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On a cyclic connectivity property of directed graphs

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Abstract

Let us call a digraph D cycle-connected if for every pair of vertices $u, v \in V(D)$ there exists a cycle containing both u and v. In this paper we study the following open problem introduced by Ádám. Let D be a cycle-connected digraph. Does there exist a universal edge in D, i.e., an edge $e \in E(D)$ such that for every $w \in V(D)$ there exists a cycle C such that $w \in V(C)$ and $e \in E(C)$?

In his 2001 paper Hetyei conjectured that cycle-connectivity always implies the existence of a universal edge. In the present paper we prove the conjecture of Hetyei for bitournaments.
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Keywords: Directed graph; Bitournament; Directed cycle; Cycle-connectivity; Universal edge

1. Introduction

A *directed graph* (or *digraph*) D consists of a non-empty finite set $V(D)$ of *vertices* and a finite set $E(D)$ of ordered pairs of distinct vertices called *edges*. Throughout this paper all digraphs are simple, i.e., contain no loops, double edges or cycles of length two. We call a pair of vertices u, v of digraph D *cyclic* if there exists a cycle C in D containing both u and v. We call an edge $e \in E(D)$ *universal* if for every $w \in V(D)$ there exists a cycle $C \subseteq D$ containing both e and w. We call a graph *cycle-connected* if any two vertices in it are cyclic. The following problem was introduced by Ádám in [1].

Problem 1.1. *Let D be a cycle-connected digraph. Does D contain a universal edge*?

Observe that all the edges of a Hamiltonian cycle are universal. The existence of Hamiltonian cycles in digraphs has been widely studied, for comprehensive surveys see for example [2,5]. It is well-known that the problem of determining whether there is a Hamiltonian cycle in an arbitrary graph is NP-complete (see [7]). A directed complete m–partite graph is called a semicomplete m-partite digraph. Bang-Jensen, Gutin and Yeo showed that the Hamiltonian cycle problem is polynomial time solvable for semicomplete m-partite digraphs (see [3]) and in particular for bitournaments. (The result in [3] extends the results of Gutin in [8] and Häggkvist and Manoussakis in [11] from which it also follows that the Hamiltonian cycle problem is polynomial time solvable for bitournaments.)

There are many well-known sufficient conditions for a graph to have a Hamiltonian cycle, which thus would guarantee the existence of universal edges for some classes of graphs. For example, strongly connected tournaments are Hamiltonian, see [6,13]. The following characterization was obtained independently by Gutin in [8] and Häggkvist and

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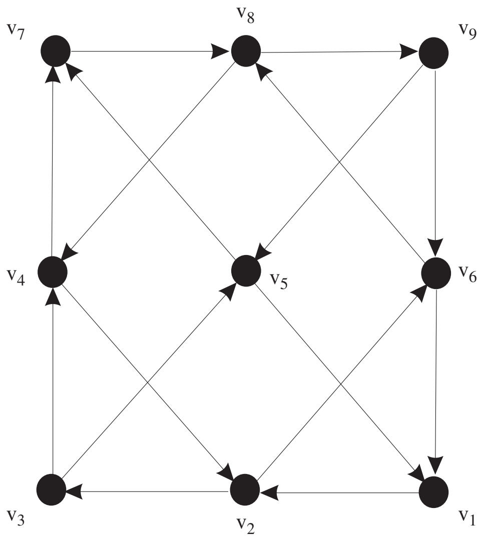

Fig. 1. Example of a cycle-connected non-Hamiltonian graph.

Manoussakis in [11]. A bitournament is Hamiltonian if and only if it is strongly connected and contains a cycle factor (see [2, p. 252]). Some sufficient conditions for the existence of a Hamiltonian cycle in a bitournament are described in the survey paper [10] by Gutin. Since cycle-connectivity implies strong connectivity, the cycle-connected tournaments have universal edges and so do cycle-connected bitournaments that contain a cycle factor.

While a Hamiltonian cycle covers all vertices of the graph, existence of a universal edge is a weaker property that implies only existence of several cycles that cover all vertices and have an edge in common. The graph in Fig. 1 shows that cycle-connectivity does not imply that the graph is Hamiltonian and neither does the existence of a universal edge. It can be easily verified that G is cycle-connected and that all its edges except the edges of the cycle $v_2v_6v_8v_4v_2$ are universal. If G had a Hamiltonian cycle, this cycle would contain paths $v_1v_2v_3$ and $v_7v_8v_9$. Indeed, v_1v_2 is the only out-edge for v_1, v_2v_3 is the only in-edge for v_3, etc. However, if we delete these two paths from G, the remaining graph will have three components, which shows that G is not Hamiltonian.

Previous works of Ádám [1] and Hetyei [12] suggest that Problem 1.1 is interesting and does not seem to be easy. In [1] Ádám defined several cyclic properties of digraphs and studied the connections between them. Among other problems, in [1] Ádám formulated Problem 1.1 (as the fourth subproblem in Problem 3). In [12] Hetyei solved most of the problems of Ádam, for Problem 1.1, however, Hetyei only formulated a conjecture that the answer is always affirmative. In the present paper we will show that the conjecture of Hetyei is true for bitournaments, defined below.

2. Definitions and notations

We call a simple digraph D *bitournament* if it is an oriented complete bipartite graph. If a digraph does not contain a cycle we call it *acyclic*. If the edge uv is in the graph, we say that vertex u *dominates* vertex v or that vertex v *is dominated* by vertex u. We use notation $u \to v$ to indicate that u dominates v. We will use the same notation to indicate that three or more vertices form a path, i.e., $x_1 \to x_2 \to x_3 \ldots$. We call *in-degree* of a vertex x the number of vertices that dominate x, similarly, the *out-degree* of x is the number of vertices dominated by x. Let C be a cycle and $x \in C$. We say that the vertex x^+ is the *successor* of x on C if $xx^+ \in E(C)$. Similarly, we define the successor of the vertex on a path P. For two vertices x_1 and x_k of cycle C let x_1Cx_k denote the path $L \subset C$ staring at x_1 and ending at x_k. We say that vertices $x_1, x_2, x_3, \ldots, x_k$ are *situated on C in this order* if this is the order in which they occur when we walk around the cycle C, starting at x_1. Let D be a digraph and H a subgraph of D. We denote by $D\backslash H$ the graph obtained by deleting all the vertices (and edges) of H from D. We call a cycle C of a digraph D *maximal* if there is no cycle C_1 in D that is longer than C and contains all vertices of C.

3. The main result

Observe that answer to Problem 1.1 is affirmative, i.e., the digraph D contains a universal edge, if D has a vertex x of degree 3 or less. Indeed, then x has in-degree or out-degree 1. Assume without loss of generality that x has out-degree 1 and $x \to x^+$. Then the edge xx^+ is contained in all cycles containing x. Since D is cycle-connected, xx^+ is universal.

Together with this observation, the next theorem shows that any cycle-connected bitournament has a universal edge.

Theorem 3.1. *Assume that*

1. *D is a bitournament*;
2. *every vertex* $x \in V(D)$ *has in-degree and out-degree at least* 2;
3. *D is cycle-connected.*

Then every maximal cycle of D has a universal edge.

From now on, let us consider only digraphs D satisfying conditions of Theorem 3.1. Throughout the paper C will always denote a maximal cycle of the digraph D.

The proof of Theorem 3.1 is based on several lemmas.

Claim 3.2. *Any vertex* $x \in D\backslash C$ *dominates a vertex of C and is dominated by a vertex of C.*

Proof. We will show that x is dominated by a vertex of C. It can be shown by a similar argument that x dominates a vertex of C.

Assume that x is not dominated by any vertex of C. Because of condition (3) of Theorem 3.1, there is a path P in $D\backslash C$ from some $y \in V(C)$ to x. We can suppose that the lengths of P is minimal.

Case 1: y and x are in the same partite class of D. Denote the successor of y on C by y^+. We can replace the edge yy^+ with the path Py^+ and increase the length of C, a contradiction to the maximality of C.

Case 2: x and y are in different partite classes. Consider the edge x_1x of x where $x_1 \in V(P)$. Clearly, $x_1 \notin V(C)$ and $x_1 \to x$. If for some $v \in V(C)$ $v \to x_1$, denote v^+ the successor of v on C. We can replace the edge vv^+ with the path vx_1xv^+ and increase the length of C, a contradiction. Thus, x_1 is not dominated by any vertex of C. Since x_1 and y are in the same partite class, we are done using the argument of Case 1. □

The next theorem has been proved by Gutin [9].

Theorem 3.3. *Let T be a strongly connected bitournament that has a collection F of disjoint cycles. The length of a longest cycle in T is at least* $|V(F)|$.

The following theorem is a corollary of Theorem 3.3 (and also a corollary of a result on cycle factors proved by Gutin [2, p. 259]).

Corollary 3.4. *If Q is a longest cycle in a bitournament D, then* $D\backslash Q$ *is acyclic*.

We will show that the above statement is true not only for a longest cycle but for any maximal cycle of D. Recall, that throughout the paper C denotes a maximal cycle of the bitournament D.

Lemma 3.5. $D\backslash C$ *is acyclic*.

Proof. Assume on the contrary, that there is a cycle K in $D\backslash C$. Let $x \in K$ be an arbitrary vertex. By Claim 3.2 x dominates a vertex of C and is dominated by a vertex of C. It is easy to see that there exist three vertices y^-, y and y^+ on C such that $y^- \to y \to y^+$ and $y^- \to x$ and $x \to y^+$. Let us denote by x^+ the successor of x on K. If $x^+ \to y$ then replacing the edge y^-y by the path y^-xx^+y we can increase the length of C, a contradiction. Thus $y \to x^+$. But in this case we can replace the edge yy^+ with the path yx^+Kxy^+, increasing the length of C, a contradiction. □

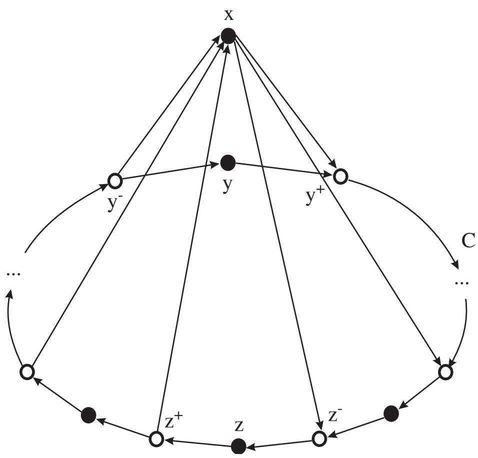

Fig. 2. Change of direction on connecting edges between x and C.

Lemma 3.6. *Any vertex $x \in D\backslash C$ is dominated by at least two vertices of C and dominates at least two vertices of C.*

Proof. Without loss of generality, suppose on the contrary that there is a vertex $x \in D\backslash C$ such that there is exactly one $y \in V(C)$ such that $x \to y$.

From (2) follows that $x \to z$ for some $z \in D\backslash C$. Let $t \in V(C)$ is such that $t \to x$, and let $tt^+ \in E(C)$. Then $zt^+ \notin E(D)$. Indeed, by substituting tt^+ by $txzt^+$ we can increase the length of C, a contradiction. Thus z dominates only vertex y^+ on C, where $yy^+ \in E(C)$. Repeating the above argument for z we find a vertex $z_1 \in D\backslash C$ such that $z \to z_1$ and dominates only one vertex of C and so on. We constructed a sequence $x \to z \to z_1 \to z_2 \to \cdots$ in $D\backslash C$. Since the graph is finite, there is a cycle in $D\backslash C$, a contradiction to Lemma 3.5. We have shown that x dominates at least two vertices of C. By symmetry, x is dominated by at least two vertices of C, which completes the proof. □

We will need the following easy, yet very useful observation.

Claim 3.7. *Let $x \in V(D\backslash C)$. Suppose that the four vertices $a_1, a_2, a_3, a_4 \in V(C)$ are situated on C in this order and $x \to a_2$, $x \to a_4$, $a_1 \to x$, $a_3 \to x$. Then for every edge e of C there is a cycle containing both e and x.*

Proof. Cycles xa_2Ca_1x, xa_2Ca_3x, xa_4Ca_3x, xa_4Ca_1x demonstrate that x is on a cycle with every edge of C. □

We call a vertex $x \in V(D\backslash C)$ *good* if there exist vertices $a_1, a_2, a_3, a_4 \in V(C)$ with the above property and *bad* otherwise. It is easy to see that if every vertex of $D\backslash C$ is good then every edge of C is universal.

From now on, we assume that the partite classes of the bitournament are colored into black and white.

In the proof of Lemma 3.8 we will see that if x is a bad vertex, then the edges connecting x and C follow a certain pattern. Direction change on the connecting edges occurs exactly twice; and in the order of C, the vertices of C dominated by x precede the vertices of C that dominate x. See Fig. 2.

To deal with the case when there are bad vertices in the graph we need the following lemma.

Lemma 3.8. *A bad vertex x is on a cycle Q with every edge of C except possibly two consecutive edges y^-y and yy^+ such that $y^- \to x$ and $x \to y^+$.*

Proof. Without loss of generality, assume that x is a black vertex. Since the graph is a bitournament, every other vertex of C (i.e., every white vertex of C) is connected with x. Let us observe the direction of the edges between x and C as we go around the cycle C. The number of direction changing is necessarily even, and it is positive by Lemma 3.6. It is easy to see that change of direction on the connecting edges occurs exactly two times, in order shown in Fig. 2, i.e., $y^- \to x$, $x \to y^+$ and $x \to z^-$, $z^+ \to x$. Otherwise we would find a configuration of four vertices on C, described in Claim 3.7, which contradicts the definition of x.

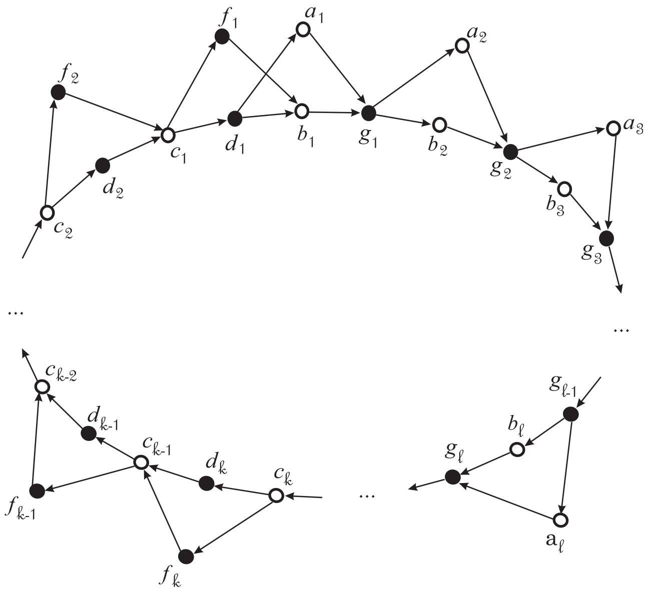

Fig. 3. The cycle C.

Denote Q the cycle $y^-xy^+Cy^-$. The cycle Q demonstrates that all the edges of C, except y^-y and yy^+ are on a cycle with x. $\square$

Using the notations of Lemma 3.8, we call a bad vertex x a *covering* vertex for the edges y^-y, yy^+ and vertices y^-, y, y^+. We say that x *covers* the vertices y^-, y, y^+ and the edges y^-y, yy^+. So, every bad vertex covers two edges and three vertices.

Lemma 3.9. *If an edge $e \in E(C)$ is not universal, there is a bad vertex x that covers it.*

Proof. Assume that edge $uv \in C$ is not universal. Consider the path $P = vCu$. If a vertex $z \in D\backslash C$ is dominated by some vertex $v_1 \in V(P)$ and dominates a vertex $v_2 \in V(P)$ such that v_2 occurs on the path P after v_1 then z is on the cycle zv_2Cv_1z with uv. So, there must be a vertex x such that all vertices of P dominated by x precede (on P) all vertices of P that dominate x. Clearly, x covers uv. $\square$

Recall, that the partite classes of the bitournament D are colored into black and white, respectively.

Assume that there is no universal edge on C. So every edge $e \in E(C)$ (and therefore every vertex in $V(C)$) has at least one covering vertex. Next we will show that if two consecutive intervals of the cycle are covered one with a set F of black vertices, another with a set A of white vertices, then all the vertices in F dominate all the vertices in A.

In Lemma 3.10, without loss of generality, we may assume that F and A are minimal monochromatic covering sets of two consecutive intervals of C.

Lemma 3.10. *Let $c_kd_kc_{k-1}d_{k-1}\ldots c_2d_2c_1d_1b_1g_1b_2g_2\ldots b_lg_l$ be an interval of C, where $\{c_k, c_{k-1}, \ldots, c_1\}$ and $\{b_1, b_2, \ldots, b_l\}$ are white vertices and the rest are black.* (*See Fig.* 3.)

Suppose that the vertices $c_k, d_k, c_{k-1}, d_{k-1}, \ldots, c_1, d_1, b_1$ are covered by black vertices $f_k, f_{k-1}, \ldots, f_1$ so that f_k covers c_k, d_k, c_{k-1}; f_{k-1} covers $c_{k-1}, d_{k-1}, c_{k-2}$; ...; f_2 covers c_2, d_2, c_1; f_1 covers c_1, d_1, b_1. Denote $F = \{f_k, f_{k-1}, \ldots, f_1\}$. Suppose that the vertices $d_1, b_1, g_1, b_2, g_2, \ldots, b_l, g_l$ are covered by white vertices $a_1, a_2, \ldots, a_l$ so that a_1 covers d_1, b_1, g_1; a_2 covers g_1, b_2, g_2; a_3 covers g_2, b_3, g_3; ...; a_l covers g_{l-1}, b_l, g_l. Denote $A=\{a_1, a_2, \ldots, a_l\}$.

Then all the vertices in F dominate all the vertices in A.

Proof. First we will prove by induction that for $j \in \{1, 2, \ldots, l\}$ $f_1 \to a_j$. For $j = 1$ we have $f_1 \to a_1$, because otherwise, we could replace the edge d_1b_1 with the path $d_1a_1f_1b_1$ that would increase the length of C, a contradiction.

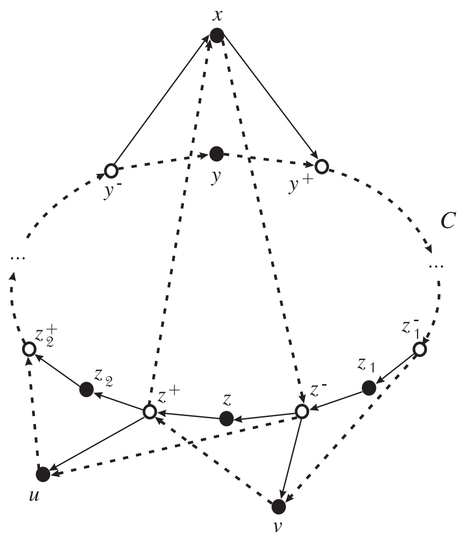

Fig. 4. The cycle K.

Provided, that we already know that for every $j < r$, $f_1 \to a_j$, suppose on the contrary that $a_r \to f_1$. From the definition of a covering vertex and Lemma 3.6 follows that $a_{r-1} \to g_r$ and therefore we can replace the path $g_{r-1}b_r g_r$ with $g_{r-1}a_r f_1 a_{r-1} g_r$ and increase the length of C, a contradiction.

Now we will prove by induction that for every $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, l\}$, $f_i \to a_j$. The case $i = 1$, $j \in \{1, 2, \ldots, l\}$ is shown above. Provided that we already know that for every $i < r$, $f_i \to a_j$, $j \in \{1, 2, \ldots, l\}$. Let q be the smallest index such that $a_q \to f_r$. Using the same argument, from the definition of a covering vertex and Lemma 3.6 it follows that $f_r \to c_{r-2}$ (in the case $r = 2$ $f_2 \to b_1$) and therefore we can replace the path $c_{r-1}d_{r-1}c_{r-2}$ (in the case $r = 2$ $c_1 d_1 b_1$) with $c_{r-1} f_{r-1} a_q f_r c_{r-2}$ (in the case $r = 2$ with $c_1 f_1 a_q f_2 b_1$) and increase the length of C, a contradiction. □

Lemma 3.11. *All vertices that cover the cycle C are of the same color.*

Proof. Assume that there is a black vertex x and a white vertex y that cover some vertices of C. Let us choose a minimal set M of covering vertices such that it covers all vertices of C and $x, y \in M$. Let us partition the cycle C into intervals, each covered with monochromatic vertices from M, such that any two consecutive intervals are covered with different colors. Observe, that as we go around C, according to Lemma 3.10, the covering vertices of each interval dominate the covering vertices of the next interval. Thus, there is a cycle in the subgraph of covering vertices, which is in $D \backslash C$, a contradiction to Lemma 3.5. □

Lemma 3.12. *If there is a monochromatic covering set for C, then every edge of C is universal.*

Proof. Without loss of generality assume that C has a set of black covering vertices. Let x be a black covering vertex, that covers path y^-yy^+ on C (i.e., $y^- \to x$ and $x \to y^+$). See Fig. 4. The cycle xy^+Cy^-x shows that x is on a cycle with every edge of C except y^-y and yy^+. We will construct a cycle K that contains x and the edges y^-y, yy^+.

Let z^-zz^+ be an interval of C such that $z^+ \to x$ and $x \to z^-$. Let us use the notations $z_1^- z_1 z^- z z^+ z_2 z_2^+$ for a path on the cycle C, as seen in Fig. 4. Denote v the covering vertex of z (and hence v covers z^- and z^+ as well). Denote the covering vertex of z_2 by u. It is easy to see that u is also a covering vertex for z^+ and z_2^+. From Lemma 3.6 $z_1^- \to v$ and $z^- \to u$. Denote by K the cycle $z^+xz^-uz_2^+Cz_1^-vz^+$ (shown in Fig. 4 by dotted line). Clearly, K contains x, y^-y and yy^+ which completes the proof. □

Proof of Theorem 3.1. Assume on the contrary that there is no universal edge on the cycle C. Then, by Lemma 3.9, all the edges of C are covered. By Lemma 3.10, all the covering vertices of C are of the same color.

It follows from Lemma 3.12 that all the edges of C are universal, a contradiction. This completes the proof of Theorem 3.1. □

4. Concluding remarks

Theorem 3.1 proves the existence of a universal edge on every maximal cycle of a bitournament. We do not know, however, how many edges of C are universal.

Problem 4.1. *Assume that G is a cycle-connected bitournament and C is a maximal cycle of G. Are all edges of C universal*?

The following problem is in the spirit of Theorem 3.1 and is a special case of Problem 1.1.

Problem 4.2. *Assume that D is a simple bipartite cycle-connected digraph. Does D have a universal edge*?

The proof of Theorem 3.1 does not work in this case, because it uses the completeness of the bipartite graph D. The author conjectures that the answer to Problem 4.2 is affirmative.

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