Interval edge-colorings of complete graphs and $n$-dimensional cubes

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ABSTRACT

An edge-coloring of a graph $G$ with colors $1, 2, \ldots, t$ is called an interval $t$-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of $G$ colored by $i$, and the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. In this paper we show that if $n = p2^q$, where $p$ is odd, $q$ is nonnegative, and $2n - 1 \leq t \leq 4n - 2 - p - q$, then the complete graph $K_{2n}$ has an interval $t$-coloring. We also prove that if $n \leq t \leq \frac{\min\{n, 2n\}}{2}$, then the $n$-dimensional cube $Q_n$ has an interval $t$-coloring.

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1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$, the maximum degree of $G$ by $\Delta(G)$, the edge-chromatic number of $G$ by $\chi'(G)$ and the diameter of $G$ by $diam(G)$. The terms and concepts that we do not define can be found in [28].

In this paper we investigate a special type of edge-coloring introduced by Asratian and Kamalian [1] in 1987. An edge-coloring of a graph $G$ with colors $1, 2, \ldots, t$ is called an interval $t$-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of $G$ colored by $i$, and the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. This concept was introduced for studying the problems that are related to constructing timetables without “gaps” for teachers and classes.

First results about interval colorings were obtained in [1, 16, 17, 26]. In [1] Asratian and Kamalian noted that if $G$ is interval colorable, then $\chi'(G) = \Delta(G)$. Also, if $G$ is $r$-regular, then $G$ has an interval coloring if and only if $G$ has a proper $r$-edge-coloring. From here and the result of Holyer [14] it follows that the problem “Is a given regular graph interval colorable or not?” is $NP$-complete. Asratian and Kamalian also proved [1] that if a triangle-free graph $G$ has an interval $t$-coloring, then $t \leq |V(G)| - 1$.

For a general graph $G$, Kamalian [17] showed that if $G$ admits an interval $t$-coloring, then $t \leq 2|V(G)| - 3$. Giaro, Kubale and Malafiejski [10] proved that if a graph $G$ has at least three vertices, then the upper bound can be improved to $2|V(G)| - 4$. Kamalian also investigated interval colorings of complete bipartite graphs and trees [16]. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval $t$-coloring if and only if $m + n - \gcd(m, n) \leq t \leq m + n - 1$, where $\gcd(m, n)$ is the greatest common divisor of $m$ and $n$.

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The smallest graph that is not interval colorable is $K_3$. Sevast'janov [26] found a bipartite graph with 28 vertices and maximum degree 21 that is not interval colorable. Other examples were obtained by Erdős (27 vertices and maximum degree 13), by Hertz and de Werra (21 vertices and maximum degree 14), and by Malafiejski (19 vertices and maximum degree 15). It is known [12] that such examples do not exist for $\Delta(G) \leq 3$. Unfortunately, we know nothing about the cases $4 \leq \Delta(G) \leq 12$. In general, it is an NP-complete problem to decide whether a bipartite graph has an interval coloring. Asratian and Kamalian [1] proved this statement for a fixed number of colors. When the number of colors is an input this result was proved by Sevast'janov [26]. In [7] Giareo showed that the problem of deciding interval $\Delta(G)$-colorability of a bipartite graph $G$ is easy if $\Delta(G) \leq 4$ and NP-complete for $\Delta(G) \geq 5$. Related to this, let us note that regular bipartite graphs [1], trees, complete bipartite graphs [16], subcubic bipartite graphs [12], doubly convex bipartite graphs [3], grids [8], outerplanar bipartite graphs [11], $(2, b)$-biregular bipartite graphs [13,15,18,20] and some classes of $(3, 4)$-biregular bipartite graphs [4,15,24] have interval colorings.

For non-bipartite graphs, the problems of existence, construction, and bounds on the numbers of colors in interval colorings were investigated in [1–3,5,10,17,19,21]. In particular, it was proved in [2] that if $G$ admits an interval $t$-coloring, then $t \leq (\text{diam}(G) + 1) (\Delta(G) - 1) + 1$, and if $G$ is a bipartite graph, then the upper bound can be improved to $t \leq \text{diam}(G) (\Delta(G) - 1) + 1$. For a planar graph $G$, Axenovich [5] showed that if $G$ has an interval $t$-coloring, then $t \leq \frac{11}{5} |V(G)|$.

Several recent papers [6,9,25] are devoted to the investigation of deficiency of a non-bipartite graph — a measure of closeness to interval colorability.

One of the important and less-investigated problems related to interval colorings is the problem of determining the exact values of the least and the greatest possible number of colors in interval colorings of graphs. The exact values of these parameters are known only for paths, even cycles, trees, complete bipartite graphs [1,16] and Möbius ladders [21]. In some papers [17,19,22] lower bounds are found for the greatest possible number of colors in interval colorings of certain graphs. Even for complete graphs and $n$-dimensional cubes, exact upper and lower bounds on the number of colors are not known.

In this paper we obtain the following results:

1. If $n = p2^q$, where $p$ is odd, $q$ is nonnegative, and $2n - 1 \leq t \leq 4n - 2 - p - q$, then the complete graph $K_{2n}$ has an interval $t$-coloring.
2. If $F$ is a set of at least $n$ edges incident to one vertex $v$ of the complete graph $K_{2n+1}$, then $K_{2n+1} - F$ has an interval coloring.
3. If $F$ is a maximum matching of the complete graph $K_{2n+1}$ with $n \geq 2$, then $K_{2n+1} - F$ has no interval coloring.
4. If $n \leq t \leq \frac{n(n+1)}{2}$, then the $n$-dimensional cube $Q_n$ has an interval $t$-coloring.

2. Main results

A partial edge-coloring of $G$ is a coloring of some of the edges of $G$ such that no two adjacent edges receive the same color. If $\alpha$ is a partial edge-coloring of $G$ and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors of colored edges incident to $v$.

Let $[a]$ denote the largest integer less than or equal to $a$. For two integers $a$ and $b$ with $a \leq b$, the set $\{a, a + 1, \ldots, b\}$ is denoted by $[a, b]$.

The set of all interval colorable graphs is denoted by $\mathcal{I}$ (see [1,17]). For a graph $G \in \mathcal{I}$, the least and the greatest values of $t$ for which $G$ has an interval $t$-coloring are denoted by $w(G)$ and $W(G)$, respectively.

Asratian and Kamalian proved the following:

**Theorem 1** ([11]). If $G$ is a regular graph, then

1. $G \in \mathcal{I}$ if and only if $\chi'(G) = \Delta(G)$.
2. If $G \in \mathcal{I}$ and $\Delta(G) \leq t \leq W(G)$, then $G$ has an interval $t$-coloring.

Vizing proved the following:

**Theorem 2** ([27]). For the complete graph $K_p$,

$$\chi'(K_p) = \begin{cases} p - 1, & \text{if } p \text{ is even}, \\ p, & \text{if } p \text{ is odd}. \end{cases}$$

From Theorems 1 and 2, it follows that $K_{2n+1} \notin \mathcal{I}$ and $K_{2n} \in \mathcal{I}$ with $w(K_{2n}) = 2n - 1$ for any $n \in \mathbb{N}$. For $W(K_{2n})$, the prior lower bound was obtained by Kamalian:

**Theorem 3** ([17]). $W(K_{2n}) \geq 2n - 1 + \lfloor \log_2 (2n - 1) \rfloor$ for any $n \in \mathbb{N}$.

We present several results improving the bound in Theorem 3.

**Theorem 4.** $W(K_{2n}) \geq 3n - 2$ for any $n \in \mathbb{N}$.

**Proof.** Let $V(K_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$.

Define a coloring $\alpha$ of the edges of $K_{2n}$.

\begin{align*}
\alpha(\{v_i, v_j\}) &= 0, \quad i < j, \\
\alpha(\{v_i, v_j\}) &= 1, \quad i > j,
\end{align*}

for all $i, j \in \{1, 2, \ldots, 2n\}$. Then $\alpha$ is an interval coloring of $K_{2n}$, and $W(K_{2n}) \geq 3n - 2$ for any $n \in \mathbb{N}$. 

\[\]
For each edge $v_i v_j \in E(K_{2n})$ with $i < j$, define a color $\alpha(v_i v_j)$ as follows:

$$
\alpha(v_i v_j) = \begin{cases} 
  i + j - 2, & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \ 2 \leq j \leq n, \ i + j \leq n + 1; \\
  i + j + n - 3, & \text{if } 2 \leq i \leq n - 1, \ \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq j \leq n, \ i + j \geq n + 2; \\
  n + j - i, & \text{if } 3 \leq i \leq n, \ n + 1 \leq j \leq 2n - 2, \ j - i \leq n - 2; \\
  j - i, & \text{if } 1 \leq i \leq n, \ n + 1 \leq j \leq 2n, \ j - i \geq n; \\
  2(i - 1), & \text{if } 2 \leq i \leq 1 + \left\lfloor \frac{n - 1}{2} \right\rfloor, \ n + 1 \leq j \leq n + \left\lfloor \frac{n - 1}{2} \right\rfloor, \ j - i = n - 1; \\
  i + j - 2, & \text{if } \left\lfloor \frac{n - 1}{2} \right\rfloor + 2 \leq i \leq n, \ n + 1 + \left\lfloor \frac{n - 1}{2} \right\rfloor \leq j \leq 2n - 1, \ j - i = n - 1; \\
  i + j - 2n, & \text{if } n + 1 \leq i \leq n + \left\lfloor \frac{n}{2} \right\rfloor - 1, \ n + 2 \leq j \leq 2n - 2, \ i + j \leq 3n - 1; \\
  i + j - n - 1, & \text{if } n + 1 \leq i \leq 2n - 1, \ n + \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq j \leq 2n, \ i + j \geq 3n.
\end{cases}
$$

Let us prove that $\alpha$ is an interval $(3n - 2)$-coloring of $K_{2n}$.

Note that

$$
S(v_1, \alpha) \cup S(v_{2n}, \alpha) = [1, 3n - 2].
$$

Therefore, for $t \in [1, 2, \ldots, 3n - 2]$, there is an edge $e \in E(K_{2n})$ with $\alpha(e) = t$.

Now let us show that the edges incident to any vertex of $K_{2n}$ are colored by $2n - 1$ consecutive colors. For example, let $i \in \left\{3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\}$.

By the definition of $\alpha$, for $3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$
S(v_i, \alpha) = [i - 1, 2i - 3] \cup [2i - 1, n - 1] \cup [2n - 1, 2n - 3 + i] \cup [2n + 1 - i, 2n - 2] \cup [2i - 2] \cup [n, 2n - i] = [i - 1, 2n - 3 + i].
$$

Similarly, it can be verified that the edges incident to other vertices of $K_{2n}$ are also colored by $2n - 1$ consecutive colors. \(\square\)

Fig. 1 shows the interval 10-coloring $\alpha$ of the graph $K_8$ described in the proof of Theorem 4.

Though we will not need this in future, let us note that

Remark 5. The algorithm described in the proof of Theorem 4 is polynomial, since it produces an interval $(3n - 2)$-coloring in time $O(n^2)$. 

\[\text{Fig. 1. The interval 10-coloring of } K_8.\]
Corollary 6. $K_{2n}$ has an interval $(3n-2)$-coloring $\alpha$ such that for each $l \in \{1, 2, \ldots, n\}$ there are vertices $v'_l, v''_l \in V(K_{2n})$ with $\min S(v'_l, \alpha) = \min S(v''_l, \alpha) = l$.

Proof. For the proof, it suffices to consider the coloring $\alpha$ presented in Theorem 4 and take
\[
\begin{align*}
&v'_1 = v_1, \quad v''_1 = v_2 \\
v'_l = v_{l+1} \quad\text{and}\quad v''_l = v_{n+l-1} \quad\text{for} \quad 2 \leq l \leq n-1 \quad\text{and}
&v''_n = v_{2n-1}, \quad v''_n = v_{2n}.
\end{align*}
\]

Remark 7. The naturally arising conjecture $W(K_{2n}) = 3n-2$ is false: the interval 11-coloring of $K_8$ shown in Fig. 2 implies that $W(K_8) \geq 11 > 10 = 3 \cdot 4 - 2$.

Theorem 8 further improves the bound in Theorem 4. Note that any integer $n$ can be expressed uniquely as $p2^q$, where $p$ is odd and $q$ is nonnegative.

Theorem 8. If $n = p2^q$, where $p$ is odd and $q$ is nonnegative, then $W(K_{2n}) \geq 4n - 2 - p - q$.

Proof. First we prove that $W(K_{4m}) - W(K_{2m}) \geq 4m - 1$ for any $m \in \mathbb{N}$. Let $V(K_{4m}) = \{v_1, v_2, \ldots, v_{4m}\}$. Let $G$ be the subgraph of $K_{4m}$ induced by $\{v_1, v_2, \ldots, v_{2m}\}$. Let $\alpha$ be an interval $W(K_{2m})$-coloring of $G$.

Now we define an edge-coloring $\beta$ of $K_{4m}$.

For $1 \leq i \leq 4m$ and $1 \leq j \leq 4m$, where $i \neq j$, we set:
\[
\beta(v_i v_j) = \begin{cases}
\alpha(v_i v_j), & \text{if } 1 \leq i \leq 2m, 1 \leq j \leq 2m;
\min S(v_i, \alpha) + 2m - 1, & \text{if } 1 \leq i \leq 2m, 2m + 1 \leq j \leq 4m, i = j - 2m;
\alpha(v_i v_{j-2m}) + 2m, & \text{if } 1 \leq i \leq 2m, 2m + 1 \leq j \leq 4m, i \neq j - 2m;
\alpha(v_{i-2m} v_{j-2m}) + 4m - 1, & \text{if } 2m + 1 \leq i \leq 4m, 2m + 1 \leq j \leq 4m.
\end{cases}
\]

Let us show that $\beta$ is an interval $(W(K_{2m}) + 4m - 1)$-coloring of $K_{4m}$. Since $\alpha$ is an interval $W(K_{2m})$-coloring of $G$, for $t \in \{1, 2, \ldots, W(K_{2m})\}$, there is an edge $e \in E(K_{4m})$ with $\beta(e) = t$. Moreover, $\max S(v_j, \alpha) - \min S(v_i, \alpha) = 2m - 2$ for $1 \leq j \leq 2m$.

By the definition of $\beta$, for $1 \leq i \leq 2m$
\[
S(v_i, \beta) = \{\alpha(v_i v_j) \mid 1 \leq j \leq 2m, i \neq j\} \cup \{\alpha(v_i v_{j-2m}) + 2m \mid 2m + 1 \leq j \leq 4m, i \neq j - 2m\}
\]
\[
\cup \{\min S(v_i, \alpha) + 2m - 1\}
\]
\[
= \{\min S(v_i, \alpha), \ldots, \max S(v_i, \alpha)\} \cup \{\min S(v_i, \alpha) + 2m, \ldots, \max S(v_i, \alpha) + 2m\}
\]
\[
\cup \{\min S(v_i, \alpha) + 2m - 1\}
\]
and for $2m + 1 \leq i \leq 4m$

$$S(v_i, \beta) = \{\alpha(v_{i-2m}v_j) + 2m \mid 1 \leq j \leq 2m, j \neq i - 2m\} \cup \{\alpha(v_{i-2m}v_{j-2m}) + 4m - 1 \mid 2m + 1 \leq j \leq 4m, i \neq j\}
\cup \{\min S(v_{i-2m}, \alpha) + 2m - 1\}
= [\min S(v_{i-2m}, \alpha) + 2m, \ldots, \max S(v_{i-2m}, \alpha) + 2m] \cup
[\min S(v_{i-2m}, \alpha) + 4m - 1, \ldots, \max S(v_{i-2m}, \alpha) + 4m - 1] \cup [\min S(v_{i-2m}, \alpha) + 2m - 1].$$

This proves that for $t \in [W(K_{2m}) + 1, \ldots, W(K_{2m}) + 4m - 1]$, there is an edge $e \in E(K_{4m})$ with $\beta(e) = t$. Next, let us prove that $\max S(v_i, \beta) - \min S(v_i, \beta) = 4m - 2$ for $1 \leq i \leq 4m$.

It is not hard to see that for $1 \leq i \leq 2m$

$$S(v_i, \beta) = \{\min S(v_i, \alpha), \ldots, \max S(v_i, \alpha) + 2m\}
= [\min S(v_i, \alpha), \ldots, \min S(v_i, \alpha) + 4m - 2]\n= [\min S(v_i, \alpha), \min S(v_i, \alpha) + 4m - 2]$$

and for $2m + 1 \leq i \leq 4m$

$$S(v_i, \beta) = \{\min S(v_{i-2m}, \alpha) + 2m - 1, \ldots, \max S(v_{i-2m}, \alpha) + 4m - 1\}
= [\min S(v_{i-2m}, \alpha) + 2m - 1, \ldots, \min S(v_{i-2m}, \alpha) + 6m - 3]\n= [\min S(v_{i-2m}, \alpha) + 2m - 1, \min S(v_{i-2m}, \alpha) + 6m - 3].$$

This shows that $\beta$ is an interval $(W(K_{2m}) + 4m - 1)$-coloring of $K_{4m}$.

Now we can conclude:

$$W(K_{p^{2p+1}}) \geq W(K_{p^{2p}}) + p2^{q+1} - 1
W(K_{p^{2p}}) \geq W(K_{p^{2p-1}}) + p2^q - 1
\ldots \ldots \ldots \ldots
W(K_{p^2}) \geq W(K_p) + p2^2 - 1.$$

By summing these inequalities, we obtain

$$W(K_{2n}) \geq W(K_{2p}) + p \sum_{i=2}^{q+1} 2^i - q.$$

Now, using Theorem 4, we have

$$W(K_{2n}) \geq 3p - 2 - q + p \sum_{i=2}^{q+1} 2^i = 3p - 2 - q + 4p(2^q - 1) = 4n - 2 - p - q \quad \square$$

**Remark 9.** Note that if $n$ is odd, then the lower bound in Theorem 8 coincides with that of Theorem 4. Also, if $n = 2^q$, then $W(K_{2^{q+1}}) \geq 2^{q+2} - 3 - q$. Moreover, the lower bound of Theorem 8 is sharp for $n \leq 4$.

From Theorems 1 and 8 we have:

**Corollary 10.** Let $n = p2^q$, where $p$ is odd and $q$ is nonnegative. If $2n - 1 \leq t \leq 4n - 2 - p - q$, then $K_{2n}$ has an interval $t$-coloring.

**Corollary 11.** For any $\varepsilon > 0$, there is a graph $G$ such that $G \in \mathcal{G}_t$ and $W(G) \geq (2 - \varepsilon) |V(G)|$.

**Proof.** For a given $\varepsilon > 0$, choose $q$ so that $\frac{q+1}{2^q} \leq \varepsilon$. Let $G = K_{2q}$. Now, using Theorem 8, we have

$$W(G) = W(K_{2q}) \geq 2^{q+1} - 2 - q \geq 2^{q+1} - q 2^q = (2 - \varepsilon) |V(G)| \quad \square$$

**Corollary 12.** For any $\varepsilon > 0$, there is a graph $G$ such that $G \in \mathcal{G}_t$ and $W(G) - w(G) \geq (1 - \varepsilon) |V(G)|$.

**Proof.** For a given $\varepsilon > 0$, choose $q$ so that $\frac{q+1}{2^q} \leq \varepsilon$. Let $G = K_{2q}$. Now, using Theorem 8, we have

$$W(G) - w(G) = W(K_{2q}) - w(K_{2q}) \geq 2^{q+1} - 2 - q - 2^q + 1 = 2^q - q - 1
\geq 2^q - 2^q = (1 - \varepsilon) |V(G)| \quad \square$$
**Remark 13.** For \( n = 1 \) and \( n = 2 \), the lower bound in **Theorem 8** coincides with the upper bounds of the result of [17] (if \( G ∈ \mathcal{H} \), then \( W(G) ≤ 2|V(G)| - 3 \)) and the result of [10] (if \( G ∈ \mathcal{H} \) with \( |V(G)| ≥ 3 \), then \( W(G) ≤ 2|V(G)| - 4 \)), respectively. **Corollary 11** implies that the coefficients in upper bounds of the results [10,17] cannot be improved.

In [23], it is shown that if we remove any \( n \) edges of \( K_{2n+1} \), then the resulting graph has a proper \( 2n \)-edge-coloring. The removal of fewer edges excludes such a possibility. The analogous statement for interval colorings is not true; more precisely the following theorem holds:

**Theorem 14.** (1) If \( F \) is a set of at least \( n \) edges incident to one vertex \( v \) of the complete graph \( K_{2n+1} \), then \( K_{2n+1} − F \) has an interval coloring.

(2) If \( F \) is a maximum matching of the complete graph \( K_{2n+1} \) with \( n ≥ 2 \), then \( K_{2n+1} − F \) has no interval coloring.

**Proof.** (1). Let \( V(K_{2n+1}) = \{v, v_1, v_2, \ldots, v_{2n}\} \), and let \( G = K_{2n+1} − v \). Clearly, \( G \) is isomorphic to the graph \( K_{2n} \), and consequently, \( G ∈ \mathcal{H} \). Since \( G \) is a complete graph with \( 2n \) vertices, we may assume for an appropriate renumbering of \( V(G) \) that there is an interval \( (3n-2) \)-coloring \( \alpha \) of \( G \) such that for each \( l ∈ \{1, 2, \ldots, 2n−|F|\} \), \( \min \left\{ v_{ij} v_{ik} \mid l, 1 ≤ l ≤ 2n−|F| \right\} \).

Now we define an edge-coloring \( β \) of \( K_{2n+1} − F \).

For any \( e ∈ E(K_{2n+1} − F) \), let

\[ β(e) = \begin{cases} \alpha(e), & \text{if } e ∈ E(G), \\ 2n−1+l, & \text{if } e = v_{i} v_{i}, 1 ≤ l ≤ 2n−|F|. \end{cases} \]

Let us show that \( β \) is an interval \( (3n-1) \)-coloring of \( K_{2n+1} − F \) if \( |F| = n \) and \( β \) is an interval \( (3n-2) \)-coloring of \( K_{2n+1} − F \) if \( n + 1 ≤ |F| ≤ 2n \).

Since \( α \) is an interval \( (3n-2) \)-coloring of \( G \), for \( t \in \{1, 2, \ldots, 3n-2\} \), there is an edge \( e ∈ E(K_{2n+1} − F) \) with \( β(e) = t \). Moreover, \( \max S(v_i, \alpha) - \min S(v_i, \alpha) = 2n-2 \) for \( 1 ≤ j ≤ 2n \). Clearly, for the proof, it suffices to show that for \( 1 ≤ l ≤ 2n−|F| \), the edges incident to the vertex \( v_i \) of \( K_{2n+1} − F \) are colored by consecutive colors.

By the definition of \( β \), for \( 1 ≤ l ≤ 2n−|F| \)

\[ S(v_i, \beta) = \{ v_i, \alpha \} \cup \{ v_i, \beta(v_i) \} = \{ \min S(v_i, \alpha) ≤ 2n−1 + l \} \cup \{ 2n−1 + l \} = \{ l, 2n−2+l \} \cup \{ 2n−1 + l \} = \{ l, 2n−1 + l \}. \]

This shows that \( β \) is an interval \( (3n-1) \)-coloring of \( K_{2n+1} − F \) if \( |F| = n \) and \( β \) is an interval \( (3n-2) \)-coloring of \( K_{2n+1} − F \) if \( n + 1 ≤ |F| ≤ 2n \).

(2). Suppose that (2) is false. Let \( V(K_{2n+1}) = \{v_0, v_1, v_2, \ldots, v_{2n}\} \) and \( \alpha \) be an interval \( t \)-coloring of \( K_{2n+1} − F \).

Without loss of generality, we may assume that \( d(v_0) = \Delta(K_{2n+1} − F) = 2n \) and \( d(v_i) = 2n−1 \) for \( 1 ≤ i ≤ 2n \). From the result of [10] (if \( G ∈ \mathcal{H} \) with \( |V(G)| ≥ 3 \), then \( W(G) ≤ 2|V(G)| - 4 \)) we have \( 2n ≤ t ≤ 4n−2 \). We distinguish two cases.

Case 1: \( 2n ≤ t ≤ 4n−3 \).

It is easy to see that

\begin{align*}
1 &≤ \min S(v_0, \alpha) ≤ 2n−2, \\
1 &≤ \min S(v_i, \alpha) ≤ 2n−1 \quad \text{for } 1 ≤ i ≤ 2n.
\end{align*}

Therefore, \( (2n−1) ≤ S(v_i, \alpha) \) for \( 0 ≤ j ≤ 2n \). This implies that the edges with color \( 2n−1 \) form a perfect matching, which contradicts the fact that \( K_{2n+1} − F \) does not have one.

Case 2: \( t = 4n−2 \).

Let \( e_i = u_i u_i \in E(K_{2n+1} − F) \) for \( i ∈ \{1, 2\} \); also \( \alpha(e_1) = 1 \) and \( \alpha(e_2) = 2n−2 \). Without loss of generality, we may assume that \( u_i u_2 \in E(K_{2n+1} − F) \). It is easy to see that either \( d(u_1) = 2n \) or \( d(u_2) = 2n \). If \( u_2 u_2 \in E(K_{2n+1} − F) \), then \( \alpha(u_2 u_2) ≤ 2n−1 \). Therefore, \( \alpha(u_2 u_2) ≤ 4n−3 \), which contradicts \( \alpha(u_2 u_2) = 4n−2 \). This implies that \( u_2 u_2 \notin E(K_{2n+1} − F) \). Analogously, it can be shown that \( u_1 u_2 \notin E(K_{2n+1} − F) \), which contradicts either \( d(u_1) = 2n \) or \( d(u_2) = 2n \). □

Finally, we consider the \( n \)-dimensional cube \( Q_n \).

**Lemma 15.** \( Q_n ∈ \mathcal{H} \) and \( w(Q_n) = n \) for any \( n ∈ \mathbb{N} \).

**Proof.** Since \( Q_n \) is a regular bipartite graph, we have \( \chi'(Q_n) = \Delta(Q_n) = n \) and, by **Theorem 1**, \( Q_n ∈ \mathcal{H} \) and \( w(Q_n) = n \). □

Kamalian proved the following:

**Theorem 16 ([17]).** \( W(Q_n) ≥ 2n−1 \) for any \( n ∈ \mathbb{N} \).

Now we derive a new lower bound for \( W(Q_n) \).

**Theorem 17.** \( W(Q_n) ≤ \frac{n(n+1)}{2} \) for any \( n ∈ \mathbb{N} \).
Proof. Let us prove that $W (Q_n) - W (Q_{n-1}) \geq n$ for $n \geq 2$.

For $i \in \{0, 1\}$, let $Q_{n-1}^{(i)}$ be the subgraph of $Q_n$ induced by the vertices

\[ \{ (i, \alpha_2, \alpha_3, \ldots, \alpha_n) | (\alpha_2, \alpha_3, \ldots, \alpha_n) \in \{0, 1\}^{n-1} \}. \]

Each $Q_{n-1}^{(i)}$ is isomorphic to $Q_{n-1}$. Lemma 15 implies that $Q_{n-1}^{(i)} \in \mathcal{H}$ for $i \in \{0, 1\}$. Let $\varphi$ be an interval $W (Q_{n-1}^{(0)})$-coloring of the graph $Q_{n-1}^{(0)}$.

Let us define an edge-coloring $\psi$ of the graph $Q_{n-1}^{(1)}$ in the following way: for every edge $(1, \tilde{\alpha}) (1, \tilde{\beta}) \in E \left(Q_{n-1}^{(1)}\right)$, let

\[ \psi \left((1, \tilde{\alpha}) (1, \tilde{\beta})\right) = \varphi \left((0, \tilde{\alpha}) (0, \tilde{\beta})\right) + n. \]

Now we define an edge-coloring $\pi$ of the graph $Q_n$.

For every edge $\tilde{\alpha} \tilde{\beta} \in E (Q_n)$, let

\[ \pi \left(\tilde{\alpha} \tilde{\beta}\right) = \begin{cases} \varphi \left(\tilde{\alpha} \tilde{\beta}\right), & \text{if } \tilde{\alpha}, \tilde{\beta} \in V \left(Q_{n-1}^{(0)}\right), \\ \min S \left(\tilde{\alpha}, \varphi\right) + n - 1, & \text{if } \tilde{\alpha} \in V \left(Q_{n-1}^{(0)}\right), \tilde{\beta} \in V \left(Q_{n-1}^{(1)}\right), \\ \psi \left(\tilde{\alpha} \tilde{\beta}\right), & \text{if } \tilde{\alpha}, \tilde{\beta} \in V \left(Q_{n-1}^{(1)}\right). \end{cases} \]

Let us show that $\pi$ is an interval $(W (Q_{n-1}) + n)$-coloring of $Q_n$. Since $\varphi$ is an interval $W (Q_{n-1})$-coloring of $Q_{n-1}^{(0)}$, for $t \in \{1, 2, \ldots, W (Q_{n-1})\}$, there is an edge $e \in E (Q_n)$ with $\pi \left(e\right) = t$. Moreover, $\max S \left(\tilde{\alpha}, \varphi\right) - \min S \left(\tilde{\alpha}, \varphi\right) = n - 2$ for $\tilde{\alpha} \in V \left(Q_{n-1}^{(0)}\right)$.

By the definitions of $\pi$ and $\psi$, for $\tilde{\alpha} \in \{0, 1\}^{n-1}$

\[ S \left((0, \tilde{\alpha}), \pi\right) = S \left((0, \tilde{\alpha}), \varphi\right) \cup \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\} \]
\[ = \{\min S \left((0, \tilde{\alpha}), \varphi\right), \ldots, \max S \left((0, \tilde{\alpha}), \varphi\right)\} \cup \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\} \]

and

\[ S \left((1, \tilde{\alpha}), \pi\right) = S \left((1, \tilde{\alpha}), \psi\right) \cup \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\} \]
\[ = \{\min S \left((1, \tilde{\alpha}), \psi\right), \ldots, \max S \left((1, \tilde{\alpha}), \psi\right)\} \cup \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\} \]
\[ = \{\min S \left((0, \tilde{\alpha}), \varphi\right), \ldots, \max S \left((0, \tilde{\alpha}), \varphi\right) + n\} \cup \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\}. \]

This proves that for $t \in \{W (Q_{n-1}) + 1, \ldots, W (Q_{n-1}) + n\}$, there is an edge $e \in E (Q_n)$ with $\pi \left(e\right) = t$.

Now let us prove that the edges incident to any vertex of $Q_n$ are colored by $n$ consecutive colors.

It is easy to see that for $\tilde{\alpha} \in \{0, 1\}^{n-1}$

\[ S \left((0, \tilde{\alpha}), \pi\right) = \{\min S \left((0, \tilde{\alpha}), \varphi\right), \ldots, \max S \left((0, \tilde{\alpha}), \varphi\right)\} \cup \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\} \]
\[ = \{\min S \left((0, \tilde{\alpha}), \varphi\right), \ldots, \min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\} \]
\[ = \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\}. \]

and

\[ S \left((1, \tilde{\alpha}), \pi\right) = \{\min \varphi + n, \ldots, \max \varphi + n\} \cup \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1\} \]
\[ = \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1, \ldots, \min S \left((0, \tilde{\alpha}), \varphi\right) + 2n - 2\} \]
\[ = \{\min S \left((0, \tilde{\alpha}), \varphi\right) + n - 1, \min S \left((0, \tilde{\alpha}), \varphi\right) + 2n - 2\}. \]

This shows that $\pi$ is an interval $(W (Q_{n-1}) + n)$-coloring of $Q_n$.

For $n \geq 2$, we have

\[ W (Q_n) \geq W (Q_{n-1}) + n \]
\[ W (Q_{n-1}) \geq W (Q_{n-2}) + n - 1 \]

\[ W (Q_2) \geq W (Q_1) + 2. \]

By summing these inequalities, we get $W (Q_n) \geq \frac{n(n+1)}{2}$. \Box

From Theorems 1 and 17 we have:

**Corollary 18.** If $n \leq t \leq \frac{n(n+1)}{2}$, then $Q_n$ has an interval $t$-coloring.
Remark 19. From the result of [2] (if $G$ is bipartite and $G \in \mathcal{G}$, then $W(G) \leq \text{diam}(G) (\Delta(G) - 1) + 1$) and taking into account $\text{diam}(Q_n) = \Delta(Q_n) = n$, we have $W(Q_n) \leq \text{diam}(Q_n) (\Delta(Q_n) - 1) + 1 = n^2 - n + 1$. This shows that the lower bound in Theorem 17 is not so far from the known upper bound. Moreover, the lower bound in Theorem 17 is sharp for $n \leq 3$. 

3. Conjectures

We conclude with the following conjectures on the exact value of the greatest possible number of colors in interval colorings of complete graphs and $n$-dimensional cubes.

Conjecture 20. If $n = p2^q$, where $p$ is odd and $q$ is nonnegative, then $W(K_{2n}) = 4n - 2 - p - q$.

Conjecture 21. $W(Q_n) = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

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References