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Fourier multiplier theorem for atomic Hardy spaces on unbounded Vilenkin groups ☆

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ABSTRACT

We characterize atomic Hardy spaces on unbounded locally compact Vilenkin groups by means of a modified maximal function. The obtained Fourier multiplier theorem is more general than the corresponding results due to Kitada, Onneweer and Quek, Daly and Phillips that were proved under the boundedness assumption on the underlying group.

1. Introduction

We denote by G a locally compact Abelian group that contains a strictly decreasing sequence of open compact subgroups $(G_n)_{n=-\infty}^{\infty}$ such that $\bigcup_{n=-\infty}^{\infty} G_n = G$, and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$. By Γ we denote the dual group of G. It is the union of the increasing sequence of its subgroups $\Gamma_n = \{\gamma \in \Gamma\colon\ \gamma(x) = 1,\ \forall x \in G_n\}$.

Let μ, λ be the Haar measures on G and Γ respectively. These are chosen so that $\mu(G_0) = \lambda(\Gamma_0) = 1$, and $\mu(G_n) = (\lambda(\Gamma_n))^{-1} = m_n^{-1}$.

Take a fixed sequence of elements $(x_n)_n$ of G such that $x_n \in G_n \backslash G_{n+1}$ for every $n \in \mathbb{Z}$. Then every $x \in G$ can be expressed in the form $x = \sum_{n=M}^{+\infty} a_n x_n$, where $a_M \neq 0$ if $x \in G_M \backslash G_{M+1}$. If we put $p_{n+1} = |G_n/G_{n+1}|$, then $m_n = p_1 p_2 \dots p_n$ and $m_{-n}^{-1} = p_0 p_{-1} p_{-2} \dots p_{-n+1}$ for $n \geqslant 1$.

The convention $G_n = G$ if $n \leqslant 0$ is used when G is compact.

The group G is said to be bounded if $\sup_n p_n < \infty$.

Definition 1.1. A complex function a is called an atom on G if

1. $\operatorname{supp}(a) \subset y + G_n$,
2. $\|a\|_\infty \leqslant \frac{1}{\mu(G_n)}$,
3. $\int_G a(x)\,dx = 0$.

The atomic Hardy space H^1 consists of integrable functions f which can be represented as $f = \sum_{i=1}^{\infty} \lambda_i a_i$, where each a_i is an atom and $\sum_{i=1}^{\infty} |\lambda_i| < +\infty$. The norm in H^1 is given by $\|f\|_{H^1} = \inf \sum_{i=1}^{\infty} |\lambda_i|$, where the infimum is taken over all such decompositions of f.

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Definition 1.2. For any distribution f, let

1. $Mf(x) = \sup_n |f * (\mu(G_n))^{-1} 1_{G_n}(x)|$, and
2. $\tilde{M}f(x) = \sup_{n,I_n} |f * (\mu(I_n))^{-1} 1_{I_n}(x)|$,

where I_n is an interval of the form $I_n = \biguplus_{i=\alpha}^{\beta} ix_n + G_{n+1}$, $0 \leqslant \alpha \leqslant \beta < p_{n+1}$.

The spaces H and $\tilde{H}$ consist of all distributions f such that $Mf \in L^1$ resp. $\tilde{M}f \in L^1$. The norms are given by $\|f\|_H = \|Mf\|_1$ resp. $\|f\|_{\tilde{H}} = \|\tilde{M}f\|_1$.

The relationship between these concepts is the matter of our attention in Section 2. In Section 3 we prove a Fourier multiplier theorem that encompasses several previous theorems of this kind.

2. On the spaces H^1, $\tilde{H}$ and H

For a short history of the Hardy spaces over Vilenkin groups one should consult [5]. Onneweer and Quek [10] proved that $H = H^1$ on bounded Vilenkin groups.

The space $\tilde{H}$ was introduced by S. Simon in [11] for compact Vilenkin groups. The maximal function used there is smaller than $\tilde{M}f$, but generates the same space. See also [6] and [12] for some further useful insights.

Considering some compact unbounded Vilenkin groups, Gat [7] constructed a function from H that does not belong to $\tilde{H}$, concluding that the atomic and maximal function definitions do not agree in this case.

In the following theorem we prove that the situation remains the same in our general setting, i.e. these relations extend to the case of locally compact groups as well.

Theorem 2.1.

1. *$H^1 = \tilde{H}$ and the respective norms are equivalent.*
2. *$\tilde{H} \subsetneq H$ on unbounded groups.*

Proof. 1. It is easily seen that $\|\tilde{M}a\|_1 \leqslant 1$ for any atom a. Thus $H^1 \subset \tilde{H}$ and $\|f\|_{\tilde{H}} \leqslant \|f\|_{H^1}$.

For the converse, we proceed as in the proof of Theorem 3.5 in [10].

Let $f \in \tilde{H}$. For each $k \in \mathbb{Z}$ we put $\Omega_k = \{x \in G\colon\ \tilde{M}f(x) > 2^k\}$. If $y \in \Omega_k$ there exist an n and an interval I_n such that $(\mu(I_n))^{-1}|f * 1_{I_n}(y)| > 2^k$. It can be seen that $y + G_{n+1} \subset \Omega_k$ since $t - I_n = y - I_n$ for every $t \in y + G_{n+1}$ and for every interval I_n. As it was noticed in [10] there exists an $\alpha(y) \in \mathbb{Z}$ such that $y + G_{\alpha(y)} \subset \Omega_k$ but $y + G_{\alpha(y)-1} \nsubseteq \Omega_k$. There exists a maximal interval $I_{\alpha(y)-1}$ containing $y + G_{\alpha(y)}$ and included in $y + G_{\alpha(y)-1} \cap \Omega_k$. We denote these intervals by $I_{k,i}$ and their union $\bigcup_i I_{k,i}$ forms the set Ω_k. This union can be taken disjoint because of the maximality of the intervals $I_{k,i}$.

For every interval $I_{k,i}$ we consider the interval $\tilde{I}_{k,i} \supset I_{k,i}$ chosen so that $\tilde{I}_{k,i} = y + \biguplus_{j=0}^{\beta} jx_n + G_{n+1}$ if $I_{k,i} = y + \biguplus_{j=1}^{\beta} jx_n + G_{n+1}$. Then $y \in \tilde{I}_{k,i} \setminus I_{k,i}$, and $\mu(\tilde{I}_{k,i}) \leqslant 2\mu(I_{k,i})$. The intervals $\tilde{I}_{k,i}$ remain mutually disjoint because of the maximality of $I_{k,i}$. Moreover, if the interval $I_{k,i}$ is of the form $y + I_n$, then the interval $\tilde{I}_{k,i}$ must be either of the form $y' + I'_n$ or $y + G_n$. It is easily seen that if $I_{k+1,j} \subset I_{k,i}$ then also $\tilde{I}_{k+1,j} \subset \tilde{I}_{k,i}$.

Let $\tilde{\Omega}_k$ be the union of the intervals $(\tilde{I}_{k,i})_i$.

For $k, n \in \mathbb{Z}$ we define the function g_k^n on G by

$$g_k^n(x) = \begin{cases} f_n(x), & x \in G \setminus \tilde{\Omega}_k; \\ P_{k,i}^n, & x \in \tilde{I}_{k,i}, \end{cases}$$

where $f_n(x) = f * (\mu(G_n))^{-1} 1_{G_n}(x)$, and $P_{k,i}^n = (\mu(\tilde{I}_{k,i}))^{-1} \int_{\tilde{I}_{k,i}} f_n(x)\,dx$.

In order to obtain the conclusion of Theorem 3.5 in [10], we need to verify the corresponding conditions on the functions g_k^n.

We first check that $|g_k^n(x)| \leqslant 2^k$ for every $x \in G$.

Suppose that $\tilde{I}_{k,i}$ is an interval of the form $y + I_l$, where $y \in \tilde{I}_{k,i} \setminus I_{k,i}$ and let $n \leqslant l$. Then f_n is constant on $y + G_l$ and therefore also on $y + I_l$, where it is obviously bounded by 2^k. Now if $n > l$, then $x + G_n \subset I_l$ for every $x \in I_l$. Hence

$$\begin{aligned} P_{k,i}^n &= \bigl(\mu(I_l)\bigr)^{-1} \int_{I_l} f_n(y+x)\,dx \\ &= \bigl(\mu(I_l)\bigr)^{-1} \int_{I_l} \bigl(\mu(G_n)\bigr)^{-1} \int_{G_n} f(y+x+t)\,dt\,dx \end{aligned}$$

$$= \big(\mu(G_n)\big)^{-1} \int_{G_n} \big(\mu(I_l)\big)^{-1} \int_{I_l} f(y+x+t)\,dx\,dt$$
$$= \big(\mu(I_l)\big)^{-1} \int_{y+I_l} f(x)\,dx.$$

The last expression is bounded by 2^k by assumption.

We notice that $\mu(\bigcap_k \tilde{\Omega}_k) = 0$ because of the integrability of $\tilde{M}f$.

Let $b^n_{k,i} = (g^n_{k+1} - g^n_k)1_{\tilde{I}_{k,i}}$, $\lambda_{k,i} = 2^{k+2}\mu(G_n)$ if $\tilde{I}_{k,i}$ is of the form $\tilde{I}_{k,i} = y + \biguplus_{j=\alpha}^{\beta} jx_n + G_{n+1}$, and $a^n_{k,i} = (\lambda_{k,i})^{-1} b^n_{k,i}$. If we prove that $\int_G b^n_{k,i}(x)\,dx = 0$, we will have all the conditions needed and used in the proof of Theorem 3.5 in [10].

We have

$$\int_G b^n_{k,i}(x)\,dx = \int_{\tilde{I}_{k,i}} \big(g^n_{k+1} - g^n_k\big)\,dx = \int_{\tilde{I}_{k,i}} g^n_{k+1}\,dx - \int_{\tilde{I}_{k,i}} f_n(x)\,dx$$
$$= \int_{\tilde{I}_{k,i}\setminus\tilde{\Omega}_{k+1}} f_n(x)\,dx + \int_{\tilde{I}_{k,i}\cap\tilde{\Omega}_{k+1}} g^n_{k+1,i} - \int_{\tilde{I}_{k,i}} f_n(x)\,dx.$$

By assumption, if the intersection $\tilde{I}_{k,i} \cap \tilde{\Omega}_{k+1}$ is not empty then it must be equal to some union of mutually disjoint intervals $(\tilde{I}_{k+1,j})_j$. If the second integral is split into integrals over the intervals $(\tilde{I}_{k+1,j})_j$ each will be of the form

$$\int_{\tilde{I}_{k+1,j}} g^n_{k+1}(x)\,dx = \int_{\tilde{I}_{k+1,j}} f_n(x)\,dx,$$

which means that their sum will be

$$\int_{\tilde{I}_{k,i}\cap\tilde{\Omega}_{k+1}} g^n_{k+1,i} = \int_{\tilde{I}_{k,i}\cap\tilde{\Omega}_{k+1}} f_n(x)\,dx,$$

and the result follows.

2. Let G be an unbounded Vilenkin group. We may assume that the sequence $(p_n)_n$ is increasing. Let the sequence of functions $(a_n)_{n\geqslant 0}$ be defined as $a_n = m_n(1_{x_n+G_{n+1}} - 1_{2x_n+G_{n+1}})$. Obviously the functions a_n are atoms and have mutually disjoint supports.

We consider two cases: when the sequence $(p_n)_n$ is such that $\sum_{n=1}^{\infty} \frac{1}{p_n} < +\infty$ and when it is such that $\sum_{n=1}^{\infty} \frac{1}{p_n} = +\infty$.

In the first case, the function f defined by $f = \sum_{n\geqslant 0} a_n$ belongs to the space H because $Mf(x) = 0$ if $x \in G_n \setminus (G_{n+1} \cup x_n + G_{n+1} \cup 2x_n + G_{n+1})$ and $Mf(x) = m_n$ if $x \in x_n + G_{n+1} \cup 2x_n + G_{n+1}$. It follows that $\int_G Mf(x)\,dx = \sum_{n=1}^{\infty} \frac{2}{p_n} < +\infty$.

In order to see that f does not belong to $\tilde{H}$ we calculate $\tilde{M}f$. For every $x \in G_n \setminus G_{n+1}$: $\tilde{M}f(x) = m_n$. This implies that $\int_G \tilde{M}f(x)\,dx = \sum_{n=1}^{\infty} m_n(\frac{1}{m_n} - \frac{1}{m_{n+1}}) = \infty$.

For the second case, we take the function $f = \sum_{n\geqslant 0} \frac{a_n}{p_n}$. $\square$

Proposition 2.2. *The maximal function $\tilde{M}$ is a bounded operator on L^2.*

Proof. It is clear that the operator $\tilde{M}$ is strongly bounded in L^∞. By interpolation theorem, we only need to show that $\tilde{M}$ is weakly bounded in L^1.

Let $f \in L^1$, $\lambda > 0$ and $E = \{x \in G\colon\ \tilde{M}f(x) > \lambda\}$. Suppose that F is a subset of finite measure of E. If we show that F is contained in a countable collection of mutually disjoint intervals $x_i + I_{n_i}$ such that $(\mu(I_{n_i}))^{-1}|\int_{x_i+I_{n_i}} f(t)\,dt| > \lambda$ for every i, then this will imply that

$$\mu(F) \leqslant \sum_i \mu(I_{n_i}) < \lambda^{-1} \sum_i \left| \int_{x_i+I_{n_i}} f(t)\,dt \right| \leqslant \lambda^{-1}\|f\|_1.$$

We construct the countable collection as follows:

Let $x \in F$. There exists some interval $x + I_l$ such that $(\mu(I_l))^{-1}|\int_{x+I_l} f(t)\,dt| > \lambda$. The collection of such intervals need not be disjoint but we can extract a countable sub-collection considering only the intersecting intervals that are contained in some coset $x + G_n$. Namely, in the other cases, two intervals only intersect when one of them is a subset of the other.

Let i_1 be the smallest index such that the coset $x + i_1 x_n + G_{n+1}$ contains at least one point, say y_1, from F. Let I_1 be such that $(\mu(I_1))^{-1}|\int_{y_1+I_1} f(t)\,dt| > \lambda$. Continuing this way, let i_2 be the smallest index such that $x + i_2 x_n + G_{n+1}$ contains an element y_2 from F and such that $x + i_2 x_n + G_{n+1}$ does not intersect $y_1 + I_1$, and let I_2 be the corresponding interval. In this way we obtain a finite collection of mutually disjoint intervals $(y_i + I_i)_i$ that contain all the elements of F in the coset $x + G_n$ and such that $(\mu(I_i))^{-1}|\int_{y_i+I_i} f(t)\,dt| > \lambda$. □

3. Fourier multipliers

The known multiplier theorems for Hardy spaces on bounded Vilenkin groups provide sufficient conditions on the kernel of the multiplier operator. These were proved in the works of Kitada [8], Onneweer and Quek [9] and then Daly and Phillips [4]. In this note we give a weaker sufficient condition in the general setting and compare it with Daly–Phillips' result.

Theorem 3.1. *Let $\phi \in L^\infty(\Gamma)$ and $\sup_N \int_{G_N^c} |(\phi - \phi_{N+1})^\vee(y)|\,dy = O(1)$, where $\phi_{N+1} = \phi 1_{\Gamma_{N+1}}$ and $\wedge$, $\vee$ denote respectively the Fourier transform and the inverse Fourier transform. Then ϕ is a multiplier on H^1, i.e $\phi \in m(H^1)$.*

Proof. In order to prove that ϕ is a multiplier we only need to prove that the operator $Tf = (\phi f^\wedge)^\vee$ is bounded on the set of atoms. Let a be an atom. We can assume that $\mathrm{supp}(a) \subseteq G_N$ for some N because the multiplier operator T is translation invariant. We split

$$\int_G \left|\tilde{M}T(a)(x)\right| dx = \int_{G_N} \left|\tilde{M}T(a)(x)\right| dx + \int_{G_N^c} \left|\tilde{M}T(a)(x)\right| dx.$$

By Proposition 2.2 and a standard L^2 argument, we obtain that the first integral on the right-hand side is bounded by $C\|\phi\|_\infty$:

$$\begin{aligned}\int_{G_N} \left|\tilde{M}T(a)(x)\right| dx &= \int \left|\tilde{M}T(a)(x)\right| 1_{G_N}(x)\,dx \\ &\leqslant \left\|\tilde{M}T(a)\right\|_2 \|1_{G_N}\|_2 \\ &\leqslant C\left\|T(a)\right\|_2 \|1_{G_N}\|_2 \\ &\leqslant C\|\phi\|_\infty \|a\|_2 \big(\mu(G_N)\big)^{\frac{1}{2}} \\ &\leqslant C\|\phi\|_\infty \big(\mu(G_N)\big)^{\frac{-1}{2}} \big(\mu(G_N)\big)^{\frac{1}{2}} = C\|\phi\|_\infty.\end{aligned}$$

To estimate the second integral, we write $T(a)$ in the form

$$T(a) = \big(\phi a^\wedge\big)^\vee = \phi^\vee * a = \left(\sum_{j=-\infty}^{\infty} \Delta_j \phi\right)^\vee * a = \sum_{j=-\infty}^{\infty} (\Delta_j \phi)^\vee * a,$$

where the equality holds in the sense of distributions. Here $\Delta_j \phi = \phi 1_{\Gamma_{j+1}} - \phi 1_{\Gamma_j}$.

It was noticed in [4] that $(\Delta_j \phi)^\vee * a = 0$ on G_N^c if $j \leqslant N$. Since $\phi_{N+1} = \sum_{j=-\infty}^{N} \Delta_j \phi$, we have

$$T(a) = \sum_{j=N+1}^{\infty} (\Delta_j \phi)^\vee * a = (\phi - \phi_{N+1})^\vee * a, \quad \text{and}$$

$$\big(\mu(I_n)\big)^{-1} T(a) * 1_{I_n} = (\phi - \phi_{N+1})^\vee * a * \big(\mu(I_n)\big)^{-1} 1_{I_n} \quad \text{hold on } G_N^c.$$

This yields the following inequality on G_N^c:

$$\left|\big(\mu(I_n)\big)^{-1} T(a) * 1_{I_n}(x)\right| \leqslant \int_G \left|(\phi - \phi_{N+1})^\vee(y)\right| \left|a * \big(\mu(I_n)\big)^{-1} 1_{I_n}(x - y)\right| dy.$$

We prove that the expression $a * (\mu(I_n))^{-1} 1_{I_n}(x - y)$ vanishes if $y \in G_N$.

Namely, if $x \in G_N^c$ and $y \in G_N$, we have

$$a * \big(\mu(I_n)\big)^{-1} 1_{I_n}(x - y) = \big(\mu(I_n)\big)^{-1} \int_{I_n} a(x - y - t)\,dt.$$

If $I_n \subseteq G_N$, then $x - y - t \in G_N^c$ for every $t \in I_n$. Thus $a(x - y - t) = 0$.

Consider now the case $I_n \supset G_N$. Then

$$\int_{I_n} a(x-y-t)\,dt = \int_{x-y-I_n} a(t)\,dt = 0,$$

because the interval $x-y-I_n$ either contains G_N or does not intersect it.

Consequently,

$$\begin{aligned}\left|\left(\mu(I_n)\right)^{-1} T(a) * 1_{I_n}(x)\right| &\leqslant \int_{G_N^c} \left|(\phi-\phi_{N+1})^\vee(y)\right| \left|a * \left(\mu(I_n)\right)^{-1} 1_{I_n}(x-y)\right| dy \\ &\leqslant \int_{G_N^c} \left|(\phi-\phi_{N+1})^\vee(y)\right| \sup_{n,I_n} \left|a * \left(\mu(I_n)\right)^{-1} 1_{I_n}(x-y)\right| dy \\ &= \int_{G_N^c} \left|(\phi-\phi_{N+1})^\vee(y)\right| \tilde{M}T(a)(x-y)\,dy,\end{aligned}$$

for every $x \in G_N^c$.

We have obtained that

$$\begin{aligned}\int_{G_N^c} \left|\tilde{M}T(a)(x)\right| dx &= \int_{G_N^c} \sup_{n,I_n} \left|T(a) * \left(\mu(I_n)\right)^{-1} 1_{I_n}(x)\right| dx \\ &\leqslant \int_{G_N^c} \left(\int_{G_N^c} \left|(\phi-\phi_{N+1})^\vee(y)\right| \tilde{M}T(a)(x-y)\,dy \right) dx \\ &= \int_{G_N^c} \left|(\phi-\phi_{N+1})^\vee(y)\right| \left(\int_{G_N^c} \tilde{M}T(a)(x-y)\,dx \right) dy \\ &\leqslant \int_{G_N^c} \left|(\phi-\phi_{N+1})^\vee(y)\right| dy < +\infty,\end{aligned}$$

which is the desired result. □

The following corollary is an extension of Daly–Phillips multiplier theorem [4, Theorem 3] from bounded to unbounded Vilenkin groups.

Corollary 3.2. *If $\phi \in L^\infty(\Gamma)$ and $\sup_N \sum_{j=N+1}^\infty \int_{G_N^c} |(\Delta_j \phi)^\vee(x)|\,dx = O(1)$, then $\phi \in m(H^1)$.*

We construct a multiplier that does not satisfy the condition of Daly and Phillips.

Example 3.3. Let G be the dyadic group and $(\omega_k)_k$ be the sequence of Walsh functions: $\omega_k = \prod_{j=0}^\infty e^{i\pi y_{j+1} k_j}$ where $k = \sum_{j=0}^\infty k_j 2^j$ and the dyadic number $x = \sum_{j=0}^\infty y_{j+1} x_j$ corresponds to the real number $\sum_{j=1}^\infty y_j 2^{-j}$ from the interval $[0,1)$.

We consider the sequence

$$R_j(x) = \sum_{k=2^{j-1}}^{2^j-1} \omega_k(x).$$

It is easily seen that

$$R_j(x) = \begin{cases} 2^{j-1}, & x \in G_j; \\ -2^{j-1}, & x \in G_{j-1} \setminus G_j; \\ 0, & x \in G \setminus G_{j-1}. \end{cases}$$

Notice that

$$R_j(x+x_0) = \sum_{k=2^{j-1}}^{2^j-1} \omega_k(x)\omega_k(x_0) = \sum_{k=2^{j-1}}^{2^j-1} (-1)^k \omega_k(x).$$

For the bounded sequence $\varphi(k)=(-1)^k$, one has

$$(\Delta_j\varphi)^\vee(x)=\sum_{k=2^j}^{2^{j+1}-1}\varphi(k)\omega_k(x)=R_{j+1}(x+x_0).$$

Let us calculate $\sup_N\sum_{j=N+1}^{\infty}\int_{G_N^c}|(\Delta_j\varphi)^\vee(x)|\,dx$.

We get

$$\int_{G_N^c}\left|(\Delta_j\varphi)^\vee(x)\right|dx=\int_{G_N^c}\left|R_{j+1}(x+x_0)\right|dx=1,$$

because $x_0+G_j\subset G_N^c$ for every $j\geqslant N+1$.

Thus $\sup_N\sum_{j=N+1}^{\infty}\int_{G_N^c}|(\Delta_j\varphi)^\vee(x)|\,dx=+\infty$.

However, using Theorem 3.1 we see that φ is a multiplier. Indeed,

$$\begin{aligned}\int_{G_N^c}\left|(\varphi-\varphi_{N+1})^\vee(x)\right|dx&=\int_{G_N^c}\left|\sum_{j=N+1}^{\infty}\sum_{k=2^j}^{2^{j+1}-1}(-1)^k\omega_k(x)\right|dx\\&=\int_{G_N^c}\left|\sum_{j=N+2}^{\infty}R_j(x+x_0)\right|dx=\sum_{i=0}^{m_N-1}\int_{x_0+e_i+G_N}\left|\sum_{j=N+2}^{\infty}R_j(t)\right|dt,\end{aligned}$$

where $G=\biguplus_{i=0}^{m_N-1}e_i+G_N$. Then

$$\begin{aligned}\int_{G_N^c}\left|(\varphi-\varphi_{N+1})^\vee(x)\right|dx&=\int_{G_N}\left|\sum_{j=N+2}^{\infty}R_j(t)\right|dt\\&=\sum_{s=N}^{\infty}\int_{G_s\setminus G_{s+1}}\left|\sum_{j=N+2}^{\infty}R_j(t)\right|dt=\sum_{s=N+1}^{\infty}\int_{G_s\setminus G_{s+1}}\left|\sum_{j=N+2}^{s+1}R_j(t)\right|dt,\end{aligned}$$

since $R_j(t)=0$ for $j>s+1$ and $t\in G_s\setminus G_{s+1}$, as well as for $j\geqslant N+2$ and $t\in G_N\setminus G_{N+1}$.

Now,

$$\sum_{j=N+2}^{s+1}R_j(t)=R_{s+1}+\sum_{j=N+2}^{s+}R_j(t)=-2^s+\sum_{j=N+2}^{s}2^{j-1}=-2^{N+1}$$

on $G_s\setminus G_{s+1}$.

Hence,

$$\sum_{s=N+1}^{\infty}\int_{G_s\setminus G_{s+1}}\left|\sum_{j=N+2}^{s+1}R_j(t)\right|dt=\sum_{s=N+1}^{\infty}2^{N+1}\frac{1}{2^{s+1}}=1.$$

Theorem 3.4. *Let G be a compact Vilenkin group. Assume that $\phi\in L^\infty(\Gamma)$ satisfies*

$$\left(\sum_{s=0}^{N-1}m_{s+1}^{\frac{1}{p'}}\log p_{s+1}\right)\left(\sum_{k=m_{N+1}}^{\infty}\left|\Delta\phi(k)\right|^p\right)^{\frac{1}{p}}=O(1),$$

for some $p\in(1,2]$, where $\frac{1}{p'}+\frac{1}{p}=1$ and $\Delta\phi(k)=\phi(k)-\phi(k+1)$. Then ϕ is a multiplier on H^1.

Proof. The proof is based on the estimation of $\sup_N\int_{G_N^c}|(\phi-\phi_{N+1})^\vee(x)|\,dx$ and application of Theorem 3.1.

Let χ_{m_k} denote the element of Γ_{k+1} for which $\chi_{m_k}(x_k)=e^{\frac{2\pi i}{p_{k+1}}}$, and put $\chi_n:=\prod_{k=0}^{s}\chi_{m_k}^{a_k}$ if $n=\sum_{k=0}^{s}a_km_k$ and $0\leqslant a_k<p_{k+1}$. We have

$$\int_{G_N^c}\left|(\phi-\phi_{N+1})^\vee(x)\right|dx=\int_{G_N^c}\left|\sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-1}\phi(k)\chi_k(x)\right|dx.$$

By Abel partial summation,

$$\sum_{k=m_j}^{m_{j+1}-1} \phi(k)\chi_k(x) = \sum_{k=m_j}^{m_{j+1}-2} \big(\phi(k)-\phi(k+1)\big)D_{k+1}(x),$$

for every $x \in G_N^c$, because $D_{m_j}(x) = D_{m_{j+1}}(x) = 0$ in this case.

Thus, we obtain

$$\int_{G_N^c} \big|(\phi-\phi_{N+1})^\vee(x)\big|\,dx = \int_{G_N^c} \left|\sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-2} \big(\phi(k)-\phi(k+1)\big)D_{k+1}(x)\right| dx$$
$$= \sum_{s=0}^{N-1} \int_{G_s \setminus G_{s+1}} \left|\sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-2} \big(\Delta\phi(k)\big)D_{k+1}(x)\right| dx = \sum_{s=0}^{N-1} F_s.$$

From this point on, we follow the argumentation of [2] that resulted in a less restrictive integrability theorem on unbounded Vilenkin groups then previously obtained in [1].

Using the formula $D_n = \chi_n(\sum_{i=0}^{N} \frac{D_{m_i}}{\chi_{m_i}^{a_i}} \cdot \frac{1-\chi_{m_i}^{a^i}}{1-\chi_{m_i}})$, for $n = \sum_{i=0}^{N} a_i m_i$ and other known properties of the Dirichlet kernel, the calculation made in the proof of Theorem 1 in [2] gives that the integrals F_s can be bounded by $F_s^{(1)} + F_s^{(2)}$, where

$$F_s^{(1)} = \int_{G_s \setminus G_{s+1}} \left|\sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-2} \big(\Delta\phi(k)\big)\sum_{i=0}^{s-1} a_i m_i \chi_{k+1}(x)\right| dx, \quad \text{and}$$

$$F_s^{(2)} = \int_{G_s \setminus G_{s+1}} \left|\sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-2} \big(\Delta\phi(k)\big) m_s \frac{1-\chi_{m_s}^{a_s}}{1-\chi_{m_s}} \overline{\chi}_{m_s}^{a_s} \chi_{k+1}(x)\right| dx.$$

Two estimations for $F_s^{(1)}$ and $F_s^{(2)}$, similar to those proved in [2] yield that $F_s^{(1)}$ is bounded by $m_s^{\frac{1}{p'}} (\sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-2} |\Delta\phi(k)|^p)^{\frac{1}{p}}$, and $F_s^{(2)}$ is bounded by $C \log p_{s+1} m_{s+1}^{\frac{1}{p'}} (\sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-2} |\Delta\phi(k)|^p)^{\frac{1}{p}}$.

This proves the result. □

Corollary 3.5. *If $\phi \in L^\infty(\Gamma)$ on a bounded compact Vilenkin group fulfills the requirement $m_N^{p-1} \sum_{k=m_{N+1}}^{m_{N+2}-1} |\Delta\phi(k)|^p = O(1)$ for some $p \in (1,2]$, then $\phi \in m(H^1)$.*

Proof. Obviously, $\sum_{s=0}^{N-1} m_{s+1}^{\frac{1}{p'}} \log p_{s+1} = O(m_N^{\frac{1}{p'}})$ in the bounded case. Hence $m_N^{p-1} \sum_{k=m_{N+1}}^{\infty} |\Delta\phi(k)|^p = O(1)$ is a sufficient condition for $\phi \in m(H^1)$ by Theorem 3.4. However,

$$\sum_{k=m_{N+1}}^{\infty} \big|\Delta\phi(k)\big|^p = \sum_{j=N+1}^{\infty}\sum_{k=m_j}^{m_{j+1}-1} \big|\Delta\phi(k)\big|^p$$
$$\leqslant C \sum_{j=N+1}^{\infty} \frac{1}{m_{j-1}^{p-1}} \leqslant C \frac{1}{m_N^{p-1}} \sum_{j=N+1}^{\infty} 2^{-(j-N-1)(p-1)} = O\left(\frac{1}{m_N^{p-1}}\right). \qquad \square$$

Remark 3.6. For H^1 on bounded Vilenkin groups, Theorem 2 in [3] provides another proof of Theorem 3.1. If the underlying group is bounded and compact, Theorem 9 in [3] gives our Corollary 3.5.

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