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# Optimal Control of a Finite Dam with Diffusion Input and State Dependent Release Rates

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**Abstract**—In this paper, we consider the optimal control of a finite dam using $P^M_{\lambda,\tau}$ policies; assuming that the dam has capacity $v$, the water input is a diffusion process reflected at 0, $v$. The release rates depend on the water content in the dam. There is a certain cost of maintaining the dam as well as a reward received. We obtain an explicit formulas for the total discounted cost over the infinite horizon as well as the long-run average cost per a unit of time. 

**Keywords**—Optimal control of dams, Diffusion processes, Potential, Infinitesimal generator, Control policies, Total discounted cost, Long-run average cost.

## 1. INTRODUCTION AND SUMMARY

Abdel-Hameed and Nakhi [1] discuss the optimal control of a finite dam using $P^M_{\lambda,\tau}$ policies (defined below). They use both the total-discounted cost as well as the long-run average cost per a unit time criterion. They assume that the water input is a Wiener process and Wiener process reflected at 0. Abdel-Hameed [2] treats the case of an infinite dam where the input process is a compound Poisson process. Bae, Kim and Lee [3] treat the case of a finite dam with a compound Poisson input; they only discuss the long-run average cost per a unit time case. Faddy [4] considers the case of a finite dam with Wiener input and $P^M_\lambda$ policy.

In all of the above papers, it is always assumed that the release rates are constants and do not depend on the water content in the dam. In this paper, we consider the case of a finite dam where the input process is a diffusion process and the release rates are state dependent. Specifically, consider a finite dam with capacity $v$ and assume that the water input $I = \{I_t;\ t \in \mathcal{R}_+\}$ is a

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diffusion process reflected at 0, $v$, and diffusion coefficients,

$$\mu(z) = \frac{\sigma^2 (z-v)^3}{v^2} + \frac{\mu (v-z)^2}{v},$$
$$\sigma^2(z) = \frac{\sigma^2 (v-z)^4}{v^2},$$

where $\mu$ is a real number and $\sigma^2$ is a nonnegative number. It follows that $I$ has state space $[0, v)$. Throughout, we will let $\mathcal{R}_+$ be the set of nonnegative real numbers. Suppose that a dam has capacity $v$. Let $Z = (Z_t;\ t \in \mathcal{R}_+)$ be the process describing the content of the dam. We restrict ourselves to policies in which the release rate is zero until the water level reaches level $\lambda$ $(0 < \lambda < v)$, when the water is released at rate

$$M(z) = \frac{M (v-z)^2}{v},$$

until it reaches level $\tau$, $(0 \le \tau < \lambda)$. Once the level $\tau$ is reached, the release rate remains zero until the level $\lambda$ is reached again, and the cycle is repeated. It is clear that the content process is a delayed regenerative process with regeneration points being the times of successive visits to state $\lambda$. During a given cycle, the water content is a diffusion process reflected at $v$ with diffusion coefficients,

$$\mu^*(z) = \frac{\sigma^2 (z-v)^3}{v^2} + \frac{\mu^* (v-z)^2}{v},$$

where $\mu^* = \mu - M$ and $\sigma^2(z)$, denoted by $I^* = \{I_t^*;\ t \in \mathcal{R}_+\}$, and remains so until it drops to level $\tau$; from then on and until it reaches $\lambda$ again the content of the dam behaves like a diffusion process with diffusion coefficients $\mu(z)$ and $\sigma^2(z)$. At any time the release rate increases from 0 to $M(z)$ a starting cost $K_1 M$ is incurred, and at any time the release rate is decreased from $M(z)$ to 0 a closing cost $K_2 M$ is incurred. Moreover, for each unit of output, a reward $A$ (which can be assumed to be 1) is received, and there is a penalty cost which accrues at a rate $f$, where $f$ is a bounded measurable function.

In Section 2, we describe the content process and give basic and main formulas for computing the cost functionals. In section 3, we give explicit expressions for the total discounted cost over the infinite horizon as well the long-run average cost per a unit of time.

## 2. BASIC AND MAIN RESULTS

Throughout, we will let $R = (R_t;\ t \in \mathcal{R}_+)$ and $Z = (Z_t;\ t \in \mathcal{R}_+)$ denote the dam content and the release rates, respectively. The content process is best described by the bivariate process $B = (Z, R)$, from the definition of the type of control policies dealt with, we have $B_0 = (0, 0)$. It should be clear that the process has state space,

$$E = ((l, \lambda) \times \{0\}) \cup ([\tau, V) \times \{M\}),$$

where $l$ denotes the lower bound of the state space of $I$.

The penalty cost occurs at a rate given by

$$f(z, r) = \begin{cases} g(z), & (z, r) \in (l, \lambda) \times \{0\}, \\ g^*(z), & (z, r) \in [\tau, V] \times \{M\}, \end{cases}$$

where $g : (l, \lambda) \to \mathcal{R}_+$ and $g^* : [\tau, V] \to \mathcal{R}_+$ are bounded measurable functions. Define the following stopping times,

$$\begin{aligned} T_0^\wedge &= \inf\{t \in \mathcal{R}_+ : Z_t = \lambda\}, \\ T_0^* &= \inf\{t \in \mathcal{R}_+ : Z_t = \tau\}, \\ T_n^\wedge &= \inf\{t \ge T_{n-1}^\wedge : Z_t = \lambda\}, \\ T_n^* &= \inf\{t \ge T_n^* : Z_t = \tau\}, \qquad n \ge 1. \end{aligned} \tag{2.1}$$

It follows that the sequence of stopping times $(T_n^\wedge)$ forms regeneration points of the content process $Z$.

Let $C_0^\alpha(x,\lambda), C_M^\alpha(\lambda,\tau)$ be the expected discounted penalty costs during the intervals $[0,T_0^\wedge)$ starting at $x$, and during the interval $[T_0^\wedge, T_0^*)$ respectively, $0 \le \alpha < \infty$. The corresponding functions, when $\alpha = 0$, are written $C_0(x,\lambda)$ and $C_M(\lambda,\tau)$. It follows that

$$\begin{aligned} C_0^\alpha(x,\lambda) &= E_x \int_0^{T_0^\wedge} e^{-\alpha t} g(I_t)\, dt, \\ C_M^\alpha(\lambda,\tau) &= E_\lambda \int_0^{T_0^*} e^{-\alpha t} g(I_t^*)\, dt, \\ C_0(x,\lambda) &= E_x \int_0^{T_0^\wedge} g(I_t)\, dt, \\ C_M(\lambda,\tau) &= E_\lambda \int_0^{T_0^*} g(I_t^*)\, dt. \end{aligned} \tag{2.2}$$

To compute the functionals indicated in (2.2) and other related functionals, we define the diffusion process killed at $\lambda$, as follows

$$X = (I_t;\ t < T_0^\wedge).$$

From the theory of Markov processes we know that the process X is a strong Markov process. It has state space $[0,\lambda)$. It follows that its generator is of the form,

$$Af(x) = \frac{\sigma^2(x)}{2} f''(x) + \mu(x) f'(x)$$

and (2.3)

$$Af(\lambda) = 0.$$

It can be shown that the domain of the generator (D(A)) is of the form,

$$D(A) = c^2[0,\lambda) \cap \{f'(0) = 0\}.$$

For any number z we let $\tilde{z} = z/(v-z)$. We note that for any $x$ in the state space,

$$C_0^\alpha(x,\lambda) = U_\alpha g(x),$$

where $U_\alpha$ is the resolvent operator of the process $X$ defined above. Let $U_\alpha(x,y)$ be $\alpha$-potential kernel of $X$. To find $U_\alpha$, we define $\phi_\alpha(x)$ as follows,

$$\phi_\alpha(x) = U_\alpha\left(e^{-\theta x}\right),$$

where $\theta$ is a fixed real number. Since the range of the resolvent operator is equal to the domain of the generator $A$, and $(\alpha I - A)\phi_\alpha(x) = e^{-\theta x}$. It follows that $\phi_\alpha(x)$ is the solution of the boundary value problem,

$$\frac{\sigma^2(x)}{2}\phi_\alpha''(x) + \mu(x)\phi_\alpha'(x) - \alpha\phi_\alpha(x) = -e^{-\theta x}, \tag{2.4}$$

where $\phi_\alpha'(0) = 0$ and $\phi_\alpha(\lambda_-) = 0$.

For the computation of $U_\alpha(x,y)$, we have the following.

THEOREM 1. *Let $U_\alpha(x,y)$ be $\alpha$-potential kernel of $X$, and $\gamma = (\mu^2 + 2\alpha\sigma^2)^{1/2}$, then for $x$ and $y$ in the state space,*

$$U_\alpha(x,y) = \frac{v}{\gamma(v-y)^2}\exp\left\{(\tilde{y}-\tilde{x})\mu/\sigma^2\right\}\left[\left(\frac{\lambda-\mu}{\lambda+\mu}\right)e^{\left(-(\gamma+\mu)\tilde{\lambda}\right)/\sigma^2} + e^{\left((\gamma-\mu)\tilde{\lambda}\right)/\sigma^2}\right]^{-1}$$
$$\times\left[\left(\frac{\gamma-\mu}{\gamma+\mu}\right)e^{\left(-\gamma(\tilde{y}+\tilde{x})+(\gamma-\mu)\tilde{\lambda}\right)/\sigma^2} - e^{\left(\gamma(\tilde{y}+\tilde{x})-(\gamma+\mu)\tilde{\lambda}\right)/\sigma^2}\right.$$
$$\left.+e^{\left(-\gamma|\tilde{y}-\tilde{x}|+(\gamma-\mu)\tilde{\lambda}\right)/\sigma^2} - \left(\frac{\gamma-\mu}{\gamma+\mu}\right)e^{\left(\gamma|\tilde{y}-\tilde{x}|-(\gamma+\mu)\tilde{\lambda}\right)/\sigma^2}\right]. \tag{2.5}$$

PROOF. It can be shown that the homogenous part of equation (2.4) is of the form,

$$\phi_\alpha^h(x) = c_1 e^{-(v(\gamma+\mu))/(\sigma^2(v-x))} + c_2 e^{(v(\gamma-\mu))/(\sigma^2(v-x))},$$

where $c_1$ and $c_2$ will be determined later. Let

$$\phi_1(x) = e^{-(v(\gamma+\mu))/(\sigma^2(v-x))},$$
$$\phi_2(x) = e^{(v(\gamma-\mu))/(\sigma^2(v-x))},$$

and

$$h(x) = -\frac{2v^2}{\sigma^2(v-x)^4}e^{-\theta x}.$$

Using differential equation techniques, the general solution of (2.4) is

$$\phi_\alpha(x) = c_1\phi_1(x) + c_2\phi_2(x) + \phi_2(x)\left[\int_0^x \frac{\phi_1(y)h(y)}{w(y)}\,dy\right] - \phi_1(x)\left[\int_0^x \frac{\phi_2(y)h(y)}{w(y)}\,dy\right], \tag{2.6}$$

where

$$w = \phi_1\phi_2' - \phi_2\phi_1' = \frac{2\gamma v}{\sigma^2(v-x)^2}\phi_1\phi_2$$

is the Wronskain of $\phi_1$ and $\phi_2$. Thus, (2.6) becomes

$$\phi_\alpha(x) = c_1\phi_1(x) + c_2\phi_2(x) + \frac{1}{\gamma}\left[\phi_1(x)\int_0^x \frac{v\phi_1^{-1}(y)e^{-\theta y}}{(v-y)^2}\,dy - \phi_2(x)\int_0^x \frac{v\phi_2^{-1}(y)e^{-\theta y}}{(v-y)^2}\,dy\right].$$

Denoting the Laplace transform of a function $f$ by $L(f)$, the above equation can be written as

$$\phi_\alpha(x) = c_1\phi_1(x) + c_2\phi_2(x)$$
$$+\frac{1}{\gamma}\left[\phi_1(x)L\left\{I(0<y<x)\frac{v\phi_1^{-1}(y)}{(v-y)^2}\right\} - \phi_2(x)L\left\{I(0<y<x)\frac{v\phi_2^{-1}(y)}{(v-y)^2}\right\}\right].$$

Using the boundary conditions $\phi_\alpha'(0) = 0$ and $\phi_\alpha(\lambda_-) = 0$, it follows that

$$c_2 = \frac{1}{\gamma}\left[\frac{\gamma-\mu}{\gamma+\mu}\phi_2(0)\phi_1^{-1}(0)\phi_1(\lambda) + \phi_2(\lambda)\right]^{-1}\left[\phi_2(\lambda)L\left(\frac{v\phi_2^{-1}(y)}{(v-y)^2}\right) - \phi_1(\lambda)L\left(\frac{v\phi_1^{-1}(y)}{(v-y)^2}\right)\right]$$

and

$$c_1 = \frac{\gamma-\mu}{\gamma+\mu}\phi_2(0)\phi_1^{-1}(0)c_2. \tag{2.7}$$

Substituting (2.7) into (2.6), we have

$$\phi_\alpha(x) = c_2\left[\frac{\gamma-\mu}{\gamma+\mu}\phi_2(0)\,\phi_1^{-1}(0)\,\phi_1(\lambda)+\phi_2(\lambda)\right] + \frac{1}{\gamma}\left[\phi_1(x)\,L\left\{I(0<y<x)\,\frac{v\phi_1^{-1}(y)}{(v-y)^2}\right\} - \phi_2(x)\,L\left\{I(0<y<x)\,\frac{v\phi_2^{-1}(y)}{(v-y)^2}\right\}\right].$$

Inverting the left-hand side of the above equation with respect to $\theta$, the result follows.

We have

$$C_0^\alpha(x,\lambda) = \int_0^\lambda g(y)\,U_\alpha(x,y)\,dy. \tag{2.8}$$

Letting $\alpha \to 0$, in (2.5), we get

$$U_0(\tau,y) = \begin{cases} \dfrac{2v}{\sigma^2(v-y)^2}\left[\tilde\lambda \vee \tilde y - \tilde\tau \vee \tilde y\right], & \text{if } \mu = 0,\\ \dfrac{v}{\mu(v-y)^2}\left[e^{-(2\mu(\tilde\tau-\tilde y)_+)/\sigma^2} - e^{-\left(2\mu(\tilde\lambda-\tilde y)_+\right)/\sigma^2}\right], & \text{if } \mu \neq 0. \end{cases}$$

We put this next as a proposition.

PROPOSITION 1. *Let $C_0(x,\lambda)$ be the nondiscounted cost in $[0,T_0^\wedge)$. Assume that the input process is a diffusion process with parameters $\mu(z)$ and $\sigma^2(z)$. Then, for $\mu > 0$,*

$$C_0(\tau,\lambda) = \begin{cases} \dfrac{2}{\sigma^2}\int_0^\lambda \dfrac{v}{(v-y)^2}g(y)\left[\tilde\lambda\vee\tilde y - \tilde\tau\vee\tilde y\right]dy, & \text{if } \mu = 0,\\ \dfrac{1}{\mu}\int_0^\lambda \dfrac{v}{(v-y)^2}g(y)\left[e^{-(2\mu(\tilde\tau-\tilde y)_+)/\sigma^2} - e^{-\left(2\mu(\tilde\lambda-\tilde y)_+\right)/\sigma^2}\right]dy, & \text{if } \mu\neq 0. \end{cases} \tag{2.9}$$

Let $1(x) = I_{[0,\lambda)}(x)$, from (2.15) of Abdel-Hameed and Nakhi [2], we know that

$$E_x e^{-\alpha T_0^\wedge} = 1 - \alpha U_\alpha 1(x) \tag{2.10}$$

and we have the following proposition.

PROPOSITION 2. *Assume that the input process is a diffusion process with parameters $\mu(z)$,and $\sigma^2(z)$. Let $T_0^\wedge$ be as defined in (2.1), and $C_0^\alpha(x,\lambda)$ be as defined in (2.2). Then,*

$$E_x e^{-\alpha T_0^\wedge} = \frac{(\gamma-\mu)\,e^{-\{(\gamma+\mu)\tilde x/\sigma^2\}} + (\gamma+\mu)\,e^{\{(\gamma-\mu)\tilde x/\sigma^2\}}}{(\gamma-\mu)\,e^{-\{(\gamma+\mu)\tilde\lambda/\sigma^2\}} + (\gamma+\mu)\,e^{\{(\gamma-\mu)\tilde\lambda/\sigma^2\}}}. \tag{2.11}$$

*From (2.10), we have that*

$$E_x T_0^\wedge = \lim_{\alpha\to 0} U_\alpha 1(x)$$

*and we have the following corollary.*

COROLLARY 1. *Assume that the input process is a diffusion process with parameters $\mu(z)$ and $\sigma^2(z)$, and let $T_0^\wedge$ be the time of first entrance in state $\lambda$. Then,*

$$E_\tau T_0^\wedge = \begin{cases} \dfrac{\tilde\lambda^2-\tilde\tau^2}{\sigma^2}, & \text{if } \mu = 0,\\ \dfrac{\tilde\lambda-\tilde\tau}{\mu} + \dfrac{\sigma^2}{2\mu^2}\left[e^{-(2\tilde\lambda\mu)/\sigma^2} - e^{-(2\tilde\tau\mu)/\sigma^2}\right], & \text{if } \mu\neq 0. \end{cases} \tag{2.12}$$

*We now compute $C_M^\alpha(\lambda,\tau)$, $E_\lambda(e^{-\alpha T_0^*})$, $C_M(\lambda,\tau)$, and $E_\lambda T_0^*$. Define*

$$X^* = (I_t^*;\ t < T_0^*).$$

It follows that $X^*$ is a standard Markov process with state space $[\tau, v)$. We note that for $x < \lambda$,

$$C_M^\alpha(\lambda,\tau) = \overset{*}{U}_\alpha g^*(\lambda),$$

where $U_\alpha^*$ is the resolvent operator of the process $X^*$ defined above. Let $\overset{*}{U}_\alpha(x,y)$ be $\alpha$-potential kernel of $X^*$. In order to determine $\overset{*}{U}_\alpha(x,y)$, we define

$$\overset{*}{\phi}_\alpha(x) = \overset{*}{U}_\alpha\left(e^{-\theta x}\right),$$

where $\theta$ lies in $[0,\infty)$. By an argument similar to the one used to establish (2.8), $\overset{*}{\phi}_\alpha(x)$ is the solution of the following boundary value problem,

$$\frac{\sigma^2(x)}{2}\overset{*}{\phi}{}''_\alpha(x) + \overset{*}{\mu}(x)\overset{*}{\phi}{}'_\alpha(x) - \alpha\overset{*}{\phi}_\alpha(x) = -e^{-\theta x}, \tag{2.13}$$

where $\overset{*}{\phi}_\alpha(\tau_-) = 0$ and $\overset{*}{\phi}{}'_\alpha(\nu_-) = 0$.

We have the following theorem.

THEOREM 2. *Let $\overset{*}{\gamma} = (\overset{*}{\mu}{}^2 + 2\alpha\sigma^2)^{1/2}$, then for $x$ and $y$ in the state space of $X^*$,*

$$\overset{*}{U}_\alpha(x,y) = \frac{v}{\overset{*}{\gamma}(v-y)^2}e^{(\tilde{y}-\tilde{x})\overset{*}{\mu}/\sigma^2}\left[e^{-\overset{*}{\gamma}|\tilde{y}-\tilde{x}|/\sigma^2} - e^{2\overset{*}{\gamma}\tau - \overset{*}{\gamma}(\tilde{y}+\tilde{x})/\sigma^2}\right]. \tag{2.14}$$

PROOF. As shown in the proof of Theorem 1, the general solution of (2.17) is

$$\overset{*}{\phi}_\alpha(x) = c_1\overset{*}{\phi}_1(x) + c_2\overset{*}{\phi}_2(x) + \frac{1}{\overset{*}{\gamma}}\left[\overset{*}{\phi}_1(x)L\left\{I(a,y)\frac{v\overset{*}{\phi}{}_1^{-1}(y)}{(v-y)^2}\right\} - \overset{*}{\phi}_2(x)L\left\{I(a,y)\frac{v\overset{*}{\phi}{}_2^{-1}(y)}{(v-y)^2}\right\}\right].$$

where

$$\overset{*}{\phi}_1(x) = e^{-\left(\overset{*}{v}\left(\overset{*}{\gamma}+\overset{*}{\mu}\right)\right)/\sigma^2(v-x)}, \qquad \phi_2(x) = e^{\left(\overset{*}{v}\left(\overset{*}{\gamma}-\overset{*}{\mu}\right)\right)/\sigma^2(v-x)},$$

$a$ is an arbitrary point in $[\tau, v)$ and $I(a,y) = I(a<y<x)$. Imposing the boundary condition $\overset{*}{\phi}{}'_\alpha(\nu_-) = 0$, it follows that $c_2 = 0$ and $a = v$. Thus,

$$\overset{*}{\phi}_\alpha(x) = c_1\overset{*}{\phi}_1(x) + \frac{1}{\overset{*}{\gamma}}\left[\overset{*}{\phi}_2(x)L\left\{I(a,y)\frac{v\overset{*}{\phi}{}_2^{-1}(y)}{(v-y)^2}\right\} - \overset{*}{\phi}_1(x)L\left\{I(a,y)\frac{v\overset{*}{\phi}{}_1^{-1}(y)}{(v-y)^2}\right\}\right]. \tag{2.15}$$

Imposing the boundary condition $\overset{*}{\phi}_\alpha(\tau_-) = 0$ on the last equation above, we get

$$c_1 = \frac{1}{\overset{*}{\gamma}}\left[\overset{*}{\phi}_1(\tau)L\left(\frac{v\overset{*}{\phi}{}_2^{-1}(y)}{(v-y)^2}\right) - \overset{*}{\phi}_2(\tau)L\left(\frac{v\overset{*}{\phi}{}_2^{-1}(y)}{(v-y)^2}\right)\right].$$

Taking the Laplace inverse with respect to $\theta$ in equation (2.15) the result follows.

The proof of the following proposition follows in a manner similar to the proof of Proposition 2, and hence, is omitted.

PROPOSITION 3. *Assume that the input process is a diffusion process with parameters $\overset{*}{\mu}(z)$ and $\sigma^2(z)$. Let $T_0^*$ be as defined in (2.1), and $C_M^\alpha(\lambda,\tau)$ be as defined in (2.2). Then,*

$$E_x e^{-\alpha T_0^*} = e^{-\left\{\left(\overset{*}{\gamma}+\overset{*}{\mu}\right)\left(\tilde{\lambda}-\tilde{\tau}\right)\right\}/\sigma^2}. \tag{2.16}$$

Letting $\alpha \to 0$ in (2.16), it follows that, for $\overset{*}{\mu} \le 0$, $T_0^* < \infty$, w.p.1, while $T_0^* = \infty$ with probability $1 - e^{-2\overset{*}{\mu}(\tilde{\lambda}-\tilde{\tau})}$, for $\overset{*}{\mu} > 0$. The proof of the following corollary is similar to the proof of Corollary 1, and is also omitted.

COROLLARY 2. *Assume that the input process is a diffusion process with parameters $\overset{*}{\mu}(z)$ and $\sigma^2(z)$, and let $T_0^*$ be the time of first entrance in state $\tau$. Then,*

$$E_\lambda T_0^* = \begin{cases} \infty, & \text{if } \overset{*}{\mu} \geq 0, \\ \dfrac{\tilde{\tau} - \tilde{\lambda}}{\overset{*}{\mu}}, & \text{if } \overset{*}{\mu} < 0. \end{cases} \tag{2.17}$$

In order to find $C_M(\lambda, \tau)$, we first have the following proposition which follows from Theorem 2.

PROPOSITION 4. *Let $\overset{*}{U}_\alpha(x, y)$ be the $\alpha$-potential kernel given in Theorem 2. Then, for any $x$, and $y \in [\tau, v]$,*

$$\overset{*}{U}_0(x,y) = \begin{cases} \dfrac{v}{\overset{*}{\mu}(v-y)^2}\left[e^{\left(-2\overset{*}{\mu}(\tilde{\lambda}-\tilde{y})_+\right)/\sigma^2} - e^{\left(-2\overset{*}{\mu}(\tilde{\lambda}-\tilde{\tau})_+\right)/\sigma^2}\right], & \overset{*}{\mu} \geq 0, \\ \dfrac{v}{\overset{*}{\mu}(v-y)^2}\left[e^{\left(-2\overset{*}{\mu}(\tilde{y}-\tilde{\tau})_+\right)/\sigma^2} - e^{\left(-2\overset{*}{\mu}(\tilde{y}-\tilde{\lambda})_+\right)/\sigma^2}\right], & \overset{*}{\mu} < 0. \end{cases}$$

Using the Lebesgue dominated convergence theorem, we have the following.

PROPOSITION 5. *Let $C_M(\lambda, \tau)$ be the cost functional defined in (2.2). Then, for $\overset{*}{\mu} < 0$,*

$$C_M(\lambda,\tau) = \frac{1}{\overset{*}{\mu}} \int_\tau^v \frac{v}{(v-y)^2} \overset{*}{g}(y) \left[e^{\left(2\overset{*}{\mu}(\tilde{y}-\tilde{\tau})_+\right)/\sigma^2} - e^{\left(2\overset{*}{\mu}(\tilde{y}-\tilde{\lambda})_+\right)/\sigma^2}\right] dy. \tag{2.18}$$

For $\overset{*}{\mu} \geq 0$, we conjecture the following.

PROPOSITION 6. *Let $C_M(\lambda, \tau)$ be the cost functional defined in (2.2). Then, for $\overset{*}{\mu} \geq 0$,*

$$C_M(\lambda,\tau) = \frac{1}{\overset{*}{\mu}} \int_\tau^v \frac{v}{(v-y)^2} \overset{*}{g}(y) \left[e^{\left(-2\overset{*}{\mu}(\tilde{\lambda}-\tilde{y})_+\right)/\sigma^2} - e^{\left(-2\overset{*}{\mu}(\tilde{\lambda}-\tilde{\tau})_+\right)/\sigma^2}\right] dy.$$

## 3. THE EXPECTED TOTAL DISCOUNTED AND LONG-RUN AVERAGE COSTS

Consider the finite dam controlled by the $P_{\lambda,\tau}^M$ with a diffusion process, reflected at 0 and $v$, as described in Section 1. Let $\alpha$ be the discounting factor. Let $C_\alpha(\lambda, \tau)$ be the expected total discounted cost over the infinite horizon, while $C_0^\alpha(0, \lambda)$ and $C_{1,\alpha}(\lambda, \tau)$ are the expected discounted costs in the intervals $[0, T_0^\wedge)$ and $[T_0^\wedge, T_1^\wedge)$, respectively. It follows that the expected total discounted cost is

$$C_\alpha(\lambda,\tau) = C_0^\alpha(0,\lambda) + \frac{E_0\left[\exp\left(-\alpha T_0^\wedge\right) E_\lambda\left[C_{1,\alpha}(\lambda,\tau)\right]\right]}{1 - E_\lambda\left[\exp\left(-\alpha W_1\right)\right]}, \tag{3.1}$$

where $W_1 = T_1^\wedge - T_0^\wedge$. It follows, from the strong Markov property, that

$$E_\lambda\left[\exp\left(-\alpha W_1\right)\right] = E_\lambda\left[\exp\left(-\alpha T_0^*\right)\right] E_\tau\left[\exp\left(-\alpha T_0^\wedge\right)\right] \tag{3.2}$$

and

$$\begin{aligned} E_\lambda\left[C_{1,\alpha}(\lambda,\tau)\right] = M\left\{K_1 + K_2 E_\lambda\left[\exp\left(-\alpha T_0^*\right)\right] - E_\lambda \int_0^{T_0^*} e^{-\alpha t}\, dt\right\} \\ + E_\lambda\left[\exp\left(-\alpha T_0^*\right)\right] C_0^\alpha(\tau,\lambda) + C_M^\alpha(\lambda,\tau). \end{aligned} \tag{3.3}$$

Now, substituting (2.5), (2.8), and (2.14) into (3.3), we obtain the expected total discounted cost over the infinite horizon $C_\alpha(\lambda, \tau)$.

Let $C_1(\lambda, \tau)$ be the cost incurred in the interval $[T_0^\wedge, T_1^\wedge)$, $C(\lambda, \tau)$ be the long-rum average cost per a unit of time and $K = K_1 + K_2$. Using the relation $C(\lambda, \tau) = \lim_{\alpha \to 0} \alpha C_\alpha(\lambda, \tau)$, we have

$$C(\lambda, \tau) = \frac{E_\lambda [C_1(\lambda, \tau)]}{E_\lambda (W_1)}. \tag{3.4}$$

From the strong Markov property, it follows that

$$E_\lambda (W_1) = E_\lambda (T_0^*) + E_\tau (T_0^\wedge),$$
$$E_\lambda [C_1(\lambda, \tau)] = M [K - E_\lambda (T_0^*)] + C_0(\tau, \lambda) + C_M(\lambda, \tau).$$

Now, substituting (2.9), (2.17), and (2.18) into (3.4), the long-run average cost per a unit of time can be determined explicitly.

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