\omega^1 - \omega^2) + 2b\omega^4_3 \wedge (\omega^1 + \omega^2) = 0,\\
&(d\beta + \alpha\omega^4_3) \wedge (\omega^1 - \omega^2) - 2(db - 2b\omega^2_1) \wedge (\omega^1 + \omega^2) = 0,\\
&(\alpha\omega^\zeta_3 + \beta\omega^\zeta_4) \wedge (\omega^1 + \omega^2) = 0,\\
&(\alpha\omega^\zeta_3 + \beta\omega^\zeta_4) \wedge (\omega^1 - \omega^2) - 2b\omega^\zeta_4 \wedge (\omega^1 + \omega^2) = 0,
\end{aligned}
$$

where $\alpha(d\alpha - \beta\omega^4_3) + \beta(d\beta + \alpha\omega^4_3) = 0$, due to $c + \alpha^2 + \beta^2 = 0$.

Subcase (ii$_3$). Existence. Let $\alpha\beta \neq 0$. Then $c < 0$. The basic forms are $\overset{*}{\omega}{}^1 = \omega^1 + \omega^2$, $\overset{*}{\omega}{}^2 = \omega^1 - \omega^2$ and a basis of secondary forms consists of $d\alpha$, ω^4_3, $db - 2b\omega^2_1$, ω^ζ_3 and ω^ζ_4. The rank of the polar system is $s_1 = 3 + 2(n-4) = 2n - 5$. Due to the Cartan's lemma

$$
\begin{aligned}
&d\alpha - \beta\omega^4_3 = P\,\overset{*}{\omega}{}^1,\\
&2b\omega^4_3 = Q\,\overset{*}{\omega}{}^1 + P\,\overset{*}{\omega}{}^2,\\
&2\beta(db - 2b\omega^2_1) = S\,\overset{*}{\omega}{}^1 + \alpha P\,\overset{*}{\omega}{}^2,\\
&\alpha\omega^\zeta_3 + \beta\omega^\zeta_4 = X^\zeta\,\overset{*}{\omega}{}^1,\\
&2b\omega^\zeta_4 = Y^\zeta\,\overset{*}{\omega}{}^1 - X^\zeta\,\overset{*}{\omega}{}^2.
\end{aligned}
$$

The number of independent coefficients on the right hand sides is the same $2n - 5$. Cartan's criterion is satisfied, this Pfaff system is compatible and determines the considered M^2_1 in $H^n_1(r)$ with arbitrariness of $2n - 5$ real functions of 1 argument.

Subcase (ii$_3$). Geometry. Due to the system of Section 4.3 for M^2_1

$$
\begin{aligned}
dx &= e_1\omega^1 + e_2\omega^2,\\
de_1 &= cx\omega^1 + e_2\omega^2_1 - \alpha e_3\omega^1 + e_4[(b - \beta)\omega^1 + b\omega^2],\\
de_2 &= -cx\omega^2 + e_1\omega^2_1 + \alpha e_3\omega^2 + e_4[b\omega^1 + (b + \beta)\omega^2],
\end{aligned}
$$

thus the lines with $\omega^2 = -\omega^1$ on M^2_1 are light-like straight lines, because for them $dx = (e_1 - e_2)\omega^1$, $d(e_1 - e_2) = -(e_1 - e_2)\omega^2_1$, $(e_1 - e_2)^2 = 0$. Hence M^2_1 is a ruled surface with light-like generators in E_2^{n+1} and lies in $H_1^n(r) \subset E_2^{n+1}$. For the corresponding parallel surface the following equations are to be added:

$$d\alpha = d\beta = \omega_3^4 = db - 2b\omega_1^2 = \omega_3^\zeta = \omega_4^\zeta = 0.$$

So the extended system is completely integrable.

This parallel surface belongs to E_2^4 spanned by x and mutually orthogonal vectors e_1, e_2, $cx - \alpha e_3$, e_4 with $e_1^2 = -1$, $e_2^2 = e_4^2 = 1$, $(cs - \alpha e_3)^2 = c + \alpha^2 = -\beta^2 < 0$. Moreover, $d(cx - \alpha e_3) = (c + \alpha^2)\,dx = -\beta^2\,dx$.

The point z with radius vector $z = x + \beta^{-2}(cx - \alpha e_3)$ is a fixed point for this parallel surface in the constant imaginary distance $i|\beta|^{-1}$ from x. Hence the surface lies on $H_1^3(\bar{r})$.

Its asymptotic lines with respect to the projective structure in $H_1^3(\bar{r})$ are determined by $h_{ij}\omega^i\omega^j = 0$, i.e., by

$$(\omega^1 + \omega^2)[(b - \beta)\omega^1 + (b + \beta)\omega^2] = 0.$$

One family consists of light-like straight lines, considered above. A line of the other family has the tangent vector $(b + \beta)e_1 - (b - \beta)e_2$ with scalar square $-2\beta b$ and with

$$d[(b + \beta)e_1 - (b - \beta)e_2] = -\beta^2(x - z)\theta + [(b + \beta)e_1 - (b - \beta)e_2]\omega_1^2,$$

hence these lines are geodesics of $H_1^3(\bar{r})$, i.e., the straight lines of the projective structure of $H_1^3(\bar{r})$, and the considered parallel surface is projectively a quadric.

The semi-parallel M_1^2 in E_1^n or $H_1^n(r)$ of the subcase (ii$_3$) is a 2nd order envelope of these parallel surfaces.

Subcase (ii$_4$). Let $\alpha = 0$, $\beta \neq 0$. Then $c + \alpha^2 + \beta^2 = 0$ gives $\beta^2 = -c = \text{const}$. In (3.3) there is suitable to replace e_4 by e_3 and thus the corresponding Pfaff system is

$$\omega_1^3 = (b - \beta)\omega^1 + b\omega^2, \quad \omega_1^\zeta = 0,$$
$$\omega_2^3 = b\omega^1 + (b + \beta)\omega^2, \quad \omega_2^\zeta = 0,$$

(ζ runs $4, \ldots, n$). Exterior differentiation gives now

$$(db - 2b\omega_1^2) \wedge (\omega^1 + \omega^2) = 0, \quad \omega_3^\zeta \wedge \omega_1^3 = 0, \quad \omega_3^\zeta \wedge \omega_2^3 = 0,$$

hence $db - 2b\omega_1^2 = P(\omega^1 + \omega^2)$, $\omega_3^\zeta = 0$, because ω_1^3 and ω_2^3 are linearly independent since $(b - \beta)(b + \beta) - b^2 = -\beta^2 \neq 0$. The next step gives $(dP - 3P\omega_1^2) \wedge (\omega^1 + \omega^2) = 0$. All this shows that in this subcase the semi-parallel surfaces M_1^2 with flat $\bar{\nabla}$ exist with arbitrariness of 1 real function of 1 argument and every of them lies in an $H_1^3(r) \subset E_2^4$, where E_2^4 is spanned by the point x and vectors e_1, e_2, cx, e_3.

The geometry of the corresponding parallel surface is the same as for the parallel surface of the subcase (ii$_3$).

The semi-parallel M_1^2 of subcase (ii$_4$) in $H_1^3(r)$ is a 2nd order envelope of the parallel quadrics, every of which has two families of straight generators, one of them consisting of light-like straight lines. This M_1^2, like in the case (ii$_3$), is a ruled surface with light-like generators.

Subcase (ii$_5$). Existence. Let $\alpha \neq 0$, $\beta = 0$. Then $c + \alpha^2 = 0$, thus $c = -\alpha^2 < 0$ and $d\alpha = 0$. The system in Section 4.3 reduces to $\omega_1^3 = -\alpha\omega^1$, $\omega_2^3 = \alpha\omega^2$, $\omega_1^4 = \omega_2^4 = b(\omega^1 + \omega^2)$ and after exterior differentiation gives

$$\omega_3^4 \wedge \overset{*}{\omega}{}^1 = 0, \qquad 2(db - 2b\omega_1^2) \wedge \overset{*}{\omega}{}^1 - \alpha\omega_3^4 \wedge \overset{*}{\omega}{}^2 = 0,$$

$$\omega_3^\zeta \wedge \overset{*}{\omega}{}^1 = 0, \qquad 2b\omega_4^\zeta \wedge \overset{*}{\omega}{}^1 - \alpha\omega_3^\zeta \wedge \overset{*}{\omega}{}^2 = 0,$$

where $\overset{*}{\omega}{}^1 = \omega^1 + \omega^2$, $\overset{*}{\omega}{}^2 = \omega^1 - \omega^2$. A basis of secondary forms consists of ω_3^4, $db - 2b\omega_1^2$, ω_3^ζ and ω_4^ζ and the rank of the polar system is $s_1 = 2 + 2(n-4) = 2(n-3)$. Due to Cartan's lemma $\omega_3^4 = Q\overset{*}{\omega}{}^1$, $2(db - 2b\omega_1^2) = U\overset{*}{\omega}{}^1 - \alpha Q\overset{*}{\omega}{}^2$, $\alpha\omega_3^\zeta = X^\zeta\overset{*}{\omega}{}^1$, $2b\omega_4^\zeta = Y^\zeta\overset{*}{\omega}{}^1 - X^\zeta\overset{*}{\omega}{}^2$; here the number of coefficients is the same $2(n-3)$. The considered M_1^2 in $H_1^n(r)$ exists with the arbitrariness of $2(n-3)$ real functions of 1 argument.

Subcase (ii_5). *Geometry.* Here the first part is the same as in subcase (ii_3). The one difference is that now among the mutually orthogonal vectors $e_1, e_2, e_4, cx - \alpha e_3$ the latter is light-like. Thus the corresponding parallel surface lies is a semi-euclidean space $E_{1,1}^4$ and for it $d(cx - \alpha e_3) = 0$.

This parallel surface is a ruled surface with light-like generators, which is an algebraic surface of $E_{1,1}^4$ (see below Section 7, Remark 1) but not a quadric.

The semi-parallel M_1^2 of the considered subcase (ii_5) is a 2nd order envelope of these algebraic ruled parallel surfaces and is a ruled surface in $H_1^n(r)$ with light-like generators.

6. Minimal semi-parallel M_1^2 in $N_1^n(c)$

6.1. Exceptional case. Let here M_1^2 be of type (i); then it is minimal iff this M_1^2 is a totally geodesic surface.

For type (ii) in exceptional case one more subcase (ii_6) is to be considered, when $\alpha = \beta = 0$ and thus $c = 0$, i.e., when M_1^2 is a minimal surface of E_1^n. Then the Pfaff system is the same as by (ii_4), but with $\beta = 0$, i.e., now $\omega_1^3 = \omega_2^3 = b(\omega^1 + \omega^2)$, $\omega_1^\zeta = \omega_2^\zeta = 0$. After exterior differentiation it gives

$$(db - 2b\omega_1^2) \wedge (\omega^1 + \omega^2) = 0, \qquad \omega_3^\zeta \wedge (\omega^1 + \omega^2) = 0.$$

This minimal M_1^2 in E_1^n exists with arbitrariness of $n-2$ real functions of 1 argument. For this M_1^2

$$dx = e_1\omega^1 + e_2\omega^2, \qquad de_1 = e_2\omega_1^2 + be_3(\omega^1 + \omega^2),$$

$$de_2 = e_1\omega_1^2 + be_3(\omega^1 + \omega^2), \qquad de_3 = b(e_1 - e_2)(\omega^1 + \omega^2),$$

thus $d(e_1 - e_2) = -(e_1 - e_2)\omega_1^2$. It follows, that this minimal M_1^2 in E_1^n is a cylinder with light-like generators in the direction of $e_1 - e_2$.

Here, recall, $\Omega_1^2 = 0$, i.e., $d\omega_1^2 = 0$. Hence at least locally $\omega_1^2 = d\lambda$ on this M_1^2. Further, $d(e^\lambda \overset{*}{\omega}{}^1) = 0$, $d(e^{-\lambda}\overset{*}{\omega}{}^2) = 0$, where $\overset{*}{\omega}{}^1 = \omega^1 + \omega^2$, $\overset{*}{\omega}{}^2 = \omega^1 - \omega^2$ and at least locally $e^\lambda \overset{*}{\omega}{}^1 = du$, $e^{-\lambda}\overset{*}{\omega}{}^2 = dv$. Denoting $\frac{1}{2}e^{-\lambda}(e_1 + e_2) = e_1^*$, $\frac{1}{2}e^\lambda(e_1 - e_2) = e_2^*$, one obtains

$$dx = e_1^*\,du + e_2^*\,dv, \qquad de_1^* = b_0 e_3\,du, \qquad de_2^* = 0,$$

where $b_0 = e^{-2\lambda}b$. It is seen that v-lines are the light-like generators of this cylinder M_1^2, but u-lines are its light-like geodesics.

For the corresponding parallel surface $db - 2b\omega_1^2 = 0$, $\omega_3^\zeta = 0$. Thus this surface lies in a E_1^3 and on it $db_0 = 0$. The system above can be now integrated. Namely, $e_2^* = \partial x/\partial v$ is a constant vector p_0, thus $x = p_0 v + q(u)$. Further, $e_1^* = \partial x/\partial u = \dot{q}$ and $\ddot{q} = \partial e_1^*/\partial u = b_0 e_3$. Since now $de_3 = 2b_0 p_0\, du$, so $\dddot{q} = 2b_0^2 p_0$ and $q = \frac{1}{3} b_0^2 p_0 u^3 + \frac{1}{2} p_1 u^2 + p_2 u + p_3$. Consequently

$$x = p_0(v + \tfrac{1}{3} b_0 u^3) + \tfrac{1}{2} p_1 u^2 + p_2 u + p_3.$$

The line, determined by $v + \frac{1}{3} b_0 u^3 = 0$ on this minimal parallel cylinder, is a parabola. Thus this cylinder with light-like generators in E_1^3 is errected on a parabola.

The minimal semi-symmetric cylindrical M_1^2 of this subcase (ii$_6$) is a 2nd order envelope of these parabolic cylinders.

6.2. Principal case. Here two subcases (ii$_1$) and (ii$_2$) are to be considered. It turns out that in both these subcases minimal semi-parallel M_1^2 in $N_1^n(c)$ exist and are parallel surfaces in $H_1^3(r)$ and $S_1^3(r)$, respectively.

For the subcase (ii$_1$) the condition $H = 0$ implies $\beta = \gamma = 0$, $b = \sqrt{-c}$ and then $T = U = P = Q = X^\zeta = Y^\zeta = 0$. The first two give $2b\omega_1^2 = S\omega^1 = V\omega^2$, thus $\omega_1^2 = 0$; the other yield $\omega_3^4 = \omega_3^\zeta = 0$. As the result $dx = e_1\omega^1 + e_2\omega^2$, $de_1 = cx\omega^1 + be_3\omega^2$, $de_2 = -cx\omega^2 + be_3\omega^1$, $de_3 = b(e_1\omega^2 - e_2\omega^1)$. Due to $b^2 = -c$ this Pfaff system is completely integrable and determines, up to congruence, a minimal parallel M_1^2 in $H_1^3(r)$. This M_1^2 is an orbit of the group $O(2) \times O^1(2)$ and carries an orthogonal net of geodesic lines of $H_1^3(r)$ (i.e., straight lines in the inner geometry of $H_1^3(r)$; one family is determined by $\omega^2 = 0$, the other by $\omega^1 = 0$). Hence M_1^2 is a quadric.

This is the particular case of (ii$_1$), namely the case of the parallel surface with $\beta = 0$ (see Section 5, Subcase (ii$_1$), Geometry).

For the subcase (ii$_2$) the condition $H = 0$ implies $\alpha = \gamma = 0$, $a = \sqrt{c}$ and then $S = T = U = V = 0$, thus $\omega_1^2 = 0$; further $a\omega_3^4 = -P\omega^1 = Q\omega^2$, $a\omega_3^\zeta = X^\zeta\omega^1 = Y^\zeta\omega^2$, hence $P = Q = X^\zeta = Y^\zeta = 0$ and $\omega_3^4 = \omega_3^\zeta = 0$. As a result $dx = e_1\omega^1 + e_2\omega^2$, $de_1 = cx\omega^1 + ae_3\omega^1$, $de_2 = -cx\omega^2 + ae_3\omega^2$, $de_3 = a(e_1\omega^1 - e_2\omega^2)$. Due to $a^2 = c$ this Pfaff system is completely integrable and determines, up to congruence, a minimal parallel M_1^2 in $S_1^3(r)$. This M_1^2 is an orbit of the group $O^1(2) \times O(2)$ and carries an orthogonal net of its geodesic curvature lines, which in the inner geometry of $S_1^3(r)$ are the plane lines of constant curvature; one family is determined by $\omega^2 = 0$, the other by $\omega^1 = 0$.

7. Propositions and concluding remarks

7.1. Surfaces with flat $\bar{\nabla}$. The results of Section 5, concerning the non-minimal surfaces M_1^2 of Theorem B, type (ii), i.e., non-minimal surfaces M_1^2 with flat $\bar{\nabla}$ in $N_1^n(c)$, can be summarized now as follows.

Proposition C. *Let M_1^2 be a non-minimal time-like surface with flat $\bar{\nabla}$ in a Lorentzian spacetime form $N_1^n(c)$. There exists an open and dense part U of M_1^2 such that every connected component of U is of one of the following types.*

(ii$_1$) *A 2nd order envelope in $H_1^n(r)$ of parallel surfaces, every of which is projectively a ruled quadric in some $H_1^3(\tilde{r})$, whose one family of generators consists of time-like, the other of space-like geodesics of $H_1^3(\tilde{r})$.*

(ii$_2$)$'$ *A 2nd order envelope in E_1^n or $S_1^n(r)$ of parallel surfaces, every of which lies in a $S_1^3(\tilde{r}) \subset E_1^4$ and is in E_1^4 a product of two plane lines of constant curvature, one time- and the other* *space-like (i.e., the latter is a circle in a E^2).*

(ii$_2$)″ *A 2nd order envelope in $H_1^n(r)$ of parallel surfaces, every of which either lies in a $S_1^3(\tilde{r}) \subset E_1^4$ or in a $H_1^3(\tilde{r}) \subset E_2^4$ and is in E_1^4 or E_2^4 a product of two plane lines of constant curvature, one time- and other space-like, or lies in a $E_{1,1}^4$ and is a translation surface of a time-like parabola with light-like diameters on a plane $E_{1,1}^2$ and of a circle on a plane E^2, orthogonal to this $E_{1,1}^2$.*

(ii$_3$) & (ii$_4$) *A 2nd order envelope in $H_1^n(r)$ of parallel surfaces, every of which is projectively a ruled quadric in some $H_1^3(\tilde{r})$, whose one family of generators consists of light-like, the other of time- or space-like geodesics of $H_1^3(\tilde{r})$. This envelope is a part of a ruled surface with light-like generators (straight lines of E_2^{n+1}, lying in $H_1^n(r)$).*

(ii$_5$) *A 2nd order envelope in $H_1^n(r)$ of parallel surfaces, every of which lies in a $E_{1,1}^4$ and is a certain ruled algebraic non-quadric surface with light-like generators (see Remark 1 below). This envelope is a part of a ruled surface with generating light-like straight lines of E_2^{n+1}.*

Remark 1. Note that Theorem B, type (i), and Proposition C together list also all cases for semi-parallel time-like surfaces in Lorentzian space forms $N_1^n(c)$, except the minimal ones, considered separately in Section 6. Some of them (e.g., lying in $E_{1,1}^4$) were determined previously in [13].

The only assertion, which is not proved above, concerns the parallel surface of the last subcase (ii$_5$). It remains to show that it is an algebraic non-quadric surface and to determine this surface. This can be done if to integrate the system of the corresponding derivation equations, which in notation of Section 6.1 is as follows (see the system of these equations of subcase (ii$_3$), where now $\alpha^2 = -c = \text{const}$, $\beta = 0$ and for a parallel surface $b_0 = e^{-2\lambda}b = \text{const}$):

$$dx = e_1^*\,du + e_2^*\,dv, \qquad d(cx - \alpha e_3) = 0,$$
$$de_1^* = \tfrac{1}{2}(cx - \alpha e_3)\,dv + b_0 e_4\,du, \qquad de_2^* = \tfrac{1}{2}(cx - \alpha e_3)\,du,$$
$$de_4 = 2b_0 e_2^*\,du.$$

It follows that $e_2^* = \partial x/\partial v$ and $\partial e_2^*/\partial v = 0$; thus $\partial x/\partial v = y(u)$ and $x = y(u)v + z(u)$. Further, $\partial e_2^*/\partial u = \frac{1}{2}(cx - \alpha e_3)$ and so $\partial^2 e_2^*/\partial u\partial u = 0$. This gives $\ddot{y} = 0$, consequently $y = p_1 u + p_2$, where p_1 and p_2 are some constant vectors of E_2^{n+1}. Similarly, $e_1^* = \partial x/\partial u$, $\partial e_1^*/\partial u = b_0 e_4$, $\partial^2 e_1^*/\partial u\partial u = 2b_0^2 y$, hence $\ddot{z} = 2b_0^2(p_1 u + p_2)$ and so

$$x = (p_1 u + p_2)v + 2b_0^2(\tfrac{1}{24}p_1 u^4 + \tfrac{1}{6}p_2 u^3 + \tfrac{1}{2}p_3 u^2 + p_4 u + p_5),$$

where p_3, p_4 and p_5 are some new constant vectors.

Here the surface is contained in $H_1^n(r)$, thus $\langle x, x\rangle = c^{-1}$ is satisfied identically with respect to u and v. This yields that the matrix of $\langle p_\varphi, p_\psi\rangle$ (φ and ψ run $1, \ldots, 5$), is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \varkappa \\ 0 & 0 & 0 & -\varkappa & 0 \\ 0 & 0 & \varkappa & 0 & 0 \\ 0 & -\varkappa & 0 & 0 & 0 \\ \varkappa & 0 & 0 & 0 & k \end{pmatrix},$$

where $k = (4cb_0^2)^{-1}$ and $\varkappa$ is a nonzero real number. The coordinates of $x - 2b_0^2 p_5$ with respect to the basis $\{p_1, p_2, p_3, p_4\}$ satisfy $4b_0^2\zeta_3 - \zeta_4^2 = 0$, $4b_0^2\zeta_1 - 2\zeta_2\zeta_4 + \zeta_3^2 = 0$, thus this parallel surface in $E^4_{1,1}$ (see Section 5.3, Subcase (ii_5). Geometry) is an algebraic surface, an intersection of two quadrics.

7.2. Minimal semi-parallel surfaces. The results in Section 6 can be summarized as follows.

Proposition D. *Let M^2_1 be a minimal semi-parallel time-like surface in a Lorentzian spacetime form $N^n_1(c)$. There exists an open and dense part U of M^2_1 such that every connected component of U is of one of the following types.*

(i)$^{\min}$ *A totally geodesic surface.*

(ii_1)$^{\min}$ *A ruled quadric in $H^3_1(r)$, whose generating net of geodesic lines of $H^3_1(r)$ is orthogonal, or its open part.*

(ii_2)$^{\min}$ *A surface in $S^3_1(r)$, whose curvature lines are geodesics and have the same constant curvature (a product of these lines in E^4_1).*

(ii_6)$^{\min}$ *A cylinder in E^n_1 with light-like generators or its open part.*

These components are parallel, except $(ii_6)^{min}$, which can be non-parallel; if parallel, every of the latter is a parabolic cylinder with light-like generators in E^3_1.

Remark 2. The fact, that a time-like surface with flat $\overline{\nabla}$ in E^3_1 is minimal iff it is a cylinder with light-like generators, is proved in [20, Theorem 3] (see also [19], where among the ruled minimal time-like surfaces in E^3_1 the so called flat B-scrolls over light-like curves are considered; actually they give these cylinders.) Now this fact is generalized to the higher codimension, i.e., to the case of E^n_1. Also the corresponding parallel surfaces are determined (described in [13] as B-scrolls over the null (i.e., light-like) cubics; cf. above Section 6.1).

Remark 3. The special study of some minimal time-like surfaces in Section 6 is motivated by the possible applications in the geometrical string theory. The minimal time-like surfaces, considered here, are simple models of strings, which play an important role in the theoretical particle physics [6] and cosmology [22].

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