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Some new common fixed point theorems under strict contractive conditions

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Abstract

The main purpose of this paper is to give some new common fixed point theorems under strict contractive conditions for mappings satisfying a new property. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

It is well known that in the setting of metric space, strict contractive condition do not ensure the existence of common fixed point unless the space is assumed compact or the strict conditions are replaced by stronger conditions as in [1–3]. In 1986, Jungck [4] introduced the notion of compatible maps. This concept was frequently used to prove existence theorems in common fixed point theory. However, the study of common fixed points of noncompatible mappings is also very interesting. Work along these lines has recently been initiated by Pant [5,6].

The aim of this paper is to define a new property which generalize the concept of noncompatible mappings, and give some common fixed point theorems under strict contractive conditions.

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We begin with some known definitions.

2. Preliminaries

In [7], Sessa introduced the notion of the weak commutativity.

Definition [7]. Two selfmappings S and T of a metric space (X,d) are said to be weakly commuting if

$$d(STx, TSx) \leqslant d(Sx, Tx), \quad \forall x \in X.$$

It is clear that two commuting mappings are weakly commuting, but the converse is not true as is shown in [7].

Jungck [4] extended this concept in the following way:

Definition [4]. Let T and S be two selfmappings of a metric space (X,d). S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Obviously, two weakly commuting mappings are compatible, but the converse is not true as is shown in [4]. Recently, Jungck introduced the concept of weakly compatible maps as follows: Two selfmapping T and S of a metric space X are said to be weakly compatible if they commute at there coicidence points; i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

It is easy to see that two compatible maps are weakly compatible.

3. Main results

Definition 1. Let S and T be two selfmappings of a metric space (X,d). We say that T and S satisfy the property (E.A) if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Examples. (1) Let $X = [0, +\infty[$. Define $T, S : X \to X$ by

$$Tx = \frac{x}{4} \quad \text{and} \quad Sx = \frac{3x}{4}, \quad \forall x \in X.$$

Consider the sequence $x_n = 1/n$. Clearly $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$. Then T and S satisfy (E.A).

(2) Let $X = [2, +\infty[$. Define $T, S : X \to X$ by

$$Tx = x + 1 \quad \text{and} \quad Sx = 2x + 1, \quad \forall x \in X.$$

Suppose that property (E.A) holds; then there exists in X a sequence (x_n) satisfying

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for some } t \in X.$$

Therefore

$$\lim_{n\to\infty} x_n = t - 1 \quad \text{and} \quad \lim_{n\to\infty} x_n = \frac{t-1}{2}.$$

Then $t = 1$, which is a contradiction since $1 \notin X$. Hence T and S do not satisfy (E.A).

Remark 1. It is clear from the Jungck's definition [4] that two selfmappings S and T of a metric space (X, d) will be noncompatible if there exists at least one sequence (x_n) in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \quad \text{for some } t \in X,$$

but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non-zero or non-existent. Therefore, two noncompatible selfmappings of a metric space (X, d) satisfy the property (E.A).

Now we state our main theorem.

Theorem 1. *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(i) *T and S satisfy the property (E.A),*
(ii)

$$d(Tx, Ty) < \max\bigl\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2, \\ [d(Ty, Sx) + d(Tx, Sy)]/2\bigr\}, \quad \forall x \neq y \in X,$$

(iii) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Proof. Since T and S satisfy the property (E.A), there exists in X a sequence (x_n) satisfying

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for some } t \in X.$$

Suppose that SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$. Also $\lim_{n\to\infty} Tx_n = Sa$.

We show that $Ta = Sa$. Suppose that $Ta \neq Sa$. Condition (ii) implies

$$d(Tx_n, Ta) < \max\bigl\{d(Sx_n, Sa), [d(Tx_n, Sx_n) + d(Ta, Sa)]/2, \\ [d(Ta, Sx_n) + d(Tx_n, Sa)]/2\bigr\}.$$

Letting $n \to +\infty$ yields

$$\begin{aligned} d(Sa, Ta) &\leqslant \max\bigl\{d(Sa, Sa), [d(Ta, Sa) + d(Sa, Sa)]/2, \\ &\qquad [d(Ta, Sa) + d(Sa, Sa)]/2\bigr\} \\ &\leqslant d(Ta, Sa)/2; \end{aligned}$$

a contradiction. Hence $Ta = Sa$.

Since T and S are weakly compatible, $STa = TSa$ and, therefore, $TTa = TSa = STa = SSa$.

Finally, we show that Ta is a common fixed point of T and S. Suppose that $Ta \neq TTa$. Then

$$\begin{aligned} d(Ta, TTa) &\leqslant \max\bigl\{d(Sa, STa), [d(Ta, Sa) + d(TTa, STa)]/2, \\ &\qquad [d(TTa, Sa) + d(Ta, STa)]/2\bigr\} \\ &\leqslant \max\bigl\{d(Ta, TTa), d(TTa, Ta)\bigr\} = d(Ta, TTa) \end{aligned}$$

which is a contradiction. Hence $TTa = Ta$ and $STa = TTa = Ta$. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$. Uniquness of the common fixed point follows easily. □

Now we give an example to support our result.

Example. Let $X = [1, +\infty[$ with the usual metric $d(x, y) = |x - y|$. Define $T, S\colon X \to X$ by

$$Tx = 2x - 1 \quad \text{and} \quad Sx = x^2, \quad \forall x \in X.$$

Then

(1) T and S satisfy the property (E.A) for the sequence $x_n = 1 + 1/n$, $n = 1, 2, \ldots,$
(2) S and T are weakly compatible,
(3) T and S satisfy for all $x \neq y$

$$d(Tx, Ty) < \max\bigl\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2, \\ [d(Ty, Sx) + d(Tx, Sy)]/2\bigr\},$$

(4) $T1 = S1 = 1$.

Since two noncompatible selfmappings of a metric space (X, d) satisfy the property (E.A), we get the following result:

Corollary 1. *Let S and T be two noncompatible weakly compatible selfmappings of a metric space (X, d) such that*

(i) $d(Tx, Ty) < \max\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2,$
$[d(Ty, Sx) + d(Tx, Sy)]/2\}, \ \forall x \neq y \in X,$
(ii) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Corollary 2. *Let S and T be two weakly compatible selfmappings of a metric space (X, d). Suppose that there exists a mapping $\phi : X \to \mathbb{R}^+$ such that*

(i) $d(Sx, Tx) < \phi(Sx) - \phi(Tx), \forall x \in X,$
(ii) $d(Tx, Ty) < \max\{d(Sx, Sy), [d(Ty, Sx) + d(Tx, Sy)]/2\}, \forall x \neq y \in X,$
(iii) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Choose $x_2 \in X$ such that $Tx_1 = Sx_2$. In general, choose $x_n \in X$ such that $Tx_{n-1} = Sx_n$. Then

$$d(Sx_n, Sx_{n+1}) = d(Sx_n, Tx_n) \leqslant \phi(Sx_n) - \phi(Tx_n) = \phi(Sx_n) - \phi(Sx_{n+1}).$$

Consider the nonnegative real sequence (a_n) defined by $a_n = \phi(Sx_n)$, $n = 1, 2, \ldots$. It is easy to see that the sequence (a_n) is nonincreasing and belowed by 0. Therefore (a_n) is a convergente sequence. On the other hand, we have

$$d(Sx_n, Sx_{n+m}) \leqslant a_n - a_{n+m}$$

which implies that the sequence (Sx_n) is a cauchy sequence in SX. Suppose that SX is a complete subspace of X. Then there exists $t \in SX$ such that $\lim_{n\to\infty} Sx_n = t$. Also, we have $\lim_{n\to\infty} Tx_n = t$. Subsequently, T and S satisfy the property (E.A). From (ii), it follows that

$$d(Tx, Ty) < \max\bigl\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2, \\ [d(Ty, Sx) + d(Tx, Sy)]/2\bigr\}, \quad \forall x \neq y \in X.$$

Therefore, all conditions of Theorem 1 are satisfied and the conclusion follows from this theorem immediately. □

In [8], Caristi proved that a selfmapping T of a complete metric space (X,d) has a fixed point if there exists a lower semi-continuous function $\phi : X \to \mathbb{R}^+$ satisfying

$$d(x, Tx) \leqslant \phi(x) - \phi(Tx).$$

However, it may be observed that T will have a fixed point if it satisfies the above inequality for arbitrary ϕ and its graph is closed. Setting $S = \mathrm{Id}_X$ in Corollary 1, we get the following result:

Corollary 3. *Let T be selfmapping of a complete metric space (X,d). Suppose that there exists a mapping $\phi : X \to \mathbb{R}^+$ such that*

(i) $d(x, Tx) \leqslant \phi(x) - \phi(Tx)$, $\forall x \in X$,
(ii) $d(Tx, Ty) < \max\{d(x, y), [d(x, Ty) + d(y, Tx)]/2\}$, $\forall x \neq y \in X$.

Then T has a unique fixed point.

Taking $T = \mathrm{Id}_X$ in Corollary 1, we have the following result:

Corollary 4. *Let S be a surjective selfmapping of a complete metric space (X,d). Suppose that there exists a mapping $\phi : X \to \mathbb{R}^+$ such that*

(i) $d(x, Sx) \leqslant \phi(Sx) - \phi(x)$, $\forall x \in X$,
(ii) $d(x, y) < \max\{d(Sx, Sy), [d(y, Sx) + d(x, Sy)]/2\}$, $\forall x \neq y \in X$.

Then S has a unique fixed point.

The next theorem involves a function F. Various conditions on F have been studied by many different authors. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the following conditions:

(F_1) F is nondecreasing on $\mathbb{R}^+$,
(F_2) $0 < F(t) < t$, for each $t \in]0, +\infty[$.

Theorem 2. *Let A, B, T and S be selfmappings of a metric space (X,d) such that*

(1) $d(Ax, By) \leqslant F(\max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\})$, $\forall (x, y) \in X^2$,
(2) *(A, S) and (B, T) are weakly compatibles,*
(3) *(A, S) or (B, T) satisfies the property* (E.A),
(4) *$AX \subset TX$ and $BX \subset SX$.*

If the range of the one of the mappings A, B, T or S is a complete subspace of X, then A, B, T and S have a unique common fixed point.

Proof. Suppose that (B,T) satisfies the property (E.A). Then there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Since $BX \subset SX$, there exists in X a sequence (y_n) such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = t$. Let us show that $\lim_{n\to\infty} Ay_n = t$. Indeed, in view of (1), we have

$$\begin{aligned} d(Ay_n, Bx_n) &\leqslant F\big(\max\big\{d(Sy_n, Tx_n), d(Sy_n, Bx_n), d(Tx_n, Bx_n)\big\}\big) \\ &\leqslant F\big(\max\big\{d(Bx_n, Tx_n), 0, d(Tx_n, Bx_n)\big\}\big) \\ &\leqslant F\big(d(Tx_n, Bx_n)\big) \leqslant d(Tx_n, Bx_n). \end{aligned}$$

Therefore $\lim_{n\to\infty} d(Ay_n, Bx_n) = 0$. Since $d(Ay_n, t\} \leqslant d(Ay_n, Bx_n) + d(Bx_n, t)$, we deduce that $\lim_{n\to\infty} Ay_n = t$. Suppose that SX is a complete subspace of X. Then $t = Su$ for some $u \in X$. Subsequently, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su$.

From (1), we have

$$d(Au, Bx_n) \leqslant F\big(\max\big\{d(Su, Tx_n), d(Su, Bx_n), d(Tx_n, Bx_n)\big\}\big).$$

Letting $n \to \infty$ and using (F_2), it follows $Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $AX \subset TX$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$. Using (1), we have

$$\begin{aligned} d(Au, Bv) &\leqslant F\big(\max\big\{d(Su, Tv), d(Su, Bv), d(Tv, Bv)\big\}\big) \\ &\leqslant F\big(\max\big\{d(Au, Bv), d(Au, Bv)\big\}\big) \\ &\leqslant F\big(d(Au, Bv)\big) \end{aligned}$$

which implies that $Au = Su = Tv = Bv$. The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Let us show that Au is a common fixed point of A, B, T and S. In view of (1), it follows

$$\begin{aligned} &d(Au, AAu) = d(AAu, Bv) \\ &\quad\leqslant F\big(\max\big\{d(SAu, Tv), d(SAu, Bv), d(Tv, Bv)\big\}\big) \\ &\quad\leqslant F\big(\max\big\{d(AAu, Au), d(AAu, Au)\big\}\big) \\ &\quad\leqslant F\big(d(AAu, Au)\big). \end{aligned}$$

Therefore $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly, we prove that Bv is a common fixed point of B and T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, T and S. The proof is similar when TX is assumed to be a complete subspace of X. The cases in which AX or BX is a complete subspace of X are similar to the cases in which TX or SX, respectively, is complete since $AX \subset TX$ and $BX \subset SX$. If $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$, then (1) gives

$$\begin{aligned} d(u, v) = d(Au, Bv) &\leqslant F\big(\max\big\{d(Su, Tv), d(Su, Bv), d(Tv, Bv)\big\}\big) \\ &\leqslant F\big(d(u, v)\big). \end{aligned}$$

Therefore $u = v$ and the common fixed point is unique. Hence we have the theorem. □

For three maps, we have the following result:

Corollary 5. *Let A, B and S be selfmappings of a metric space (X, d) such that*

(1) $d(Ax, By) < F(\max\{d(Sx, Sy), d(Sx, By), d(Sy, By)\}), \forall (x, y) \in X^2$,
(2) (A, S) *and* (B, S) *are weakly compatibles,*
(3) (A, S) *or* (B, S) *satisfies the property* (E.A),
(4) $AX \subset SX$ *and* $BX \subset SX$.

If the range of the one of the mappings A, B or S is a complete subspace of X, then A, B and S have a unique common fixed point.

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