Morphing polyhedra with parallel faces: Counterexamples✩

Therese Biedl, Anna Lubiw∗, Michael J. Spriggs

David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada

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1. Introduction

This paper studies polyhedra that are parallel with each other. To define parallel polyhedra, we first study the notion of parallel drawings of graphs. We say that two straight-line drawings of a graph in $\mathbb{R}^d$ are parallel if, for every edge $\{u, v\}$ of the graph, the vector from $u$ to $v$ is non-zero and has the same direction in both drawings. A parallel morph is a sequence of drawings determined by a continuous transformation of the vertex positions, such that every intermediate drawing in the sequence is simple (i.e. non-self-intersecting) and parallel with the original drawing.

Let us extend this idea to polyhedra: two polyhedra in $\mathbb{R}^3$ are parallel if they share a common edge graph and face set—where a face is represented by a cycle of vertices in the edge graph—and the drawings of the edge graph determined by the vertex positions in the two polyhedra are parallel. In this setting, a parallel morph is a continuous sequence of polyhedra determined by a continuous transformation of the vertex positions, such that every intermediate polyhedron is simple and parallel with the original.

We have been investigating the existence of parallel morphs, and the complexity of finding them for various classes of graphs and polyhedra [4–6]. These questions are about connectivity within a parallel family of polygons/graphs/polyhedra: can we go from one member of the family to any other via continuous changes that keep us within the family?

Previous results are existence results for the case of cycles in the plane: Guibas et al. [13] and independently, Grenander et al. [12] prove that there exists a parallel morph between any two simple parallel polygons in the plane. Before that Thomassen [23] had proved the same for the special case of orthogonal polygons (see Section 2 for precise definitions).

We have explored algorithmic issues for the case of graphs in the plane. We prove that every pair of parallel orthogonal drawings in the plane admits a parallel morph, and give an algorithm that computes such a morph in polynomial time with respect to the complexity of the graph [6]. (This algorithm is a key subroutine that we use in an algorithm for computing a morph between orthogonal drawings of a graph that are not necessarily parallel [15]; the goal here is to keep
all intermediate drawings simple and orthogonal.) For pairs of parallel non-orthogonal drawings there does not always exist a parallel morph, and the decision problem is NP-hard [6].

In three dimensions, even cycles present an interesting challenge. For one thing, cycles may be “knotted” in different ways, precluding the possibility of a continuous morph altogether. Restricting to cycles representing the trivial knot is not enough to guarantee a parallel morph even for the case of orthogonal cycles [4]. We conjecture that deciding parallel morphability for parallel drawings of a cycle in \( \mathbb{R}^3 \) is computationally intractable.

Although cycles in \( \mathbb{R}^3 \) seem complicated, we had hoped that genus-0 (i.e., sphere-like) polyhedra would be simpler, at least for the orthogonal case. The purpose of this brief note is to prove otherwise: we give examples of parallel orthogonal polyhedra of genus 0 that do not admit a parallel morph. In the PhD thesis of the third author [20], the second counterexample is used to prove that it is PSPACE-hard to decide parallel morphability for orthogonal polyhedra of genus 0.

In this paper we present two counterexamples. The first demonstrates that parallel orthogonal drawings on the surface of a cuboid (i.e., a box) do not always admit a parallel morph. One might think of these drawings as degenerate orthogonal polyhedra in which faces may lie coplanar. It follows that not all parallel orthogonal polyhedra admit a parallel morph. We can modify the example to eliminate coplanar faces and create orthogonally convex parallel polyhedra that do not morph.

Our first counterexample requires that some of the faces are not rectangles. In fact, parallel drawings on a cuboid in which all faces are rectangles always admit a parallel morph. Our second counterexample demonstrates that—even when we restrict all faces to be rectangles—there exist pairs of parallel orthogonal genus-0 polyhedra that do not admit a parallel morph. We leave open the question of whether parallel orthogonally convex polyhedra always morph when all faces are rectangles.

1.1. Background

1.1.1. Morphing

For a general survey on morphing see Gomes et al. [10] or, for a more theoretical perspective, see the relevant section of Alt and Guibas [1]. Most morphing algorithms that assume a specified correspondence between the source and target (as we do) are not able to maintain simplicity during the morph—we will use the term “non-intersecting morph” for those that do.

The earliest result about non-intersecting morphs predates the coining of the term “morph”\(^1\): Cairns in 1944 [7] showed that there is a non-intersecting morph from any planar triangulation to any isomorphic one with the same fixed triangle as a boundary. The result was strengthened in 1983 by Thomassen [23] in two ways. First, he showed that any planar subdivision with convex faces can be morphed to any isomorphic one with the same fixed boundary via a morph that preserves convexity. Secondly, he showed that any straight line graph has a non-intersecting morph to any other straight line graph that is isomorphic and embedded with the same oriented faces. He proved this second result by arguing that the source and target graphs can be simultaneously augmented to isomorphic triangulations with a fixed boundary, a result that is of interest in its own right, and is now known as “compatible triangulation”, due to Aronov, Seidel and Souvaine [2]. Both Cairns and Thomassen’s results are constructive, but algorithmic issues are not explored. Thomassen’s morphs move only one vertex (or a cluster of vertices that have been contracted together) at a time, along a straight line.

Independently, Floater and Gotsman [9] proved Thomassen’s first result (on convex morphing) using an entirely different approach based on Tutte’s graph embedding method. In their morph, all vertices move at once, and what can be computed is not the individual trajectories of the vertices, but snapshots of the graph at any intermediate time during the morph. Gotsman and Surazhsky [11,19] then combined this result with the compatible triangulation result from Aronov et al. [2] to show that simple polygons have non-intersecting morphs, thus re-proving Thomassen’s second result with a different morphing technique.

Our exploration of parallel morphs should be viewed as being about connectivity within a parallel family of polygons: can we go from any polygon in the family to any other one via continuous changes that keep us within the family?

1.1.2. Connection to linkage reconfiguration

Closely related to parallel morphs, there have been studies done on transforming configurations while restricting the amount of change to angles and/or edge lengths. The most stringent requirement on edge lengths is that they not change at all, i.e., the source and target configuration have the same edge lengths. This is the problem of linkage reconfiguration, which can be done (while maintaining simplicity) only for a limited class of graphs: between any two simple chains in the plane with corresponding edges of the same length, there is a transformation that preserves simplicity and edge lengths [8,22]. The result extends to cycles (or equivalently, polygons) but not to trees [3].

1.1.3. Connection to parallel redrawings and rigidity

As defined by Whiteley [24] a drawing of a graph has a parallel redrawing if the vertices can be moved such that all edges remain parallel to those in the original drawing, and the resultant drawing is neither a translation nor a scaling of the original. Here, maintaining the simplicity of the drawing is not an issue.

\(^1\) 1975, short for “metamorphose”, according to the Merriam Webster Dictionary.
Whiteley [24], and Servatius and Whiteley [18] study questions of the existence of a parallel redrawing. This turns out to be directly related to questions in rigidity theory. In rigidity theory, frameworks are composed of rigid bars, idealized as straight-line edges and attached at vertices. Where two bars meet at a vertex, the angle between them is allowed to change freely. The fundamental problem is that of deciding whether or not a framework is rigid. That is, is there a non-trivial \textit{infinitesimal} motion that moves the vertices while keeping the lengths of the bars fixed? Simplicity is not an issue. To contrast the two situations: in parallel redrawings edge lengths may change but angles are fixed, whereas in rigidity theory, edge lengths are fixed and angles may change. The answers are the same however: a configuration has a parallel redrawing if and only if it is not rigid, since the vectors orthogonal to an \textit{infinitesimal} motion provide a parallel redrawing [18].

Linkage reconfiguration problems also deal with rigid bars and flexible angles, however—in contrast with rigidity theory—simplicity must be maintained, and the questions are about reachability: given two structures that are composed of the same set of rigid bars and with the same combinatorial structure, can we morph from one to the other preserving incidence relationships and the lengths of the bars?

There is a clear relationship between this pair of problems and the morphing problems discussed above: linkage reconfiguration problems are to rigidity theory as parallel morphs are to parallel redrawings. In both cases we impose simplicity, and we ask questions of reachability between a pair of configurations, rather than about infinitesimal motions of a single configuration. Although the equivalence between rigidity theory and parallel redrawings does not carry over, this relationship seems tantalizing.

Also related is the recent work of Streinu [21], who studies \textit{kinetic graphs}: straight-line drawings of a graph in which vertices move with constant velocities. She defines a \textit{parallel redrawing graph} as a kinetic graph in which each edge maintains its slope throughout the motion. Of particular interest are planar graphs that maintain non-crossing edges throughout the motion (unlike our notion of \textit{parallel}, a parallel redrawing allows edges to shrink to zero length, and reverse direction). Streinu characterizes this class of kinetic graphs.

1.1.4. Connection to knot theory

A \textit{knot} is defined as a closed, non-self-intersecting curve embedded in \(\mathbb{R}^3\). Two knots are equivalent if one knot can be continuously deformed to the other without self-intersection. Deciding whether two knots are equivalent is a central problem of \textit{knot theory} (see Prasolov and Sossinsky [17] for an introduction). The complexity of deciding knot equivalence has not yet been completely determined. It has been shown that the problem is in \textsc{PSPACE} [14]. A related problem, that of deciding whether a knot can be deformed to lie in a plane, is in \textsc{NP}. There exist algorithms for both of these problems with running times that are exponential with respect to the number of crossings in an orthogonal projection of the knot(s) [14].

Suppose that we are given non-self-intersecting parallel drawings of a cycle graph in \(\mathbb{R}^3\). Each drawing is a closed non-self-intersecting curve (i.e., a knot). If the drawings admit a parallel morph then they correspond to equivalent knots. However, the converse might not be true [4].

2. Definitions

Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\), and let \(p : V(G) \rightarrow \mathbb{R}^d\) be a function. This vertex mapping \(p\) uniquely determines a \textit{drawing} \(P\) of \(G\) in which each vertex \(v \in V(G)\) is represented by the point \(p(v)\), and each edge \((u, v) \in E(G)\) is represented by the open line segment between \(p(u)\) and \(p(v)\). In this paper, we will concern ourselves mainly with drawings in \(\mathbb{R}^2\) and \(\mathbb{R}^3\).

A \(d\)-dimensional drawing is said to \textit{self-intersect} if there is some \(\alpha \in \mathbb{R}^d\) such that \(\alpha\) belongs to two or more elements (i.e., vertices or edges) of the drawing. A drawing that does not self-intersect is called \textit{simple}. Two drawings of a graph \(G\) determined by mappings \(p\) and \(q\) are called \textit{parallel} if for every \((u, v) \in E(G)\), there exists some \(\lambda > 0\) such that \(p(u) - p(v) = \lambda(q(u) - q(v))\).

We define a \textit{morph} as a sequence of drawings of a graph determined by a continuous transformation of the vertex positions. Given a morph \(R\), \(R(t)\) denotes the drawing at time \(t\), \(t \in [0, 1]\). Suppose that drawings \(P\) and \(Q\) are parallel drawings of a graph. A morph \(R\) is a \textit{parallel morph} from \(P\) to \(Q\) if \(R(0) = P, R(1) = Q\) and for all \(t \in [0, 1]\), \(R(t)\) is simple and is parallel with \(P\) and \(Q\).

We assume familiarity with polygons and polyhedra; see O’Rourke’s book [16] for precise definitions. The notions of \textit{parallel} and \textit{parallel morph} generalize easily from graph drawings to polyhedra (not necessarily convex). A polyhedron may be thought of as being composed of a drawing of its edge graph, together with its planar faces which are represented by the open interiors of polygons. A polyhedron is \textit{simple} if no point belongs to more than one of the vertices/edges/faces of the polyhedron. We require that polyhedra have a connected edge graph, and we allow adjacent faces to be coplanar. In this paper, we are interested only in simple genus-0 polyhedra in \(\mathbb{R}^3\) and polyhedral surfaces that are topological disks. To distinguish these from drawings of a graph, we write their labels in boldface.

Let \(P, Q\) be a pair of polyhedra with the same edge-graph \(G\) and the same face set. We say that \(P\) and \(Q\) are \textit{parallel} if the two drawings of \(G\)—induced by their respective vertex mappings—are parallel. Observe that parallel polyhedra have corresponding faces lying in parallel planes. A \textit{parallel morph} between parallel polyhedra \(P\) and \(Q\) continuously transforms \(P\) to \(Q\) such that every intermediate polyhedron of the morph is simple and parallel with \(P\) and \(Q\).

A drawing/polyhedron is called \textit{orthogonal} if each edge is parallel with one of the axes. An orthogonal polygon is \textit{orthogonally convex} if every line parallel with an axis either does not intersect the polygon, or else intersects it in a single line.
segment. An orthogonal polyhedron is orthogonally convex if every plane that is perpendicular to one of the coordinate axes either does not intersect the polyhedron, or else intersects it in an orthogonally convex polygon.

3. Main results

3.1. Unmorphable drawings on a cuboid

We say that an orthogonal drawing is a cuboid drawing if it is the edge graph of a cuboid (i.e. a box) in $\mathbb{R}^3$, augmented by the addition of orthogonal drawings on each face; we allow that edges of the cuboid are subdivided by vertices.

**Theorem 1.** Parallel cuboid drawings do not always admit a parallel morph.

**Proof.** Consider the orthogonal drawings in Fig. 1. Observe that $A$ is parallel with $B$, and $C$ is parallel with $D$. In any parallel morph from $A$ to $B$, the $y$-order—i.e. the vertical order in the figure—of $b$ and $d$ can change only if prior to that, the $x$-order of $a$ and $c$ changes. Similarly, in any parallel morph from $C$ to $D$, the $x$-order of $a$ and $c$ can change only if prior to that, the $y$-order of $b$ and $d$ has changed.

We use the drawings of Fig. 1 to construct parallel unmorphable cuboid drawings. For the first drawing, place a copy of $C$ at $z = 0$ and a copy of $A$ at $z = 1$. Using segments parallel with the $z$-axis, connect each vertex at $z = 0$ to the corresponding vertex at $z = 1$; see Fig. 2. Construct the second drawing similarly, except with a copy of $D$ at $z = 0$ and a copy of $B$ at $z = 1$.

It is not difficult to verify that these two drawings are parallel and cuboid. If there exists a parallel morph between these drawings, then this morphs $A$ to $B$ and simultaneously $C$ to $D$. During the morph from $A$ to $B$, the $y$-order of $b$ and $d$ changes. Therefore the $x$-order of $a$ and $c$ must have changed even earlier. However, this is the same as in the drawings of $C$ and $D$. Hence this necessitates that the $y$-order of $b$ and $d$ has changed even earlier, a contradiction. Therefore, there does not exist a parallel morph.

In the drawings in Fig. 2 every face is a simple polygon, and so the drawings are orthogonal polyhedra. From this counterexample, we may conclude that not all pairs of parallel orthogonal polyhedra admit a parallel morph. If the reader objects to coplanar faces, it is still possible to build parallel orthogonal polyhedra that do not admit a parallel morph; see Fig. 3. We note that these polyhedra are orthogonally convex.

What if we further restrict the shapes that faces may take? A drawing in the plane is called rectangular if (the boundary of) every face—including the external face—is a rectangle. Note that each side of a rectangular face might be composed of several graph edges and vertices.

For parallel rectangular drawings a linear morph—i.e. a morph in which each vertex moves from its source position to its target position at constant velocity—is a parallel morph [6]. From this, it is easy to see that for parallel cuboid drawings in which all faces are rectangles, a linear morph is a parallel morph. The question remains whether restricting faces to be rectangular means that orthogonal polyhedra will admit a parallel morph. We solve this problem in the next section.

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**Fig. 1.** Orthogonal drawings in the $x$-$y$ plane.

**Fig. 2.** Parallel unmorphable cuboid drawings.
3.2. Unmorphable rectangular-faced polyhedra

In this section, we give another pair of parallel orthogonal polyhedra that cannot be morphed. This pair adds new features as follows: (1) The faces can be made rectangular (at least after allowing coplanar faces), and (2) the polyhedra are defined from parallel cycles. In particular, we first show that there exist parallel orthogonal drawings of a cycle graph which form the trivial knot, and yet cannot be morphed—a result that is interesting in its own right. (Also interesting is the fact that the two projections of these unmorphable drawings have the same writhe [20].) We then show how to augment the cycles into parallel orthodisks in which all faces are rectangles. Here, an orthodisk is a simple polyhedral surface that is topologically equivalent to a closed disk, such that faces are perpendicular to a coordinate axis and the boundary is an orthogonal drawing of a cycle. These orthodisks can then be “fattened” into parallel orthogonal genus-0 polyhedra, also with all faces rectangles.

**Lemma 3.1.** The parallel drawings $P$ and $Q$ illustrated in Fig. 4 do not admit a parallel morph.

**Proof.** Suppose $P$ and $Q$ are determined by vertex mappings $p$ and $q$, respectively. We introduce new notation. For each $v \in V$, let $p(v) = (p_x(v), p_y(v), p_z(v))$ and $q(v) = (q_x(v), q_y(v), q_z(v))$. Notice that vertices $e, \ldots, l$ must lie in a common $x$-$z$ plane in any drawing that is parallel with $P$ and $Q$. We use the notation $p_y(e, \ldots, l)$ to denote the $y$-value of the plane containing vertices $e, \ldots, l$ in mapping $p$.

Observe that $p_y(b, c) < p_y(e, \ldots, l)$ and $q_y(b, c) > q_y(e, \ldots, l)$. For the sake of contradiction, assume that there exists a parallel morph $R$ from $P = R(0)$ to $Q = R(1)$, where for each $t \in [0, 1]$, $R(t)$ is determined by the mapping $(r_x, r_y, r_z)$. There must exist some $t \in [0, 1]$ such that $r_y(b, c) = r_y(e, \ldots, l)$. Let $t_0$ denote the smallest $t$ for which equality holds.

In $R(t_0)$, $b$ and $c$ lie in the same $x$-$z$ plane as do $e, \ldots, l$. However, before all these vertices can be made coplanar, the edge $(b, c)$ must be moved to a position that does not overlap with edges $(e, f)$ and $(k, l)$ with respect to $x$ and $z$ axes. We will show that this cannot happen in any $R(t)$, where $t < t_0$. Therefore, there exists no parallel morph between $P$ and $Q$. 

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**Fig. 3.** Parallel polyhedra that do not admit a parallel morph; no two faces are coplanar.

**Fig. 4.** Parallel orthogonal drawings of a cycle that do not admit a parallel morph.
Observe, for all \( t \in [0, 1] \) the following inequality holds:
\[
\max(r^t_x(e), r^t_y(l)) < r^t_z(a, b, c, d)
\]
(1)

Let us restrict our attention to drawings \( R(t) \) where \( t < t_0 \). By definition, when \( t < t_0 \)
\[
r^t_y(b, c) < r^t_y(e, \ldots, l) < r^t_y(a, d)
\]
(2)

Therefore, for all \( t < t_0 \) edge \((a, b)\) in drawing \( R(t) \) intersects the \( x-z \) plane through \( r^t_y(e, \ldots, l) \) at some point. Let \( \alpha = (\alpha^t_x, \alpha^t_y, \alpha^t_z) \) denote this point. Observe that \( \alpha^t_z = r^t_z(a, b, c, d) \). Thus, by Eq. (1), \( \alpha^t_z \) must be larger than \( r^t_z(l) \). Since the path of \( \alpha \) must be continuous, and remains in the same \( x-z \) plane as \( e, \ldots, l \) for \( t < t_0 \), we can bound \( \alpha^t \) by the following.
\[
r^t_x(l) < \alpha^t_z < r^t_x(j, k)
\]
(3)

and
\[
r^t_x(j, l) < \alpha^t_z < r^t_y(k, l)
\]
(4)

Symmetrically, let \( \beta = (\beta^t_x, \beta^t_y, \beta^t_z) \) where \( t < t_0 \) denotes the point of intersection in \( R(t) \) between edge \((c, d)\) and the \( x-z \) plane through \( r^t_y(e, \ldots, l) \). Then,
\[
r^t_x(e) < \beta^t_z < r^t_x(f, g)
\]
(5)

and
\[
r^t_x(e, f) < \beta^t_z < r^t_y(g, h)
\]
(6)

Notice that \( \alpha^t_z = \beta^t_z = r^t_z(a, b, c, d) \), where \( t < t_0 \). Putting this together with Eqs. (1), (3) and (5) we have that
\[
\max(r^t_x(e), r^t_y(l)) < r^t_z(a, b, c, d) < \min(r^t_x(j, k), r^t_x(f, g))
\]
(7)

We claim that for all \( t < t_0 \),
\[
r^t_x(k, l) < r^t_x(e, f)
\]
(8)

Suppose that this is not true. Then there must exist some \( t < t_0 \), such that either \( r^t_x(j, k) < r^t_x(e) \) or \( r^t_x(f, g) < r^t_x(l) \). However, by Eq. (7) neither of these can hold. So, by contradiction we have that Eq. (8) holds for all \( t < t_0 \).

Now, for \( t < t_0 \), \( \alpha^t_z = r^t_x(a, b) \) and \( \beta^t_z = r^t_x(c, d) \). Putting these facts together with Eqs. (4), (6) and (8), we have that for all \( t < t_0 \),
\[
r^t_x(a, b) < r^t_x(k, l) < r^t_x(e, f) < r^t_x(c, d)
\]
(9)

By Eqs. (7) and (9), we conclude that for all \( t < t_0 \) in \( R(t) \) edge \((b, c)\) will intersect both \((k, l)\) and \((e, f)\) with respect to \( x \) and \( z \) coordinates. Hence, it is not possible that in \( R(t_0) \) vertices \( b, c, e, \ldots, l \) lie in the same \( x-z \) plane. By contradiction we conclude that \( P \) and \( Q \) do not admit a parallel morph. \( \square \)

**Theorem 2.** There exist parallel orthodisks that do not admit a parallel morph, even when all faces are rectangles.

**Proof.** We construct parallel orthodisks \( P \) and \( Q \) whose boundaries are the drawings \( P \) and \( Q \) of Fig. 4, respectively. By Lemma 3.1, \( P \) and \( Q \) do not admit a parallel morph. Therefore, the orthodisks \( P \) and \( Q \) do not admit a parallel morph.

We begin our construction of \( P \) and \( Q \) by adding new vertices, edges and faces to each of \( P \) and \( Q \), as illustrated by the topmost two drawings in Fig. 5. In particular, to both \( P \) and \( Q \) we add a lower structure that looks like a box without a top, called the tray. Attached to the tray is the band, which consists of three rectangles and connects the tray to the ring, which is simply the boundary of a rectangle. It should be clear that these parallel structures do not admit a parallel morph. However, due to the presence of the ring, the structures are not orthodisks.

To convert these unmorphable structures to orthodisks, we incorporate new parallel orthodisks called gloves. The lowermost drawings in Fig. 5 depict the glove for each of \( P \) and \( Q \). The boundary of each glove is a rectangle. In both \( P \) and \( Q \) the boundary of the glove is arranged to coincide with the ring. With the addition of the glove, the construction is complete.

For the sake of visualizing our construction, imagine that in \( Q \) the glove is a rubber surface stretched over the ring. To get to \( P \) from \( Q \), the tray passes through the ring, extending the rubber surface around the tray. Hence, in \( P \) the glove encloses the tray, while in \( Q \) the tray is not enclosed by the glove (see Fig. 6). To complete the construction the gloves of both \( P \) and \( Q \) must be transformed from rubber surfaces to parallel orthodisks.

We are not yet done. The glove has five faces that are not rectangles (three of these are visible in Fig. 5). However, in each of them the \( x \)-order of the vertices is the same in both polyhedra. Therefore, we can subdivide these faces into rectangles by adding, in both orthodisks, lines perpendicular to the \( x \)-axis from reflex vertices; the other endpoint of each of these lines hits the same edge in both orthodisks since the \( x \)-order is the same, and hence the orthodisks stay parallel. See Fig. 7. \( \square \)
Corollary 3. Pairs of parallel orthogonal genus-0 polyhedra do not always admit a parallel morph, even when faces are restricted to be rectangles.

Proof. The construction above produced parallel polyhedral surfaces, which we now turn into polyhedra. The simplest approach is, for each surface, to glue together two identical copies of the surface, producing a degenerate polyhedron of 0 volume. To avoid degeneracy we can “shrink” one copy of the surface away from the other keeping it combinatorially the same. It is not difficult to construct parallel polyhedra such that, in each, the band connecting the two copies is composed of rectangular faces. \qed
4. Conclusion

Our brief exploration of parallel morphing in \( \mathbb{R}^3 \) raises some interesting questions. We were surprised to discover that parallel orthogonally convex polyhedra do not always morph. Is a parallel morph always possible between parallel orthogonally convex polyhedra when all faces are rectangles? In subsequent work [20] the example of Section 3.2 is used to prove that it is PSPACE-hard to decide whether parallel orthogonal polyhedra of genus 0 admit a parallel morph. It is still open to determine the complexity of deciding whether parallel (orthogonal) drawings of a cycle in \( \mathbb{R}^3 \) admit a parallel morph, though the case of orthogonal drawings of a graph is shown to be PSPACE-hard [20].

References