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Cayley graphs of order 30p are Hamiltonian

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ABSTRACT

Suppose G is a finite group, such that $|G| = 30p$, where p is prime. We show that if S is any generating set of G, then there is a Hamiltonian cycle in the corresponding Cayley graph $\mathrm{Cay}(G; S)$.

1. Introduction

There is a folklore conjecture that every connected Cayley graph has a Hamiltonian cycle. (See the surveys [3,13,15] for some background on this question.) The papers [9,11] began a systematic study of this conjecture in the case of Cayley graphs for which the number of vertices has a prime factorization that is small and easy. In particular, combining several of the results in [11] with [4,5] and this paper shows:

> If $|G| = kp$, where p is prime, with $1 \leq k < 32$ and $k \neq 24$, then every connected Cayley graph on G has a Hamiltonian cycle.

This paper's contribution to the project is the case $k = 30$:

Theorem 1.1. *If $|G| = 30p$, where p is prime, then every connected Cayley graph on G has a Hamiltonian cycle.*

2. Preliminaries

Additional details of some of the proofs in this paper can be found in an expanded version that has been posted on the arxiv [6].

Before proving Theorem 1.1, we present some useful facts about Hamiltonian cycles in Cayley graphs.

Notation. Throughout this paper, G is a finite group.

- For any subset S of G, $\mathrm{Cay}(G; S)$ denotes the *Cayley graph* of G with respect to S. Its vertices are the elements of G, and there is an edge joining g to gs for every $g \in G$ and $s \in S$.
- For $x, y \in G$:
  - $[x, y]$ denotes the *commutator* $x^{-1}y^{-1}xy$, and
  - y^x denotes the *conjugate* $x^{-1}yx$.

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- $\langle A\rangle$ denotes the subgroup generated by a subset A of G.
- G' denotes the *commutator subgroup* $[G, G]$ of G.
- $Z(G)$ denotes the *center* of G.
- $G \ltimes H$ denotes a *semidirect product* of the groups G and H.
- D_{2n} denotes the *dihedral group* of order $2n$.
- For $S \subset G$, a sequence $(s_1, s_2, \ldots, s_n)$ of elements of $S \cup S^{-1}$ specifies the walk in the Cayley graph $\mathrm{Cay}(G; S)$ that visits (in order) the vertices

  $$e, s_1, s_1s_2, s_1s_2s_3, \ldots, s_1s_2 \ldots s_n.$$

  If N is a normal subgroup of G, we use $(\overline{s_1}, \overline{s_2}, \ldots, \overline{s_n})$ to denote the image of this walk in the quotient $\mathrm{Cay}(G/N; S)$.
- If the walk $(\overline{s_1}, \overline{s_2}, \ldots, \overline{s_n})$ in $\mathrm{Cay}(G/N; S)$ is closed, then its *voltage* is the product $s_1s_2 \ldots s_n$. This is an element of N.
- For $k \in \mathbb{Z}^+$, we use $(s_1, \ldots, s_m)^k$ to denote the concatenation of k copies of the sequence $(s_1, \ldots, s_m)$. Abusing notation, we often write s^k and s^{-k} for

  $$(s)^k = (s, s, \ldots, s) \quad \text{and} \quad (s^{-1})^k = (s^{-1}, s^{-1}, \ldots, s^{-1}),$$

  respectively. Furthermore, we often write $\big((s_1, \ldots, s_m), (t_1, \ldots, t_n)\big)$ to denote the concatenation $(s_1, \ldots, s_m, t_1, \ldots, t_n)$. For example, we have

  $$\big((a^2, b)^2, c^{-2}\big)^2 = (a, a, b, a, a, b, c^{-1}, c^{-1}, a, a, b, a, a, b, c^{-1}, c^{-1}).$$

Theorem 2.1 (*Marušič, Durnberger, Keating–Witte [10]*)**.** *If G' is a cyclic group of prime-power order, then every connected Cayley graph on G has a Hamiltonian cycle.*

Lemma 2.2 (*"Factor Group Lemma" [15, Section 2.2]*)**.** *Suppose*

- *S is a generating set of G,*
- *N is a cyclic, normal subgroup of G,*
- *$\overline{C} = (\overline{s_1}, \overline{s_2}, \ldots, \overline{s_n})$ is a Hamiltonian cycle in $\mathrm{Cay}(G/N; S)$, and*
- *the voltage of $\overline{C}$ generates N.*

Then $(s_1, \ldots, s_n)^{|N|}$ is a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

The following easy consequence of the Factor Group Lemma 2.2 is well known (and is implicit in [12]).

Corollary 2.3. *Suppose*

- *S is a generating set of G,*
- *N is a normal subgroup of G, such that $|N|$ is prime,*
- *$s \equiv t \pmod{N}$ for some $s, t \in S \cup S^{-1}$ with $s \neq t$, and*
- *there is a Hamiltonian cycle in $\mathrm{Cay}(G/N; S)$ that uses at least one edge labeled s.*

Then there is a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Theorem 2.4 (*Alspach [1, Corollary 5.2]*)**.** *If $G = \langle s\rangle \ltimes \langle t\rangle$, for some elements s and t of G, then $\mathrm{Cay}\big(G; \{s, t\}\big)$ has a Hamiltonian cycle.*

Lemma 2.5 (*[11, Lemma 2.27]*)**.** *Let S generate the finite group G, and let $s \in S$, such that $\langle s\rangle \lhd G$. If $\mathrm{Cay}\big(G/\langle s\rangle; S\big)$ has a Hamiltonian cycle, and either*

1. *$s \in Z(G)$, or*
2. *$Z(G) \cap \langle s\rangle = \{e\}$,*

then $\mathrm{Cay}(G; S)$ has a Hamiltonian cycle.

Lemma 2.6. *Suppose*

- *$G = \langle a\rangle \ltimes \langle S_0\rangle$, where $\langle S_0\rangle$ is an abelian subgroup of odd order,*
- *$\#(S_0 \cup S_0^{-1}) \geq 3$, and*
- *$\langle S_0\rangle$ has a nontrivial subgroup H, such that $H \lhd G$ and $H \cap Z(G) = \{e\}$.*

Then $\mathrm{Cay}\big(G; S_0 \cup \{a\}\big)$ has a Hamiltonian cycle.

Proof. Since $\langle S_0\rangle$ is abelian of odd order, and $\#(S_0 \cup S_0^{-1}) \geq 3$, we know that $\mathrm{Cay}\big(\langle S_0\rangle; S_0\big)$ is Hamiltonian connected [2]. Therefore, it has a Hamiltonian path $(s_1, s_2, \ldots, s_m)$, such that $s_1s_2 \ldots s_m \in H$. Then

$$\big(s_1, s_2, \ldots, s_m, a\big)^{|a|}$$

is a Hamiltonian cycle in $\mathrm{Cay}\big(G; S_0 \cup \{a\}\big)$. $\square$

Lemma 2.7 ([4, Corollary 4.4]). *If $a, b \in G$, such that $G = \langle a, b \rangle$, then $G' = \langle [a, b] \rangle$.*

Lemma 2.8 ([14, Proposition 5.5]). *If p, q, and r are prime, then every connected Cayley graph on the dihedral group D_{2pqr} has a Hamiltonian cycle.*

Lemma 2.9. *If $G = D_{2pq} \times \mathbb{Z}_r$, where p, q, and r are distinct odd primes, then every connected Cayley graph on G has a Hamiltonian cycle.*

Proof. Let S be a minimal generating set of G, let $\varphi \colon G \to D_{2pq}$ be the natural projection, and let T be the group of rotations in D_{2pq}, so $T = \mathbb{Z}_p \times \mathbb{Z}_q$.

For $s \in S$, we may assume:

- If $\varphi(s)$ has order 2, then $s = \varphi(s)$ has order 2. (Otherwise, Corollary 2.3 applies with $t = s^{-1}$.)
- $\varphi(s)$ is nontrivial. (Otherwise, $s \in \mathbb{Z}_r \subset Z(G)$, so Lemma 2.5(1) applies.)

Since $\varphi(S)$ generates D_{2pq}, it must contain at least one reflection (which is an element of order 2). So $S \cap D_{2pq}$ contains a reflection.

Case 1. *Assume $S \cap D_{2pq}$ contains only one reflection.* Let $a \in S \cap D_{2pq}$, such that a is a reflection.

Let $S_0 = S \setminus \{a\}$. Since $\langle S_0 \rangle$ is a subgroup of the cyclic, normal subgroup $T \times \mathbb{Z}_r$, we know $\langle S_0 \rangle$ is normal. Therefore $G = \langle a \rangle \ltimes \langle S_0 \rangle$, so:

- If $\#S_0 = 1$, then Theorem 2.4 applies.
- If $\#S_0 \geq 2$, then Lemma 2.6 applies with $H = T$, because $T \times \mathbb{Z}_r$ is abelian of odd order.

Case 2. *Assume $S \cap D_{2pq}$ contains at least two reflections.* Since no minimal generating set of D_{2pq} contains three reflections, the minimality of S implies that $S \cap D_{2pq}$ contains exactly two reflections; say a and b are reflections.

Let $c \in S \setminus D_{2pq}$, so $\mathbb{Z}_r \subset \langle c \rangle$. Since $|c| > 2$, we know $\varphi(c)$ is not a reflection, so $\varphi(c) \in T$. The minimality of S (combined with the fact that $\#S > 2$) implies $\langle \varphi(c) \rangle \neq T$. Since $\varphi(c)$ is nontrivial, this implies we may assume $\langle \varphi(c) \rangle = \mathbb{Z}_p$ (by interchanging p and q if necessary). Hence, we may write

$$c = wz \quad \text{with } \langle w \rangle = \mathbb{Z}_p \text{ and } \langle z \rangle = \mathbb{Z}_r.$$

We now use the argument of [10, Case 5.3, p. 96], which is based on ideas of Marušič [12]. Let

$$\overline{G} = G/\mathbb{Z}_p = \overline{D_{2pq}} \times \mathbb{Z}_r = \overline{D_{2pq}} \times \langle \overline{c} \rangle.$$

Then $\overline{D_{2pq}} \cong D_{2q}$, so $(a, b)^q$ is a Hamiltonian cycle in $\mathrm{Cay}\big(\overline{D_{2pq}}; a, b\big)$. With this in mind, it is easy to see that

$$\Big(c^{r-1}, a, \big((b, a)^{q-1}, c^{-1}, (a, b)^{q-1}, c^{-1}\big)^{(r-1)/2}, (b, a)^{q-1}, b\Big)$$

is a Hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$. This contains the string

$$\big(c, a, (b, a)^{q-1}, c^{-1}, a\big),$$

which can be replaced with the string

$$\big(b, c, (b, a)^{q-1}, b, c^{-1}\big)$$

to obtain another Hamiltonian cycle. Since

$$\begin{aligned} ca(ba)^{q-1}c^{-1}a &= \big(cac^{-1}a\big)(ba)^{-(q-1)} \quad (ba \in T \text{ is inverted by } a) \\ &= \big((wz)a(wz)^{-1}a\big)(ba)^{-(q-1)} \\ &= \big(w^2\big)(ba)^{-(q-1)} \quad (a \text{ inverts } w \text{ and centralizes } z) \\ &\neq \big(w^{-2}\big)(ba)^{-(q-1)} \\ &= \big(b(wz)b(wz)^{-1}\big)(ba)^{-(q-1)} \quad (b \text{ inverts } w \text{ and centralizes } z) \\ &= \big(bcbc^{-1}\big)(ba)^{-(q-1)} \\ &= bc(ba)^{q-1}bc^{-1}, \quad (ba \in T \text{ is inverted by } b) \end{aligned}$$

these two Hamiltonian cycles have different voltages. Therefore at least one of them must have a nontrivial voltage. This nontrivial voltage must generate $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

Proposition 2.10. *Suppose*

- *$|G| = 30p$, where p is prime, and*
- *$|G|$ is not square-free (i.e., $p \in \{2, 3, 5\}$).*

Then every Cayley graph on G has a Hamiltonian cycle.

Proof. We know $|G|$ is either 60, 90, or 150, and it is known that every connected Cayley graph of any of these three orders has a Hamiltonian cycle. This can be verified by exhaustive computer search, or see [11, Propositions 7.2 and 9.1] and [7]. □

Lemma 2.11. *Suppose*

- $|G| = 30p$, *where p is prime, and*
- $p \geq 7$.

Then

1. G' *is cyclic,*
2. $G' \cap Z(G) = \{e\}$,
3. $G \cong \mathbb{Z}_n \ltimes G'$, *for some* $n \in \mathbb{Z}^+$, *and*
4. *if b is a generator of $\mathbb{Z}_n$, and we choose $\tau \in \mathbb{Z}$, such that $x^b = x^\tau$ for all $x \in G'$, then* $\gcd\big(\tau - 1, |a|\big) = 1$.

Proof. Since $|G|$ is square-free (because $p \geq 7$), we know that every Sylow subgroup of G is cyclic. Therefore the conclusions follow from [8, Theorem 9.4.3, p. 146].[1] □

3. Proof of the main theorem

Proof of Theorem 1.1. Because of Proposition 2.10, we may assume

$$p \geq 7,$$

so the conclusions of Lemma 2.11 hold.

We may also assume $|G'|$ is not prime (otherwise Theorem 2.1 applies). Furthermore, if $|G'| = 15p$, then G is a dihedral group, so Lemma 2.8 applies. In addition, if $|G'| = 15$, then $G \cong D_{30} \times \mathbb{Z}_p$, so Lemma 2.9 applies. Thus, we may assume $|G'| = pq$, where $q \in \{3, 5\}$. So

$$G = \mathbb{Z}_{2r} \ltimes \mathbb{Z}_{pq}, \quad \text{with } \{q, r\} = \{3, 5\} (\text{and } G' = \mathbb{Z}_{pq}).$$

Note that $\mathbb{Z}_r$ centralizes $\mathbb{Z}_q$, because there is no nonabelian group of order 15, so $\mathbb{Z}_2$ must act nontrivially on $\mathbb{Z}_q$. Therefore

$$y^x = y^{-1} \quad \text{whenever } y \in \mathbb{Z}_q \text{ and } \langle x \rangle = \mathbb{Z}_{2r}.$$

We also assume

$$\mathbb{Z}_r \quad \text{does not centralize } \mathbb{Z}_p,$$

because otherwise $G \cong D_{2pq} \times \mathbb{Z}_r$, so Lemma 2.9 applies.

Given a minimal generating set S of G, we may assume

$$S \cap G' = \emptyset,$$

for otherwise Lemma 2.5(2) applies.

Case 1. *Assume* $\#S = 2$. Write $S = \{a, b\}$.

Subcase 1.1. *Assume $|a|$ is odd.* This implies a has order r in G/G', so $(a^{-(r-1)}, b^{-1}, a^{r-1}, b)$ is a Hamiltonian cycle in $\mathrm{Cay}(G/G'; S)$. Its voltage is

$$a^{-(r-1)} b^{-1} a^{r-1} b = [a^{r-1}, b].$$

Since $\gcd(r-1, |a|) \mid \gcd(r-1, 15p) = 1$, we know $\langle a^{r-1}, b \rangle = \langle a, b \rangle = G$. So $\langle [a^{r-1}, b] \rangle = G'$ (see Lemma 2.7). Therefore the Factor Group Lemma 2.2 applies.

Subcase 1.2. *Assume a and b both have even order.*

Subsubcase 1.2.1. *Assume a has order* 2 *in* G/G'. Note that $q \nmid |a|$, since $\mathbb{Z}_2$ does not centralize $\mathbb{Z}_q$. Also, if $|a| = 2p$, then Corollary 2.3 applies. Therefore, we may assume $|a| = 2$.

Now b must generate G/G' (since $\langle a, b \rangle = G$, and b has even order), so b has trivial centralizer in $\mathbb{Z}_{pq}$. Then, since $|a| = 2$ and $\langle a, b \rangle = G$, it follows that a must also have trivial centralizer in $\mathbb{Z}_{pq}$. Therefore (up to isomorphism), we must have either:

1. $a = x^3$ and $b = xyw$, in $G = \mathbb{Z}_6 \ltimes (\mathbb{Z}_5 \times \mathbb{Z}_p) = \langle x \rangle \ltimes \big(\langle y \rangle \times \langle w \rangle\big)$, with $y^x = y^{-1}$ and $w^x = w^d$, where d is a primitive 6th root of 1 in $\mathbb{Z}_p$ (so $d^2 - d + 1 \equiv 0 \pmod p$), or
2. $a = x^5$ and $b = xyw$, in $G = \mathbb{Z}_{10} \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_p) = \langle x \rangle \ltimes \big(\langle y \rangle \times \langle w \rangle\big)$ with $y^x = y^{-1}$ and $w^x = w^d$, where d is a primitive 10th root of 1 in $\mathbb{Z}_p$ (so $d^4 - d^3 + d^2 - d + 1 \equiv 0 \pmod p$).

[1] The condition $[(r-1), nm] = 1$ in the statement of [8, Corollary 9.4.3, p. 146] suffers from a typographical error—it should say $\gcd\Big((r-1)n, m\Big) = 1$.

For (1), we note that the sequence $\big((a, b^{-5})^4, a, b^5\big)$ is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$
\begin{array}{cccccccccccc}
\overline{e} & \xrightarrow{a} & \overline{x^3} & \xrightarrow{b^{-1}} & \overline{x^2y} & \xrightarrow{b^{-1}} & \overline{x} & \xrightarrow{b^{-1}} & \overline{y} & \xrightarrow{b^{-1}} & \overline{x^5} \\
 & \xrightarrow{b^{-1}} & \overline{x^4y} & \xrightarrow{a} & \overline{xy^4} & \xrightarrow{b^{-1}} & \overline{y^2} & \xrightarrow{b^{-1}} & \overline{x^5y^4} & \xrightarrow{b^{-1}} & \overline{x^4y^2} \\
 & \xrightarrow{b^{-1}} & \overline{x^3y^4} & \xrightarrow{b^{-1}} & \overline{x^2y^2} & \xrightarrow{a} & \overline{x^5y^3} & \xrightarrow{b^{-1}} & \overline{x^4y^3} & \xrightarrow{b^{-1}} & \overline{x^3y^3} \\
 & \xrightarrow{b^{-1}} & \overline{x^2y^3} & \xrightarrow{b^{-1}} & \overline{xy^3} & \xrightarrow{b^{-1}} & \overline{y^3} & \xrightarrow{a} & \overline{x^3y^2} & \xrightarrow{b^{-1}} & \overline{x^2y^4} \\
 & \xrightarrow{b^{-1}} & \overline{xy^2} & \xrightarrow{b^{-1}} & \overline{y^4} & \xrightarrow{b^{-1}} & \overline{x^5y^2} & \xrightarrow{b^{-1}} & \overline{x^4y^4} & \xrightarrow{a} & \overline{xy} \\
 & \xrightarrow{b} & \overline{x^2} & \xrightarrow{b} & \overline{x^3y} & \xrightarrow{b} & \overline{x^4} & \xrightarrow{b} & \overline{x^5y} & \xrightarrow{b} & \overline{e}.
\end{array}
$$

Calculating modulo the normal subgroup $\langle y\rangle$, its voltage is

$$
\begin{aligned}
(ab^{-5})^4(ab^5) &= (ab)^4(ab^{-1}) \quad (b^6 = e) \\
&\equiv \big(x^3\,(xw)\big)^4\,\big(x^3\,(xw)^{-1}\big) \\
&= \big(x^4w\big)^4\,\big((xw^{-1})^{-1}\,x^3\big) \quad (x^3 \text{ inverts } w) \\
&= \big(x^{16}w^{d^{12}+d^8+d^4+1}\big)\,\big((wx^{-1})\,x^3\big) \\
&= x^{-2}w^{1+d^2-d+2}x^2 \quad \begin{pmatrix} x^6 = e \text{ and} \\ d^3 \equiv -1 \pmod p \end{pmatrix} \\
&= x^{-2}w^{d^2+2}x^2 \\
&= x^{-2}w^{d+1}x^2 \quad \big(d^2 - d + 1 \equiv 0 \pmod p\big),
\end{aligned}
$$

which is nontrivial. Therefore, the voltage generates $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

For (2), here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$
\begin{array}{cccccccccccc}
\overline{e} & \xrightarrow{a} & \overline{x^5} & \xrightarrow{b} & \overline{x^6y} & \xrightarrow{b} & \overline{x^7} & \xrightarrow{b} & \overline{x^8y} & \xrightarrow{b} & \overline{x^9} \\
 & \xrightarrow{a} & \overline{x^4} & \xrightarrow{b} & \overline{x^5y} & \xrightarrow{a} & \overline{y^2} & \xrightarrow{b} & \overline{xy^2} & \xrightarrow{b} & \overline{x^2y^2} \\
 & \xrightarrow{b} & \overline{x^3y^2} & \xrightarrow{b} & \overline{x^4y^2} & \xrightarrow{a} & \overline{x^9y} & \xrightarrow{b^{-1}} & \overline{x^8} & \xrightarrow{b^{-1}} & \overline{x^7y} \\
 & \xrightarrow{b^{-1}} & \overline{x^6} & \xrightarrow{a} & \overline{x} & \xrightarrow{b^{-1}} & \overline{y} & \xrightarrow{a} & \overline{x^5y^2} & \xrightarrow{b} & \overline{x^6y^2} \\
 & \xrightarrow{b} & \overline{x^7y^2} & \xrightarrow{a} & \overline{x^2y} & \xrightarrow{b} & \overline{x^3} & \xrightarrow{b} & \overline{x^4y} & \xrightarrow{a} & \overline{x^9y^2} \\
 & \xrightarrow{b^{-1}} & \overline{x^8y^2} & \xrightarrow{a} & \overline{x^3y} & \xrightarrow{b^{-1}} & \overline{x^2} & \xrightarrow{b^{-1}} & \overline{xy} & \xrightarrow{b^{-1}} & \overline{e}.
\end{array}
$$

Calculating modulo $\langle y\rangle$, its voltage is

$$
\begin{aligned}
&ab^4(aba)b^4(ab^{-3}a)b^{-1}(ab^2)^2(ab^{-1}a)b^{-3} \\
&\quad \equiv x^5(xw)^4\big(x^5(xw)x^5\big)(xw)^4\big(x^5(xw)^{-3}x^5\big)\cdot(xw)^{-1}\big(x^5(xw)^2\big)^2\big(x^5(xw)^{-1}x^5\big)(xw)^{-3} \\
&\quad = x^5(xw)^4\big(xw^{-1}\big)(xw)^4\big(xw^{-1}\big)^{-3}\cdot(xw)^{-1}\big((xw^{-1})^2(xw)^2\big)\big(xw^{-1}\big)^{-1}(xw)^{-3} \\
&\quad = x^5(x^4w^{d^3+d^2+d+1})\big(xw^{-1}\big)(x^4w^{d^3+d^2+d+1})\big(w^{d^2+d+1}x^{-3}\big)\cdot(w^{-1}x^{-1})\big(x^4w^{-d^3-d^2+d+1}\big)\big(wx^{-1}\big)(w^{-(d^2+d+1)}x^{-3}) \\
&\quad = w^{d(d^3+d^2+d+1)}w^{-1}w^{d^6(d^3+d^2+d+1)}w^{d^6(d^2+d+1)}\cdot w^{-d^9}w^{d^6(-d^3-d^2+d+1)}w^{d^6}w^{-d^7(d^2+d+1)} \\
&\quad = w^{-2d^9+2d^7+4d^6+d^4+d^3+d^2+d-1}.
\end{aligned}
$$

Modulo p, the exponent of w is:

$$
\begin{aligned}
-2d^9+2d^7+4d^6+d^4+d^3+d^2+d-1 &\equiv 2d^4-2d^2-4d+d^4+d^3+d^2+d-1 \quad (\text{because } d^5 \equiv -1) \\
&= 3d^4+d^3-d^2-3d-1 \\
&= 3(d^4-d^3+d^2-d+1)+4(d^3-d^2-1) \\
&\equiv 3(0)+4(d^3-d^2-1) \\
&= 4(d^3-d^2-1).
\end{aligned}
$$

This is nonzero (mod p), because $d^4-d^3+d^2-d+1 \equiv 0 \pmod p$ and

$$(d^3-d^2)(d^3-d^2-1)-(d^2-d-1)(d^4-d^3+d^2-d+1) = 1.$$

Therefore the voltage generates $\langle w\rangle = \mathbb{Z}_p$, so the Factor Group Lemma 2.2 applies.

Subsubcase 1.2.2. *Assume a and b both have order $2r$ in G/G'*. Then $|a| = |b| = 2r$ (because $\mathbb{Z}_{2r}$ has trivial centralizer in $\mathbb{Z}_{pq}$).

We have $a \in b^i G'$ for some i with $\gcd(i, 2r) = 1$. We may assume $1 \le i < r$ by replacing a with its inverse if necessary. Here is a Hamiltonian cycle in $\mathrm{Cay}(G/G'; S)$:

$$\big((a, b, a^{-1}, b)^{(i-1)/2}, a, b^{2r+1-2i}\big).$$

To calculate its voltage, write $a = b^i yw$, where $\langle y\rangle = \mathbb{Z}_q$ and $\langle w\rangle = \mathbb{Z}_p$. We have $y^b = y^{-1}$ and $w^b = w^d$, where d is a primitive rth or $(2r)$th root of unity in $\mathbb{Z}_p$. Then the voltage of the walk is:

$$\begin{aligned}
(aba^{-1}b)^{(i-1)/2}ab^{2r+1-2i} &= \big((b^iyw)b(b^iyw)^{-1}b\big)^{(i-1)/2}(b^iyw)b^{1-2i}\\
&= \big((b^iyw)b(w^{-1}y^{-1}b^{-i})b\big)^{(i-1)/2}(b^iyw)b^{1-2i}\\
&= \big(b^2y^{-2}w^{(d-1)d^{1-i}}\big)^{(i-1)/2}(b^iyw)b^{1-2i}\\
&= \big(b^{i-1}y^{-(i-1)}w^{(d-1)d^{1-i}(d^{i-3}+d^{i-5}+\cdots+d^2+1)}\big)(b^iyw)b^{1-2i}\\
&= b^{2i-1}y^{(i-1)+1}w^{(d-1)d(d^{i-3}+d^{i-5}+\cdots+d^2+1)+1}b^{1-2i}.
\end{aligned}$$

Now:

- The exponent of y is $(i-1)+1 = i$. If $q \mid i$, then, since $i < r$, we must have $q = 3$, $r = 5$, and $i = 3$.
- The exponent of w is

$$\begin{aligned}
&(d-1)d(d^{i-3}+d^{i-5}+\cdots+d^2+1)+1 = d(d-1)\frac{d^{i-1}-1}{d^2-1}+1\\
&= d\frac{d^{i-1}-1}{d+1}+1 = \frac{d^i-d}{d+1}+\frac{d+1}{d+1} = \frac{d^i+1}{d+1}.
\end{aligned}$$

  This is not divisible by p, because d is a primitive rth or $(2r)$th root of 1 in $\mathbb{Z}_p$, and $\gcd(i, 2r) = 1$.

Thus, the voltage generates G' (so the Factor Group Lemma 2.2 applies) unless $q = 3$, $r = 5$, and $i = 3$.

In this case, since $i = 3$, we have $a = b^3yw$. Also, we may assume $b = x$. Then a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$ is:

$$\begin{array}{ccccccccccc}
\overline{e} & \xrightarrow{a^{-1}} & \overline{x^7y} & \xrightarrow{a^{-1}} & \overline{x^4} & \xrightarrow{a^{-1}} & \overline{xy} & \xrightarrow{a^{-1}} & \overline{x^8} & \xrightarrow{a^{-1}} & \overline{x^5y}\\
 & \xrightarrow{a^{-1}} & \overline{x^2} & \xrightarrow{a^{-1}} & \overline{x^9y} & \xrightarrow{a^{-1}} & \overline{x^6} & \xrightarrow{a^{-1}} & \overline{x^3y} & \xrightarrow{b} & \overline{x^4y^2}\\
 & \xrightarrow{a} & \overline{x^7y^2} & \xrightarrow{a} & \overline{y^2} & \xrightarrow{a} & \overline{x^3y^2} & \xrightarrow{a} & \overline{x^6y^2} & \xrightarrow{a} & \overline{x^9y^2}\\
 & \xrightarrow{a} & \overline{x^2y^2} & \xrightarrow{a} & \overline{x^5y^2} & \xrightarrow{a} & \overline{x^8y^2} & \xrightarrow{a} & \overline{xy^2} & \xrightarrow{b} & \overline{x^2y}\\
 & \xrightarrow{a} & \overline{x^5} & \xrightarrow{a} & \overline{x^8y} & \xrightarrow{a} & \overline{x} & \xrightarrow{a} & \overline{x^4y} & \xrightarrow{a} & \overline{x^7}\\
 & \xrightarrow{a} & \overline{y} & \xrightarrow{a} & \overline{x^3} & \xrightarrow{a} & \overline{x^6y} & \xrightarrow{a} & \overline{x^9} & \xrightarrow{b} & \overline{e}.
\end{array}$$

Calculating modulo $\langle y\rangle$, and noting that $|a| = 2r = 10$, its voltage is

$$\begin{aligned}
a^{-9}b(a^9b)^2 &= ab(a^{-1}b)^2 \equiv \big((x^3w)x\big)\big(w^{-1}x^{-2}\big)^2\\
&= \big(x^4w^d\big)\big(w^{-1-d^2}x^{-4}\big) = x^4w^{-(d^2-d+1)}x^{-4}.
\end{aligned}$$

Since d is a primitive 5th or 10th root of 1 in $\mathbb{Z}_p$, we know that it is not a primitive 6th root of 1, so $d^2 - d + 1 \not\equiv 0 \pmod p$. Therefore the voltage is nontrivial, and hence generates $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 applies.

Case 2. *Assume $\#S = 3$, and S remains minimal in $G/\mathbb{Z}_p = \overline{G}$*. Since $G = \mathbb{Z}_{2r} \ltimes \mathbb{Z}_{pq}$ and $\mathbb{Z}_r$ centralizes $\mathbb{Z}_q$, we know $\overline{G} \cong (\mathbb{Z}_2 \ltimes \mathbb{Z}_q) \times \mathbb{Z}_r$. Also, since $\mathbb{Z}_2$ inverts $\mathbb{Z}_q$, we have $\mathbb{Z}_2 \ltimes \mathbb{Z}_q \cong D_{2q}$. Therefore, $\overline{G} \cong D_{2q} \times \mathbb{Z}_r$, so we may write $S = \{a, b, c\}$ with $\langle \overline{a}, \overline{b}\rangle = D_{2q}$ and $\langle \overline{c}\rangle = \mathbb{Z}_r$. Since $S \cap G' = \emptyset$, we know that $\overline{a}$ and $\overline{b}$ are reflections, so they have order 2 in $G/\mathbb{Z}_p$. Therefore, we may assume $|a| = |b| = 2$, for otherwise Corollary 2.3 applies. Also, since $\mathbb{Z}_r$ does not centralize $\mathbb{Z}_p$, we know that $|c| = r$. Replacing c by a conjugate, we may assume $\langle c\rangle = \mathbb{Z}_r$.

We may assume $\mathbb{Z}_r \not\subset Z(G)$ (otherwise Lemma 2.9 applies), so we may assume $[a, c] \ne e$ (by interchanging a and b if necessary). Let

$$W = \big((b, a)^{q-1}, c, (c^{r-2}, a, c^{-(r-2)}, b)^{q-1}\big).$$

Then

$$\big(W, c^{r-2}, a, c^{-(r-1)}, a\big) \quad \text{and} \quad \big(W, c^{r-3}, a, c^{-(r-1)}, a, c\big)$$

are Hamiltonian cycles in $\mathrm{Cay}(G/G'; S)$. Let v be the voltage of the first of these, and let $\gamma = [a, c]\,[a, c]^{ac}$. Then the voltage of the second is

$$\begin{aligned} v \cdot (c^{r-2}ac^{-(r-1)}a)^{-1}(c^{r-3}ac^{-(r-1)}ac) &= v \cdot \left(ac^{r-1}ac^{-(r-2)}\right)(c^{r-3}ac^{-(r-1)}ac\right) \\ &= v \cdot \left(ac^{-1}ac^{-1}acac\right) \\ &= v \cdot \left(ac^{-1}[a, c]ac\right) \\ &= v \cdot \left(ac^{-1}ac[a, c]^{ac}\right) \\ &= v \cdot \left([a, c]\,[a, c]^{ac}\right) \\ &= v\gamma. \end{aligned}$$

Since $[a, c]$ generates $\mathbb{Z}_p$, and ac does not invert $\mathbb{Z}_p$ (this is because a inverts $\mathbb{Z}_p$, and c does not centralize $\mathbb{Z}_p$), we know $\gamma \neq e$. Therefore v and $v\gamma$ cannot both be trivial, so at least one of them generates $\mathbb{Z}_p$. Then the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Case 3. *Assume* $\#S = 3$, *and* S *does not remain minimal in* $G/\mathbb{Z}_p$. Choose a 2-element subset $\{a, b\}$ of S that generates $G/\mathbb{Z}_p$. As in Case 2, we have $G/\mathbb{Z}_p \cong D_{2q} \times \mathbb{Z}_r$. From the minimality of S, we see that $\langle a, b\rangle = D_{2q} \times \mathbb{Z}_r$ (up to a conjugate). The projection of $\{a, b\}$ to D_{2q} must be of the form $\{f, y\}$ or $\{f, fy\}$, where f is a reflection and y is a rotation. Thus, using z to denote a generator of $\mathbb{Z}_r$ (and noting that $y \notin S$, because $S \cap G' = \emptyset$), we see that $\{a, b\}$ must be of the form

1. $\{f, yz\}$, or
2. $\{f, fyz\}$, or
3. $\{fz, yz^{\ell}\}$, with $\ell \not\equiv 0 \pmod r$, or
4. $\{fz, fyz^{\ell}\}$, with $\ell \not\equiv 0 \pmod r$.

Let c be the final element of S. We may write

$$c = f^i y^j z^k w \quad \text{with } 0 \le i < 2,\ 0 \le j < q,\ \text{and } 0 \le k < r.$$

Note that, since $S \cap G' = \emptyset$, we know that i and k cannot both be 0. Let d be a primitive rth root of unity in $\mathbb{Z}_p$, such that

$$w^z = w^d \quad \text{for } w \in \mathbb{Z}_p.$$

Subcase 3.1. *Assume* $a = f$ *and* $b = yz$. From the minimality of S, we know $\langle b, c\rangle \neq G$, so $i = 0$, so we must have $k \neq 0$.

Subsubcase 3.1.1. *Assume* $k = 1$. Then $b \equiv c \pmod{G'}$, so we have the Hamiltonian cycles $(a, b^{-(r-1)}, a, b^{r-2}, c)$ and $(a, b^{-(r-1)}, a, b^{r-3}, c^2)$ in $\mathrm{Cay}(G/G'; S)$. The voltage of the first is

$$\begin{aligned} ab^{-(r-1)}ab^{r-2}c &= \left(ab^{-(r-1)}ab^{r-1}\right)\left(b^{-1}c\right) \\ &= \left((f)(yz)^{-(r-1)}(f)(yz)^{r-1}\right)\left((yz)^{-1}(y^j zw)\right) \\ &= \left(y^{2(r-1)}\right)\left(y^{j-1}w\right) \\ &= \begin{cases} y^{j+3}w & \text{if } r = 3 \text{ and } q = 5, \\ y^{j+7}w & \text{if } r = 5 \text{ and } q = 3 \end{cases} \\ &= y^{j-2}w, \end{aligned}$$

which generates $\mathbb{Z}_q \times \mathbb{Z}_p = G'$ if $j \neq 2$.

So we may assume $j = 2$ (for otherwise the Factor Group Lemma 2.2 applies). In this case, the voltage of the second Hamiltonian cycle is

$$\begin{aligned} ab^{-(r-1)}ab^{r-3}c^2 &= \left(ab^{-(r-1)}ab^{r-1}\right)\left(b^{-2}c^2\right) \\ &= \left((f)(yz)^{-(r-1)}(f)(yz)^{r-1}\right)\left((yz)^{-2}(y^2zw)^2\right) \\ &= \left(y^{2(r-1)}\right)\left(y^2w^{d+1}\right) \\ &= \begin{cases} y^6w^{d+1} & \text{if } r = 3 \text{ and } q = 5, \\ y^{10}w^{d+1} & \text{if } r = 5 \text{ and } q = 3 \end{cases} \\ &= yw^{d+1}, \end{aligned}$$

which generates $\mathbb{Z}_q \times \mathbb{Z}_p = G'$. So the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Subsubcase 3.1.2. *Assume* $k > 1$. We may replace c with its inverse, so we may assume $k \le (r-1)/2$. Therefore $r \neq 3$, so we must have $r = 5$ and $k = 2$. So $a = f$, $b = yz$, and $c = y^j z^2 w$.

Subsubsubcase 3.1.2.1. *Assume* $j = 0$. Here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$\begin{array}{cccccccccccc}
\overline{e} & \xrightarrow{a} & \overline{f} & \xrightarrow{b} & \overline{fyz} & \xrightarrow{a} & \overline{y^2z} & \xrightarrow{b} & \overline{z^2} & \xrightarrow{a} & \overline{fz^2} \\
 & \xrightarrow{b} & \overline{fyz^3} & \xrightarrow{a} & \overline{y^2z^3} & \xrightarrow{b} & \overline{z^4} & \xrightarrow{a} & \overline{fz^4} & \xrightarrow{b^{-1}} & \overline{fy^2z^3} \\
 & \xrightarrow{a} & \overline{yz^3} & \xrightarrow{b} & \overline{y^2z^4} & \xrightarrow{c^{-1}} & \overline{y^2z^2} & \xrightarrow{a} & \overline{fyz^2} & \xrightarrow{c} & \overline{fyz^4} \\
 & \xrightarrow{b^{-1}} & \overline{fz^3} & \xrightarrow{a} & \overline{z^3} & \xrightarrow{b} & \overline{yz^4} & \xrightarrow{a} & \overline{fy^2z^4} & \xrightarrow{c^{-1}} & \overline{fy^2z^2} \\
 & \xrightarrow{a} & \overline{yz^2} & \xrightarrow{c^{-1}} & \overline{y} & \xrightarrow{a} & \overline{fy^2} & \xrightarrow{b} & \overline{fz} & \xrightarrow{a} & \overline{z} \\
 & \xrightarrow{b^{-1}} & \overline{y^2} & \xrightarrow{a} & \overline{fy} & \xrightarrow{b} & \overline{fy^2z} & \xrightarrow{a} & \overline{yz} & \xrightarrow{b^{-1}} & \overline{e}.
\end{array}$$

Letting $\epsilon \in \{\pm 1\}$, such that $w^f = w^\epsilon$, and calculating modulo $\langle y \rangle$, its voltage is

$$\begin{aligned}
&(ab)^4(ab^{-1}ab)(c^{-1}ac)(b^{-1}ab)(ac^{-1})^2(abab^{-1})^2 \\
&\quad \equiv (fz)^4(fz^{-1}fz)(w^{-1}z^{-2}fz^2w)(z^{-1}fz)(fw^{-1}z^{-2})^2(fzfz^{-1})^2 \\
&\quad = (z^4)(e)(w^{\epsilon-1}f)(f)(w^{-(\epsilon+d^2)}z^{-4})(e) \\
&\quad = z^4w^{-(d^2+1)}z^{-4}.
\end{aligned}$$

Since d is a primitive 5th root of unity in $\mathbb{Z}_p$, we know that $d^2 + 1 \not\equiv 0 \pmod{p}$, so the voltage is nontrivial, and hence generates $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 applies.

Subsubsubcase 3.1.2.2. *Assume* $j \neq 0$. Since $\langle a, c \rangle \neq G$, this implies f centralizes $\mathbb{Z}_p$, so $G = D_6 \times (\mathbb{Z}_5 \ltimes \mathbb{Z}_p)$.

If $j = 1$ (so $c = yz^2w$), here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$\begin{array}{cccccccccccc}
\overline{e} & \xrightarrow{a} & \overline{f} & \xrightarrow{b} & \overline{fyz} & \xrightarrow{a} & \overline{y^2z} & \xrightarrow{b} & \overline{z^2} & \xrightarrow{a} & \overline{fz^2} \\
 & \xrightarrow{b} & \overline{fyz^3} & \xrightarrow{a} & \overline{y^2z^3} & \xrightarrow{b} & \overline{z^4} & \xrightarrow{b} & \overline{y} & \xrightarrow{a} & \overline{fy^2} \\
 & \xrightarrow{b} & \overline{fz} & \xrightarrow{a} & \overline{z} & \xrightarrow{b^{-1}} & \overline{y^2} & \xrightarrow{a} & \overline{fy} & \xrightarrow{b} & \overline{fy^2z} \\
 & \xrightarrow{a} & \overline{yz} & \xrightarrow{b} & \overline{y^2z^2} & \xrightarrow{a} & \overline{fyz^2} & \xrightarrow{c} & \overline{fy^2z^4} & \xrightarrow{a} & \overline{yz^4} \\
 & \xrightarrow{b^{-1}} & \overline{z^3} & \xrightarrow{a} & \overline{fz^3} & \xrightarrow{b} & \overline{fyz^4} & \xrightarrow{a} & \overline{y^2z^4} & \xrightarrow{b^{-1}} & \overline{yz^3} \\
 & \xrightarrow{a} & \overline{fy^2z^3} & \xrightarrow{b} & \overline{fz^4} & \xrightarrow{c^{-1}} & \overline{fy^2z^2} & \xrightarrow{a} & \overline{yz^2} & \xrightarrow{c^{-1}} & \overline{e}.
\end{array}$$

Calculating modulo the normal subgroup $D_6 = \langle f, y \rangle$, its voltage is

$$\begin{aligned}
(ab)^4(ba)^2(b^{-1}a)(ba)^2(c)(ab^{-1}ab)^2(c^{-1}ac^{-1}) &\equiv (ez)^4(ze)^2(z^{-1}e)(ze)^2(z^2w)(ez^{-1}ez)^2(w^{-1}z^{-2}ew^{-1}z^{-2}) \\
&= z^7w^{-1}z^{-2} \\
&= z^2w^{-1}z^{-2},
\end{aligned}$$

because $|z| = r = 5$. Since this voltage generates $\mathbb{Z}_p$, the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

If $j = 2$ (so $c = y^2z^2w$), here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$\begin{array}{cccccccccccc}
\overline{e} & \xrightarrow{b^{-1}} & \overline{y^2z^4} & \xrightarrow{a} & \overline{fyz^4} & \xrightarrow{b} & \overline{fy^2} & \xrightarrow{b} & \overline{fz} & \xrightarrow{a} & \overline{z} \\
 & \xrightarrow{b} & \overline{yz^2} & \xrightarrow{a} & \overline{fy^2z^2} & \xrightarrow{b} & \overline{fz^3} & \xrightarrow{a} & \overline{z^3} & \xrightarrow{c} & \overline{y^2} \\
 & \xrightarrow{b^{-1}} & \overline{yz^4} & \xrightarrow{a} & \overline{fy^2z^4} & \xrightarrow{b} & \overline{f} & \xrightarrow{b} & \overline{fyz} & \xrightarrow{a} & \overline{y^2z} \\
 & \xrightarrow{b} & \overline{z^2} & \xrightarrow{a} & \overline{fz^2} & \xrightarrow{b} & \overline{fyz^3} & \xrightarrow{a} & \overline{y^2z^3} & \xrightarrow{c} & \overline{y} \\
 & \xrightarrow{b^{-1}} & \overline{z^4} & \xrightarrow{a} & \overline{fz^4} & \xrightarrow{b} & \overline{fy} & \xrightarrow{b} & \overline{fy^2z} & \xrightarrow{a} & \overline{yz} \\
 & \xrightarrow{b} & \overline{y^2z^2} & \xrightarrow{a} & \overline{fyz^2} & \xrightarrow{b} & \overline{fy^2z^3} & \xrightarrow{a} & \overline{yz^3} & \xrightarrow{c} & \overline{e}.
\end{array}$$

Calculating modulo the normal subgroup $D_6 = \langle f, y \rangle$, its voltage is

$$\left(b^{-1}ab^2(ab)^2(ac)\right)^3 \equiv \left(z^{-1}ez^2(ez)^2(ez^2w)\right)^3 = \left(z^5w\right)^3 = w^3,$$

because $|z| = r = 5$. Since this voltage generates $\mathbb{Z}_p$, the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Subcase 3.2. *Assume* $a = f$ *and* $b = fyz$. Since $\langle b, c \rangle \neq G$, we must have $c \in \langle fy, z \rangle w$, so

$$c = (fy)^iz^kw \quad \text{with } 0 \le i < 2 \text{ and } 0 \le k < r.$$

Subsubcase 3.2.1. *Assume* $k = 0$. Then $c = fyw$, so we have $c \equiv a \pmod{G'}$. Therefore $(b^{-(r-1)}, a, b^{r-1}, c)$ is a Hamiltonian cycle in $\mathrm{Cay}(G/G'; S)$. Since

$$b^{r-1} = (fyz)^{r-1} = (fy)^{r-1}(z^{r-1}) = (e)(z^{-1}) = z^{-1},$$

its voltage is

$$b^{-(r-1)}ab^{r-1}c = (b^{-(r-1)}ab^{r-1}a)(ac) = [b^{r-1}, a](ac) = [z^{-1}, f](yw) = yw,$$

which generates $\mathbb{Z}_q \times \mathbb{Z}_p = G'$, so the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Subsubcase 3.2.2. *Assume* $i = 0$. Then $c = z^k w$, and we know $k \neq 0$, because $S \cap G' = \emptyset$.

If $k = 1$, then $\big((a, c)^{r-1}, a, b\big)$ is a Hamiltonian cycle in $\mathrm{Cay}(G/G'; S)$. Letting $\epsilon \in \{\pm 1\}$, such that $w^f = w^\epsilon$, its voltage is

$$\begin{aligned}
(ac)^{r-1}\, a\, b &= (ac)^r\, (c^{-1}\, b) \\
&= (fzw)^r \big((zw)^{-1}(fyz)\big) \\
&= \big(f^r z^r w^{(\epsilon d)^{r-1}+(\epsilon d)^{r-2}+\cdots+1}\big)\big(w^{-1}z^{-1}fyz\big) \\
&= f\, w^{(\epsilon d)^{r-1}+(\epsilon d)^{r-2}+\cdots+\epsilon d}\, fy \\
&= w^{\epsilon((\epsilon d)^{r-1}+(\epsilon d)^{r-2}+\cdots+\epsilon d)} y \\
&= w^{d((\epsilon d)^{r-2}+(\epsilon d)^{r-3}+\cdots+1)} y.
\end{aligned}$$

Since ϵd is a primitive rth or $(2r)$th root of unity in $\mathbb{Z}_p$, it is clear that the exponent of w is nonzero (mod p). Therefore the voltage generates $\mathbb{Z}_p \times \mathbb{Z}_q = G'$, so the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

We may now assume $k \geq 2$. However, we may also assume $k \leq (r-1)/2$ (by replacing c with its inverse if necessary). So $r = 5$ and $k = 2$. In this case, here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$\begin{array}{ccccccccccc}
\bar{e} & \xrightarrow{a} & \overline{f} & \xrightarrow{b} & \overline{fyz} & \xrightarrow{a} & \overline{y^2z} & \xrightarrow{b^{-1}} & \overline{y} & \xrightarrow{a} & \overline{fy^2} \\
& \xrightarrow{b} & \overline{fz} & \xrightarrow{a} & \overline{z} & \xrightarrow{b^{-1}} & \overline{y^2} & \xrightarrow{a} & \overline{fy} & \xrightarrow{b} & \overline{fy^2z} \\
& \xrightarrow{a} & \overline{yz} & \xrightarrow{b} & \overline{y^2z^2} & \xrightarrow{a} & \overline{fyz^2} & \xrightarrow{b} & \overline{fy^2z^3} & \xrightarrow{a} & \overline{yz^3} \\
& \xrightarrow{b} & \overline{y^2z^4} & \xrightarrow{a} & \overline{fyz^4} & \xrightarrow{b^{-1}} & \overline{fz^3} & \xrightarrow{a} & \overline{z^3} & \xrightarrow{b} & \overline{yz^4} \\
& \xrightarrow{c^{-1}} & \overline{yz^2} & \xrightarrow{a} & \overline{fy^2z^2} & \xrightarrow{c} & \overline{fy^2z^4} & \xrightarrow{b^{-1}} & \overline{fyz^3} & \xrightarrow{a} & \overline{y^2z^3} \\
& \xrightarrow{b} & \overline{z^4} & \xrightarrow{a} & \overline{fz^4} & \xrightarrow{c^{-1}} & \overline{fz^2} & \xrightarrow{a} & \overline{z^2} & \xrightarrow{c^{-1}} & \bar{e}.
\end{array}$$

Its voltage is

$$(abab^{-1})^2(ab)^4(ab^{-1}ab)(c^{-1}ac)(b^{-1}ab)(ac^{-1})^2.$$

Since the voltage is in $\mathbb{Z}_p$, it is a power of w, and it is clear that the only terms that contribute a power of w to the product are contained in the last three parenthesized expressions (because c does not appear anywhere else). Choosing $\epsilon \in \{\pm 1\}$, such that $w^f = w^\epsilon$, we calculate the product of these three expressions modulo $\langle y \rangle$:

$$\begin{aligned}
(c^{-1}ac)(b^{-1}ab)(ac^{-1})^2 &\equiv \big((z^2w)^{-1}f(z^2w)\big)\big((fz)^{-1}f(fz)\big)\big(f(z^2w)^{-1}\big)^2 \\
&= \big(w^{\epsilon-1}f\big)\big(f\big)\big(w^{-(\epsilon+d^2)}z^{-4}\big) \\
&= w^{-(d^2+1)}z^{-4}.
\end{aligned}$$

Since the power of w is nonzero, the voltage generates $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Subsubcase 3.2.3. *Assume* i *and* k *are both nonzero*. Since $\langle a, c\rangle \neq G$, this implies that f centralizes w. Therefore $G = D_{2q} \times (\mathbb{Z}_r \ltimes \mathbb{Z}_p)$. Also, since $0 \leq i < 2$, we know $i = 1$, so $c = fyz^k w$. We may assume $k \neq 1$ (for otherwise $b \equiv c$ (mod $\mathbb{Z}_p$), so Corollary 2.3 applies). Since we may also assume that $k \leq (r-1)/2$ (by replacing c with its inverse if necessary), then we have $r = 5$ and $k = 2$.

Here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$\begin{array}{ccccccccccc}
\bar{e} & \xrightarrow{a} & \overline{f} & \xrightarrow{b} & \overline{yz} & \xrightarrow{a} & \overline{fy^2z} & \xrightarrow{b} & \overline{y^2z^2} & \xrightarrow{a} & \overline{fyz^2} \\
& \xrightarrow{c} & \overline{z^4} & \xrightarrow{a} & \overline{fz^4} & \xrightarrow{b^{-1}} & \overline{yz^3} & \xrightarrow{a} & \overline{fy^2z^3} & \xrightarrow{c} & \overline{y^2} \\
& \xrightarrow{a} & \overline{fy} & \xrightarrow{b} & \overline{z} & \xrightarrow{a} & \overline{fz} & \xrightarrow{b} & \overline{yz^2} & \xrightarrow{a} & \overline{fy^2z^2} \\
& \xrightarrow{c} & \overline{y^2z^4} & \xrightarrow{a} & \overline{fyz^4} & \xrightarrow{b^{-1}} & \overline{z^3} & \xrightarrow{a} & \overline{fz^3} & \xrightarrow{c} & \overline{y} \\
& \xrightarrow{a} & \overline{fy^2} & \xrightarrow{b} & \overline{y^2z} & \xrightarrow{a} & \overline{fyz} & \xrightarrow{b} & \overline{z^2} & \xrightarrow{a} & \overline{fz^2} \\
& \xrightarrow{c} & \overline{yz^4} & \xrightarrow{a} & \overline{fy^2z^4} & \xrightarrow{b^{-1}} & \overline{y^2z^3} & \xrightarrow{a} & \overline{fyz^3} & \xrightarrow{c} & \bar{e}.
\end{array}$$

Calculating modulo the normal subgroup $D_6 = \langle f, y\rangle$, its voltage is

$$\begin{aligned}\big((ab)^2acab^{-1}ac\big)^3 &\equiv \big((ez)^2e(z^2w)ez^{-1}e(z^2w)\big)^3\\ &= \big(z^4wzw\big)^3\\ &= w^{3(d+1)},\end{aligned}$$

which generates $\langle w\rangle = \mathbb{Z}_p$, so the Factor Group Lemma 2.2 applies.

Subcase 3.3. *Assume $a = fz$ and $b = yz^\ell$, with $\ell \neq 0$.* Since $\langle a, c\rangle \neq G$ and $\langle b, c\rangle \neq G$, we must have $c \in \langle f, z\rangle w$ and $c \in \langle y, z\rangle w$. So $c \in \langle z\rangle w$; write $c = z^k w$ (with $k \neq 0$, because $S \cap G' = \emptyset$).

Subsubcase 3.3.1. *Assume $\ell = k$.* Then $b \equiv c \equiv z^\ell \pmod{G'}$, so

$$\big(a^{-1}, b^{-(r-1)}, a, b^{r-2}, c\big)$$

is a Hamiltonian cycle in $\mathrm{Cay}(G/G'; S)$. Its voltage is

$$\begin{aligned}a^{-1}b^{-(r-1)}ab^{r-2}c &= (fz)^{-1}(yz^\ell)^{-(r-1)}(fz)(yz^\ell)^{r-2}(z^\ell w)\\ &= (f^{-1}y^{-(r-1)}f)y^{r-2}w \quad \begin{pmatrix}z\text{ commutes}\\ \text{with } f \text{ and } y\end{pmatrix}\\ &= (y^{r-1})y^{r-2}w \quad (f \text{ inverts } y)\\ &= y^{2r-3}w.\end{aligned}$$

Since $2(3) - 3 \not\equiv 0 \pmod 5$ and $2(5) - 3 \not\equiv 0 \pmod 3$, we have $2r - 3 \not\equiv 0 \pmod q$, so y^{2r-3} is nontrivial, and hence generates $\mathbb{Z}_q$. Therefore, this voltage generates $\mathbb{Z}_q \times \mathbb{Z}_p = G'$. So the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Subsubcase 3.3.2. *Assume $\ell \neq k$.* We may assume $\ell, k \le (r-1)/2$ (perhaps after replacing b and/or c by their inverses). Then we must have $r = 5$ and $\{\ell, k\} = \{1, 2\}$.

For $(\ell, k) = (1, 2)$, here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$\begin{array}{ccccccccccc}
\overline{e} & \xrightarrow{a} & \overline{fz} & \xrightarrow{b} & \overline{fyz^2} & \xrightarrow{a^{-1}} & \overline{y^2z} & \xrightarrow{a^{-1}} & \overline{fy} & \xrightarrow{b^{-1}} & \overline{fz^4}\\
 & \xrightarrow{a^{-1}} & \overline{z^3} & \xrightarrow{a^{-1}} & \overline{fz^2} & \xrightarrow{a^{-1}} & \overline{z} & \xrightarrow{a^{-1}} & \overline{f} & \xrightarrow{b^{-1}} & \overline{fy^2z^4}\\
 & \xrightarrow{a} & \overline{y} & \xrightarrow{a} & \overline{fy^2z} & \xrightarrow{a} & \overline{yz^2} & \xrightarrow{a} & \overline{fy^2z^3} & \xrightarrow{a} & \overline{yz^4}\\
 & \xrightarrow{a} & \overline{fy^2} & \xrightarrow{a} & \overline{yz} & \xrightarrow{a} & \overline{fy^2z^2} & \xrightarrow{a} & \overline{yz^3} & \xrightarrow{b} & \overline{y^2z^4}\\
 & \xrightarrow{a^{-1}} & \overline{fyz^3} & \xrightarrow{a^{-1}} & \overline{y^2z^2} & \xrightarrow{a^{-1}} & \overline{fyz} & \xrightarrow{a^{-1}} & \overline{y^2} & \xrightarrow{a^{-1}} & \overline{fyz^4}\\
 & \xrightarrow{a^{-1}} & \overline{y^2z^3} & \xrightarrow{b} & \overline{z^4} & \xrightarrow{a^{-1}} & \overline{fz^3} & \xrightarrow{a^{-1}} & \overline{z^2} & \xrightarrow{c^{-1}} & \overline{e}.
\end{array}$$

Its voltage is

$$aba^{-2}b^{-1}a^{-4}b^{-1}a^9ba^{-6}ba^{-2}c^{-1}.$$

Since there is precisely one occurrence of c in this product, and therefore only one occurrence of w, it is impossible for this appearance of w to cancel. So the voltage is nontrivial, and therefore generates $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G; S)$.

For $(\ell, k) = (2, 1)$, here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p; S)$:

$$\begin{array}{ccccccccccc}
\overline{e} & \xrightarrow{a^{-1}} & \overline{fz^4} & \xrightarrow{a^{-1}} & \overline{z^3} & \xrightarrow{a^{-1}} & \overline{fz^2} & \xrightarrow{a^{-1}} & \overline{z} & \xrightarrow{a^{-1}} & \overline{f}\\
 & \xrightarrow{a^{-1}} & \overline{z^4} & \xrightarrow{b} & \overline{yz} & \xrightarrow{a^{-1}} & \overline{fy^2} & \xrightarrow{a^{-1}} & \overline{yz^4} & \xrightarrow{c} & \overline{y}\\
 & \xrightarrow{a^{-1}} & \overline{fy^2z^4} & \xrightarrow{a^{-1}} & \overline{yz^3} & \xrightarrow{a^{-1}} & \overline{fy^2z^2} & \xrightarrow{c} & \overline{fy^2z^3} & \xrightarrow{a^{-1}} & \overline{yz^2}\\
 & \xrightarrow{a^{-1}} & \overline{fy^2z} & \xrightarrow{b} & \overline{fz^3} & \xrightarrow{a^{-1}} & \overline{z^2} & \xrightarrow{a^{-1}} & \overline{fz} & \xrightarrow{b} & \overline{fyz^3}\\
 & \xrightarrow{a^{-1}} & \overline{y^2z^2} & \xrightarrow{a^{-1}} & \overline{fyz} & \xrightarrow{c} & \overline{fyz^2} & \xrightarrow{a^{-1}} & \overline{y^2z} & \xrightarrow{a^{-1}} & \overline{fy}\\
 & \xrightarrow{a^{-1}} & \overline{y^2z^4} & \xrightarrow{c} & \overline{y^2} & \xrightarrow{a^{-1}} & \overline{fyz^4} & \xrightarrow{a^{-1}} & \overline{y^2z^3} & \xrightarrow{b} & \overline{e}.
\end{array}$$

Choosing $\epsilon \in \{\pm 1\}$, such that $w^f = w^\epsilon$, we calculate the voltage, modulo $\langle y\rangle$:

$$\begin{aligned}a^{-4}\Big(\big(a^{-2}ba^{-2}\big)ca^{-3}c\big(a^{-2}b\big)\Big)^2 &\equiv (fz)^{-4}\Big(\big((fz)^{-2}z^2(fz)^{-2}\big)(zw)(fz)^{-3}(zw)\big((fz)^{-2}z^2\big)\Big)^2\\ &= z^{-4}\big((z^{-2})(zw)(fz^{-3})(zw)(e)\big)^2\end{aligned}$$

$$
\begin{aligned}
&= z^{-4}\left(z^{-1}wfz^{-2}w\right)^2\\
&= z^{-4}(w^{d^6+\epsilon d^4+\epsilon d^3+d}z^{-6})\\
&= z^{-4}(w^{d(\epsilon d^3+\epsilon d^2+2)}z^4).
\end{aligned}
$$

Since d is a primitive rth root of unity in $\mathbb{Z}_p$, and $r=5$, we know $d^4+d^3+d^2+d+1\equiv 0 \pmod 5$. Combining this with the fact that

$$-(d^3+d^2-1)(d^3+d^2+2)+(d^2+d-1)(d^4+d^3+d^2+d+1)=1,$$

and

$$(d^3+d^2+3)(-d^3+-d^2+2)+(d^2+d-1)(d^4+d^3+d^2+d+1)=5\not\equiv 0 \pmod p,$$

we see that $\epsilon d^3+\epsilon d^2+2$ is nonzero in $\mathbb{Z}_p$. Therefore the voltage is nontrivial, so it generates $\mathbb{Z}_p$. Hence, the Factor Group Lemma 2.2 provides a Hamiltonian cycle in $\mathrm{Cay}(G;S)$.

Subcase 3.4. *Assume $a=fz$ and $b=fyz^\ell$, with $\ell\neq 0$.* Since $\langle a,c\rangle\neq G$ and $\langle b,c\rangle\neq G$, we must have $c\in\langle f,z\rangle w$ and $c\in\langle fy,z\rangle w$. So $c\in\langle z\rangle w$; write $c=z^kw$ (with $k\neq 0$ because $S\cap G'=\emptyset$).

We may assume $k,\ell\leq(r-1)/2$, by replacing either or both of b and c with their inverses if necessary. We may also assume $\ell\neq 1$, for otherwise $a\equiv b \pmod{\langle y\rangle}$, so Corollary 2.3 applies. Therefore, we must have $r=5$ and $\ell=2$. We also have $k\in\{1,2\}$.

For $k=1$, here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p;S)$:

$$
\begin{array}{ccccccccccc}
\overline{e} & \xrightarrow{a} & \overline{fz} & \xrightarrow{b^{-1}} & \overline{yz^4} & \xrightarrow{a^{-1}} & \overline{fy^2z^3} & \xrightarrow{a^{-1}} & \overline{yz^2} & \xrightarrow{b} & \overline{fz^4}\\
 & \xrightarrow{a^{-1}} & \overline{z^3} & \xrightarrow{a^{-1}} & \overline{fz^2} & \xrightarrow{a^{-1}} & \overline{z} & \xrightarrow{a^{-1}} & \overline{f} & \xrightarrow{b^{-1}} & \overline{yz^3}\\
 & \xrightarrow{a} & \overline{fy^2z^4} & \xrightarrow{a} & \overline{y} & \xrightarrow{a} & \overline{fy^2z} & \xrightarrow{c^{-1}} & \overline{fy^2} & \xrightarrow{a} & \overline{yz}\\
 & \xrightarrow{a} & \overline{fy^2z^2} & \xrightarrow{b} & \overline{y^2z^4} & \xrightarrow{a^{-1}} & \overline{fyz^3} & \xrightarrow{a^{-1}} & \overline{y^2z^2} & \xrightarrow{a^{-1}} & \overline{fyz}\\
 & \xrightarrow{a^{-1}} & \overline{y^2} & \xrightarrow{a^{-1}} & \overline{fyz^4} & \xrightarrow{a^{-1}} & \overline{y^2z^3} & \xrightarrow{a^{-1}} & \overline{fyz^2} & \xrightarrow{a^{-1}} & \overline{y^2z}\\
 & \xrightarrow{a^{-1}} & \overline{fy} & \xrightarrow{b} & \overline{z^2} & \xrightarrow{a} & \overline{fz^3} & \xrightarrow{a} & \overline{z^4} & \xrightarrow{c} & \overline{e}.
\end{array}
$$

Its voltage is

$$ab^{-1}a^{-2}ba^{-4}b^{-1}a^3c^{-1}a^2ba^{-9}ba^2c.$$

Calculating modulo y, the product between the occurrence of c^{-1} and the occurrence of c is

$$a^2ba^{-9}ba^2\equiv(fz)^2(fz^2)(fz)^{-9}(fz^2)(fz)^2=z^{-1},$$

which does not centralize w. So the occurrence of w^{-1} in c^{-1} does not cancel the occurrence of w in c. Therefore the voltage is nontrivial, so it generates $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 applies.

For $k=2$, here is a Hamiltonian cycle in $\mathrm{Cay}(G/\mathbb{Z}_p;S)$:

$$
\begin{array}{ccccccccccc}
\overline{e} & \xrightarrow{a} & \overline{fz} & \xrightarrow{b} & \overline{yz^3} & \xrightarrow{b} & \overline{f} & \xrightarrow{a} & \overline{z} & \xrightarrow{a} & \overline{fz^2}\\
 & \xrightarrow{a} & \overline{z^3} & \xrightarrow{a} & \overline{fz^4} & \xrightarrow{b^{-1}} & \overline{yz^2} & \xrightarrow{a} & \overline{fy^2z^3} & \xrightarrow{a} & \overline{yz^4}\\
 & \xrightarrow{a} & \overline{fy^2} & \xrightarrow{a} & \overline{yz} & \xrightarrow{a} & \overline{fy^2z^2} & \xrightarrow{c} & \overline{fy^2z^4} & \xrightarrow{a} & \overline{y}\\
 & \xrightarrow{a} & \overline{fy^2z} & \xrightarrow{b} & \overline{y^2z^3} & \xrightarrow{a} & \overline{fyz^4} & \xrightarrow{a} & \overline{y^2} & \xrightarrow{a} & \overline{fyz}\\
 & \xrightarrow{a} & \overline{y^2z^2} & \xrightarrow{a} & \overline{fyz^3} & \xrightarrow{a} & \overline{y^2z^4} & \xrightarrow{a} & \overline{fy} & \xrightarrow{a} & \overline{y^2z}\\
 & \xrightarrow{a} & \overline{fyz^2} & \xrightarrow{b} & \overline{z^4} & \xrightarrow{a^{-1}} & \overline{fz^3} & \xrightarrow{a^{-1}} & \overline{z^2} & \xrightarrow{c^{-1}} & \overline{e}.
\end{array}
$$

Its voltage is

$$ab^2a^4b^{-1}a^5ca^2ba^9ba^{-2}c^{-1}.$$

Calculating modulo y, the product between the occurrence of c and the occurrence of c^{-1} is

$$a^2ba^9ba^{-2}\equiv(fz)^2(fz^2)(fz)^9(fz^2)(fz)^{-2}=fz^{13}=fz^3,$$

which does not centralize w. So the occurrence of w^{-1} in c^{-1} does not cancel the occurrence of w in c. Therefore the voltage is nontrivial, so it generates $\mathbb{Z}_p$, so the Factor Group Lemma 2.2 applies.

Case 4 Assume $\#S\geq 4$. Write $S=\{s_1,s_2,\ldots,s_\ell\}$, and let $G_i=\langle s_1,\ldots,s_i\rangle$ for $i=1,2,\ldots,\ell$. Since S is minimal, we know

$$\{e\}\subsetneq G_1\subsetneq G_2\subsetneq\cdots\subsetneq G_\ell\subseteq G.$$

Therefore, the number of prime factors of $|G_i|$ is at least i. Since $|G| = 30p$ is the product of only 4 primes, and $\ell = \#S \geq 4$, we conclude that $|G_i|$ has exactly i prime factors, for all i. (In particular, we must have $\#S = 4$.) By permuting the elements of $\{s_1, s_2, \ldots, s_\ell\}$, this implies that if S_0 is any subset of S, then $|\langle S_0 \rangle|$ is the product of exactly $\#S_0$ primes. In particular, by letting $\#S_0 = 1$, we see that every element of S must have prime order.

Now, choose $\{a, b\} \subset S$ to be a 2-element generating set of $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_r$. From the preceding paragraph, we see that we may assume $|a| = 2$ and $|b| = r$ (by interchanging a and b if necessary). Since $|\langle a, b \rangle|$ is the product of only two primes, we must have $|\langle a, b \rangle| = 2r$, so $\langle a, b \rangle \cong G/G'$. Therefore

$$G = \big(\langle a \rangle \times \langle b \rangle\big) \ltimes G'.$$

Since $\langle S \rangle = G$, we may choose $s_1 \in S$, such that $s_1 \notin \langle a, b \rangle \mathbb{Z}_p$. Then $\langle a, b, s_1 \rangle = \langle a, b \rangle \mathbb{Z}_q$. Since a centralizes both a and b, but does not centralize $\mathbb{Z}_q$, which is contained in $\langle a, b, s_1 \rangle$, we know that $[a, s_1]$ is nontrivial. Therefore $\langle a, s_1 \rangle$ contains $\langle a, b, s_1 \rangle' = \mathbb{Z}_q$. Then, since $|\langle a, s_1 \rangle|$ is only divisible by two primes, we must have $|\langle a, s_1 \rangle| = 2q$. Also, since $S \cap G' = \emptyset$, we must have $|s_1| \neq q$; therefore $|s_1| = 2$. Hence $2r \mid |\langle b, s_1 \rangle|$, so we must have $|\langle b, s_1 \rangle| = 2r$. Therefore

$$[b, s_1] \in \langle b, s_1 \rangle \cap \langle a, b, s_1 \rangle' = \langle b, s_1 \rangle \cap \mathbb{Z}_q = \{e\},$$

so b centralizes s_1. It also centralizes a, so b centralizes $\langle a, s_1 \rangle = \mathbb{Z}_2 \ltimes \mathbb{Z}_q$.

Similarly, if we choose $s_2 \in S$ with $s_2 \notin \langle a, b \rangle \mathbb{Z}_q$, then a centralizes $\langle b, s_2 \rangle = \mathbb{Z}_r \ltimes \mathbb{Z}_p$.

Therefore $G = \langle a, s_1 \rangle \times \langle b, s_2 \rangle$, so

$$\mathrm{Cay}(G; S) \cong \mathrm{Cay}\big(\langle a, s_1 \rangle; \{a, s_1\}\big) \times \mathrm{Cay}\big(\langle b, s_2 \rangle; \{b, s_2\}\big).$$

This is a Cartesian product of Hamiltonian graphs and therefore is Hamiltonian. □

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