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Semi-idempotent and Semi-strongly Continuous Measures

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INTRODUCTION

In this paper G is a nondiscrete compact Abelian group with character group Γ and $M(G)$ is the usual convolution algebra of finite Borel measures on G. The Fourier–Stieltjes transform of $\mu \in M(G)$ is the function $\hat{\mu}$ defined on Γ by

$$\hat{\mu}(\gamma) = \int_G \gamma(-x)\, d\mu(x).$$

We say Γ is *ϕ-ordered* if there exists a nontrival group homomorphism $\phi: \Gamma \to \mathbb{R}$, where $\mathbb{R}$ is the additive group of real numbers. If Γ is ϕ-ordered, we put $\mathscr{P} = \phi^{-1}([0, \infty))$. The discrete group Γ is said to be *fully ordered* if there exists a semi-group $\mathscr{P}$, such that $\mathscr{P} \cup -\mathscr{P} = \Gamma$ and $\mathscr{P} \cap -\mathscr{P} = \{0\}$. We define Γ to be ordered if either Γ is ϕ-ordered or if Γ is fully ordered. For Γ fully ordered, the semi-group $\mathscr{P}^+ = \{\gamma \in \Gamma: \gamma > 0\} = \mathscr{P} \backslash \{0\}$ is called the positive cone in Γ. For Γ ϕ-ordered $\mathscr{P}^+ = \{\gamma \in \Gamma: \phi(\gamma) > 0\}$.

In Section 1 we prove a generalized version of a theorem of Cohen and Davenport [1]. We give applications of the result in Sections 2, 3 and 4.

Let Γ be ordered. A measure $\mu \in M(G)$ is said to be semi-idempotent if $\hat{\mu}(\gamma) = \hat{\mu}^2(\gamma)$ for all $\gamma \in \mathscr{P}^+$. Kessler announced in [6] that if Γ is fully ordered and μ is semi-idempotent, then there exists an idempotent measure $\nu \in M(G)$ such that $\hat{\nu}(\gamma) = \hat{\mu}(\gamma)$ for all $\gamma \in \mathscr{P}^+$. A more detailed discussion of the literature concerning the semi-idempotent problem will be given at the end of this section.

In Section 2 we prove the following result: Let Γ be ϕ-ordered and let S be a Sidon subset of Γ. If $\hat{\mu}(\gamma) = \hat{\mu}^2(\gamma)$ for all $\gamma \in \mathscr{P}^+ \backslash S$, then there exists an idempotent measure $\nu \in M(G)$ such that $\hat{\nu}(\gamma) = \hat{\mu}(\gamma)$ for all $\gamma \in \mathscr{P}^+ \backslash S$. Futhermore, an upper bound for the norm of ν depending only on the norm of μ is obtained if S is empty.

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Let $\mathscr{H}$ be the family of all closed subgroups of G with infinite index in G. Put $\{H_\alpha\}$ equal to the set of all cosets of H and define for any Borel set $E \subset G$, $\mu_H(E) = \sum_\alpha \mu(E \cap H_\alpha)$. A measure $\mu \in M(G)$ is said to be *strongly continuous* if $\mu_H = 0$ for all $H \in \mathscr{H}$. Ramsey proved in [12] that if Γ has a finite torsion subgroup and if μ is strongly continuous and satisfies the condition

$$\{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geqslant 1\} \cup \{\gamma \in \Gamma: |\hat{\mu}(\gamma)| < \varepsilon\} = \Gamma,$$

then provided ε is small enough (as a function of the norm of μ), $\operatorname{card}\{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geqslant 1\}$ is finite. Ramsey also proved that an upper bound for the cardinality of $\{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geqslant 1\}$ depends only on the norm of μ and the cardinality of the torsion subgroup of Γ. Subsequently, Ramsey and Wells [13] obtained the above result for all compact Abelian groups G except that in the general case no such upper bound on the cardinality of $\{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geqslant 1\}$ is possible.

Let Γ be ϕ-ordered and let $\mu \in M(G)$. We define $\mathscr{K}$ to be the family of all subgroups $K \in \mathscr{H}$ such that K does not contain the annihilator in G of the kernel of ϕ. Then μ *is said to be ϕ-continuous if $\mu_K = 0$ for all $K \in \mathscr{K}$.*

In Section 3 we prove that if μ is ϕ-continuous and satisfies the condition

$$\{\gamma \in \mathscr{P}^+: |\hat{\mu}(\gamma)| \geqslant 1\} \cup \{\gamma \in \mathscr{P}^+: |\hat{\mu}(\gamma)| \leqslant \varepsilon\} = \mathscr{P}^+,$$

then provided ε is small enough (as a function of the norm of μ), $\{\gamma \in \mathscr{P}^+: |\hat{\mu}(\gamma)| \geqslant 1\}$ is contained in a finite number of cosets of the kernel of ϕ. Futhermore, an upper bound for the number of cosets depending only on the norm of μ is obtained.

Let Γ be ordered. A measure $\mu \in M(G)$ is *semi-strongly continuous* if for all $H \in \mathscr{H}$, $\hat{\mu}_H(\gamma) = 0$ for all $\gamma \in \mathscr{P}^+$. As a consequence of the result cited above we prove that if μ is semi-strongly continuous and satisfies the condition

$$\{\gamma \in \mathscr{P}^+: |\hat{\mu}(\gamma)| \geqslant 1\} \cup \{\gamma \in \mathscr{P}^+: |\hat{\mu}(\gamma)| \leqslant \varepsilon\} = \mathscr{P}^+,$$

then provided ε is small enough (as a function of the norm of μ), $\operatorname{card}\{\gamma \in \mathscr{P}^+: |\hat{\mu}(\gamma)| \geqslant 1\}$ is finite. Moreover, if Γ is fully ordered, an upper bound for the cardinality of $\{\gamma \in \mathscr{P}^+: |\hat{\mu}(\gamma)| \geqslant 1\}$ depending only on the norm of μ is obtained.

In Section 4 we establish a connection between ϕ-continuous measures and semi-idempotents. In particular, we prove the semi-idempotent theorem for ordered groups and obtain as a special case the result announced by Kessler in [6].

The semi-idempotent theorem for empty Sidon set and $G = \mathbb{T}$ was first proved by Helson in [5]. Kessler announced the semi-idempotent theorem for fully ordered groups and empty Sidon set in [6] but as far as we know never published a proof. Meyer in [9] gave a proof of the semi-idempotent theorem with empty Sidon set and Γ a subgroup of the reals. The Archimedean case with Sidon pertubation was proved by Pigno in [10]. The methods of [6] do not apply when Γ is ϕ-ordered even if the Sidon set is empty, and for infinite Sidon sets and Γ fully ordered the methods of [6] are in general inapplicable.

1. A Generalized Cohen–Davenport Theorem

Theorems A and B stated below are essentially from [1]. Our formulation of these theorems closely follows that of [4]. The reader should compare Theorem B with the technical lemma of [13].

Theorem A. *Suppose Γ is fully ordered and $r, N \in \mathbb{Z}^+$ with $r \leqslant (\log N/(4 \log\log N))^{1/2}$. Let $\mathscr{B} \subset \Gamma$ such that $N \leqslant \operatorname{card} \mathscr{B} < \infty$. Then there is a subset of $\mathscr{B}$, $\{\gamma_0\} \cup \{\gamma_{ks}: 1 \leqslant k \leqslant r^2, 1 \leqslant s \leqslant r\}$ satisfying: Let $P_0 = \{\gamma_0\}$. For $1 \leqslant k \leqslant r^2$ put*

$$P_k = P_{k-1} \cup \{p + \gamma_{ks} - \gamma_{kt}: p \in P_{k-1}, 1 \leqslant s < t \leqslant r\} \\ \cup \{\gamma_{ks}: 1 \leqslant s \leqslant r\}.$$

Then

(1) $\gamma_{ks} > \gamma_{kt}$ if $s < t$,

(2) $p + \gamma_{ks} - \gamma_{kt} \notin \mathscr{B}$ if $p \in P_{k-1}$ and $1 \leqslant s < t \leqslant r$.

Theorem B. *Let $r \in \mathbb{Z}^+$, $r \geqslant 31$. Let $\mu \in M(G)$. Let $\mathscr{B}(\mu) = \{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geqslant 1\}$. Suppose we can find a set $\{\gamma_0\} \cup \{\gamma_{ks}: 1 \leqslant k \leqslant r^2, 1 \leqslant s \leqslant r\} \subset \mathscr{B}(\mu)$ satisfying (2). Suppose $|\hat{\mu}(\gamma)| \leqslant e^{-r}$ for $\gamma \in P_{r^2} \backslash \mathscr{B}(\mu)$. Then $\|\mu\| \geqslant r^{1/2}/4$.*

Theorem 1. *Let Γ^* denote a translate of a subgroup of Γ and let $\phi: \Gamma \to \mathbb{R}$ be a nontrivial group homomorphism of Γ into the additive group of real numbers. Let $\mu \in M(G)$, $\|\mu\| < r^{1/2}/4$. Let $\mathscr{B} = \{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geqslant 1\}$ and $\mathscr{S} = \{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \leqslant e^{-r}\}$. Suppose there exists an interval $I = [-\infty, b]$ or $I = [a, b] \subset \mathbb{R}$ such that*

$$\phi^{-1}((b, \infty)) \cap \Gamma^* \subset \mathscr{S}$$

and

$$\phi^{-1}(I) \cap \Gamma^* \subset \mathscr{B} \cup \mathscr{S}.$$

Then

$$\operatorname{card}\{\phi(\Gamma^* \cap \mathscr{B}) \cap I\} < N, \qquad \textit{where} \quad r \leqslant (\log N/(4 \log\log N))^{1/2}.$$

Proof. We prove the present theorem by modifying the counting argument of the proof of Theorem A.

For simplicity we suppose $I = [0, b]$.

Let $B = \phi(\Gamma^* \cap \mathscr{B}) \cap I$. Suppose B is infinite. Let $M \in \mathbb{Z}^+$ be large and to be chosen later.

For $L \in \mathbb{Z}^+$ let $\rho_1 = b/L$. Let $\Theta_i = (i-1)\rho_1$ for $i = 1,..., L$. We choose L sufficiently large so that

$$\operatorname{card}\{i\colon B \cap [\theta_i, \theta_i + \rho_1) \neq \phi\} \geqslant M$$

is satisfied. Let $\rho_2 = b/L \cdot l$, where $l = 2r^2 + 2$.

Put

$${}_k\theta_i = \theta_i + (k-1)\rho_2 \qquad (k = 1, 2,..., l).$$

We distinguish certain ${}_k\theta_i$'s as follows: If $[{}_k\theta_i - \rho_1/2,\ {}_k\theta_i + \rho_1/2) \cap B \neq \phi$, then we write ${}_k\theta_i = {}_kx_i$. For each fixed value of k, we define B_k to be the set of all ${}_kx_i$'s and we write $M_k = \operatorname{card} B_K$.

Notice that for k such that $(k-1)/l \leqslant \frac{1}{2}$, we have (if $i \neq L$)

$$[\theta_i, \theta_i + \rho_1) \subset [{}_k\theta_i - \rho_1/2, {}_k\theta_i + \rho_1/2) \cup [{}_k\theta_{i+1} - \rho_1/2, {}_k\theta_{i+1} + \rho_1/2).$$

Also, for $(k-1)/l > \frac{1}{2}$ we have (if $i \neq 1$)

$$[\theta_i, \theta_i + \rho_1) \subset [{}_k\theta_{i-1} - \rho_1/2, {}_k\theta_{i-1} + \rho_1/2) \cup [{}_k\theta_i - \rho_1/2, {}_k\theta_i + \rho_1/2).$$

Thus, we see that, for all $k = 1, 2,..., l$, $2M_k \geqslant M - 2$.

For any real number α and any $k = 1, 2,..., l$ we define $N_k(\alpha)$ to be the number of elements in B_k which are greater than or equal to α. We call ${}_kx_i$ *good* if $[{}_kx_i - \rho_2/2,\ {}_kx_i + \rho_2/2) \cap B \neq \phi$. We call ${}_kx_i$ *useful* and *i a place* if $[\theta_i - \rho_2/2,\ \theta_i + \rho_1 - \rho_2/2) \cap B \neq \phi$. Also, if i is a place, we say that *the interval* $[\theta_i - \rho_2/2,\ \theta_i + \rho_1 - \rho_2/2)$ *is useful.*

For ${}_kx_i$ useful we define $M({}_kx_i)$ to be the number of places $j \geqslant i$. It follows, as before, that if $j \neq 1, L$, then $[{}_kx_j - \rho_1/2,\ {}_kx_j + \rho_1/2)$ is a subset of either

$$[\theta_{j-1} - \rho_2/2,\ \theta_{j-1} + \rho_1 - \rho_2/2) \cup [\theta_j - \rho_{2/}/2,\ \theta_j + \rho_1 - \rho_2/2)$$

or

$$[\theta_j - \rho_2/2, \theta_j + \rho_1 - \rho_2/2) \cup [\theta_{j+1} - \rho_2/2, \theta_{j+1} + \rho_1 - \rho_2/2).$$

Thus, $N_k({}_kx_i) \leqslant 2M({}_kx_i) + 2$.

Notice that for all i,

$$[\theta_i - \rho_2/2, \theta_i + \rho_1 - \rho_2/2) = \bigcup_{k=1}^{l} [{}_k\theta_i - \rho_2/2, {}_k\theta_i + \rho_2/2).$$

We gather from this that if i is a place then ${}_kx_i$ is good for at least one k. Indeed, it follows that since there are at least $M-1$ places, there are at least $M-1$ ${}_kx_i$'s which are good.

For at least one value of k, we will construct a system

$$\mathscr{P}_k = \{P_{k,j}: j = -1, 0, 1, 2,..., r^2\},$$

such that

$$P_{k,-1} = \phi$$

and such that the system $\mathscr{P}_k$ is generated from good ${}_kx_i$ in the manner of Theorem A and such that (1) and (2) are satisfied with respect to B_k.

We first let $P_{k,-1} = \phi$ for all k and, as in the definition of the function M, we order the useful intervals from right to left. We look at the first useful interval and choose any good ${}_sx_i$ in it, $1 \leqslant s \leqslant l$. We let

$$P_{s,0} = \{{}_sx_i\}$$

and observe that (1) and (2) are vacuously satisfied since $P_{s,-1} = \phi$.

Although we may begin our induction here, it may be helpful to do another step in our construction. We have already selected from the first useful interval and obtained ${}_sx_i$. We now select the largest good ${}_tx_j$ from the next useful interval. If $t \neq s$, we set $P_{t,0} = \{{}_tx_j\}$ and we have adjoined one more set $P_{t,0}$ to the system $\mathscr{P}_t$. If $t = s$ (so ${}_tx_j = {}_sx_j$), we hold ${}_tx_j$ in abeyance and we search through at most the next $N_s({}_sx_i) + 1$ useful intervals to find the largest good ${}_ux_v < {}_sx_j$ such that

$${}_sx_i + {}_sx_j - {}_ux_v \notin B_s. \tag{*}$$

Continue in this way looking at all statements of the form (*) (where ${}_ux_v < {}_sx_j$ represents the variable and ${}_sx_j$ represents any one of the good elements held in abeyance) and after at most r steps we have either adjoined some set $P_{y,0}$ to $\mathscr{P}_y$, where $y \neq s$ or we have found r good elements of B_s which by construction generate $P_{s,1}$. Notice that in either case we have accomplished this after searching through at most $(r-1)\,N_s({}_sx_i) + 1$ useful intervals.

Now, in general suppose that for each k we have partially constructed the system $\mathscr{P}_k$ with the sets $P_{k,j}$, $j = -1, 0, 1, 2,..., j_k < r^2$, where $j_k \geqslant 0$ for at least one k. Indeed, we do know $j_s \geqslant 0$. We look at the next useful interval and choose any good ${}_{w}x_z$ in it. Vacuously, ${}_{w}x_z$ satisfies (1) and (2) for the set P_{w,j_w}. We have a simultaneous system of statements of the form

$$p + {}_{w}x_z - {}_{c}x_d \notin B_w, \qquad (\#)$$

where p runs through P_{w,j_w} and ${}_{c}x_d < {}_{w}x_z$ represents a good element. After inspection of at most $\sum N(p) + 1 (p \in P_{w,j_w})$ useful intervals, we have found ${}_{c}x_d$. We continue in this way each time choosing a new good element which satisfies all simultaneous systems (#) for previously chosen good elements at this stage. After at most $(r-1)\,l+1$ choices of good elements requiring inspection of at most $(r-1)\sum_{k=1}^{l}\sum_{p\in P_{k,j_k}} N(p)+1$ more useful intervals we can adjoin one more set $P_{k,j_{k+1}}$ to some system $\mathscr{P}_k$.

If M is chosen large enough, we will be able to complete construction of at least one system $\mathscr{P}_k$ which is generated by good elements. We list these elements as

$$\{{}_{k}x_h\} \cup \{{}_{k}x_{i,j}: i = 1, 2,..., r^2, j = 1, 2,..., r\},$$

where $\{{}_{k}x_h\} = P_{k,0}$.

To each of these good elements, g, we associate an element $\gamma \in \mathscr{B} \cap \Gamma^*$ such that

$$\phi(\gamma) \in [g - \rho_2/2, \ + \rho_2/2).$$

Notice that for any element, δ, generated from any of the γ's, $\phi(\delta)$ is within $\rho_1/2$ of the number d_δ generated by the corresponding g's. (This follows from the inequality $\rho_2 < \rho_1/(2r^2+1)$.)

Consider any such number d_δ. Then by the definitions of ${}_{k}\theta_i$ and θ_i

$$\begin{aligned} d_\delta = {} & (i-1)\rho_1 + (k-1)\rho_2 + ((j-1)\rho_1 + (k-1)\rho_2 - (l-1)\rho_1 - (k-1)\rho_2) \\ & + \cdots + ((m-1)\rho_1 + (k-1)\rho_2 - (n-1)\ \rho_1 - (k-1)\rho_2). \end{aligned}$$

So,

$$\begin{aligned} d_\delta &= ((i-1) + (j-1) - (l-1) + \cdots + (m-1) - (n-1))\rho_1 + (k-1)\rho_2 \\ &= (p-1)\rho_1 + (k-1)\rho_2. \end{aligned}$$

If $(p-1)\rho_1 < b$, then $d_\delta = {}_{k}\theta_p$. Since, by construction $d_\delta \notin B_k$, we see that $[d_\delta - \rho_1/2, \ d_\delta + \rho_1/2) \cap B = \phi$. If $(p-1)\rho_1 > b$, then we also have $[d_\delta - \rho_1/2, d_\delta + \rho_1/2] \cap B = \phi$ because $B \subset [0, b]$. Finally, if $(p-1)\rho_1 = b$, then for certain values of k, we may have the unpleasant situation that $[d_\delta - \rho_1/2, \ d_\delta + \rho_1/2) \cap B \neq \phi$. However, if we reperform the entire construction on the interval $[0, 2b]$ instead of $[0, b]$, we also obtain $[d_\delta - \rho_1/2, d_\delta + \rho_1/2) \cap B = \phi$.

Thus for all δ's, $\phi(\delta) \in [d_\delta - \rho_1/2, \ d_\delta + \rho_1/2)$ and $[d_\delta - \rho_1/2, d_\delta + \rho_1/2) \cap B = \phi$. Thus, $\phi(\delta) \notin B$ and (2) is satisfied with respect to B. By Theorem B, $\|\mu\| \geqslant r^{1/2}/4$ and this is a contradiction. We conclude that B is finite.

We now apply Theorems A and B to see that card $B < N$.

COROLLARY. *Suppose Γ is ϕ-ordered. Let $\mu \in M(G)$. Let r and N be related to $\|\mu\|$ as in Theorem* 1. *Suppose $\mathscr{P} \cap (\mathscr{B} \cup \mathscr{S}) = \mathscr{P}$. If $\phi(\mathscr{B})$ is bounded above, then* card$\{\phi(\mathscr{B} \cap \mathscr{P})\} < N$. *If ϕ is an isomorphism (so that Γ is Archimedean) and if $\phi(\mathscr{B})$ is bounded above, then* card$\{\mathscr{B} \cap \mathscr{P}\} < N$.

2. Semi-idempotent Measures

Given any finite set of integers $\{N_1, ..., N_n\}$ put $\delta_i = N_i \delta_0$, where δ_0 is the identity measure in $M(G)$. We say that $\hat{\mu}$ vanishes at infinity in the direction of $\phi: \Gamma \to \mathbb{R}$ if whenever $\phi(\gamma_j) \to +\infty$ then $\hat{\mu}(\gamma_j) \to 0$. The set of all $\mu \in M(G)$ which vanish at infinity in the direction of ϕ will be designated by $M_\phi(G)$. Let $M_\phi^\perp(G) = \{\rho \in M(G): \rho \perp \tau$ for each $\tau \in M_\phi(G)\}$. We begin by proving the following theorem:

THEOREM C. *Let Γ be ϕ-ordered and $\mu \in M(G)$. Suppose the convolution product satisfies*

$$\prod_{i=1}^{n} (\mu - \delta_i) \in M_\phi(G),$$

where $N_1, ..., N_n$ are given integers and $\delta_i = N_i \delta_0$.
Then

(a) $\hat{\mu}_\perp(\Gamma) \subset \mathbb{Z}$, *where* $\mu_\perp \in M_\phi^\perp(G)$, *and*

(b) *the support of $\prod_{i=1}^{n} (\mu_\perp - \delta_i)\hat{}$ is contained in a finite number (depending only on $\|\mu\|$) of cosets of* ker ϕ.

Proof. The first part of Theorem C was proved in [10]. To prove part (b) we observe that if $\prod_{i=1}^{n} (\mu - \delta_i) \in M_\phi(G)$, then

$$\prod_{i=1}^{n} (\mu_\perp - \delta_i) \in M_\phi(G). \tag{c.1}$$

Let $\phi^*(x) = \phi(-x) = -\phi(x)$. It follows from [2, p. 220] that (c.1) implies

$$\prod_{i=1}^{n} (\mu_\perp - \delta_i) \in M_{\phi^*}(G). \tag{c.2}$$

Inasmuch as $\ker \phi^* = \ker \phi$, (c.1), (c.2) and Theorem 1 of Section 1 in combination with part (a) of the present theorem yield the desired result.

A subset S of Γ is called a Sidon set if whenever $f \in L^\infty(G)$ and $\hat{f}$ is spectral in S we have $\sum |\hat{f}(\gamma)| < \infty$. For $A \subset \Gamma$ and $\mu \in M(G)$ put $\mu \in F(N_1, ..., N_j; A)$ if $\hat{\mu}|_A \subset \{N_1, ..., N_j\}$ and $N_i \in \mathbb{Z}$.

THEOREM 2a. *Let Γ be ϕ-ordered and suppose $\mu \in F(N_1, ..., N_j; \mathscr{P}^+ \backslash S$ with S a Sidon set in Γ. Then there exists $\nu \in M(G)$ such that $\nu \in F(N_1, ..., N_j; \Gamma)$ satisfying*

(a) *$\hat{\nu} = \hat{\mu}$ on $\mathscr{P}^+ \backslash S$;*

(b) *$\|\nu\|$ is bounded by a constant depending only on $\|\mu\|$ if S is empty;*

(c) *if ϕ is an isomorphism, or Γ is torsion free, then $\|\nu\|$ is bounded by a constant depending only on $\|\mu\|$ and the Sidon constant of S.*

Proof. Let $\mu \in F(N_1, ..., N_j; \mathscr{P}^+ \backslash S)$. We claim

$$\prod_{i=1}^{j} (\mu - \delta_i) \in M_\phi(G). \tag{2a.1}$$

By Drury's result [8, p. 42] there is a measure $\omega \in M(G)$ such that

$$\hat{\omega}(S^+) = 0, \qquad \text{where} \quad S^+ = \mathscr{P}^+ \cap S, \tag{2a.2}$$

and

$$\hat{\omega}(\Gamma \backslash S^+) > 1. \tag{2a.3}$$

Observe that by (2a.2)

$$\omega * \prod_{i=1}^{j} (\mu - \delta_i) \in M_\phi(G), \tag{2a.4}$$

since $\mu \in F(N_1, ..., N_j; \Gamma \backslash -\mathscr{P} \cup S)$. Put $\phi^*(x) = \phi(-x)$. As a consequence of (2a.4), and [2] we gather that

$$\omega * \prod_{i=1}^{j} (\mu - \delta_i) \in M_{\phi^*}(G). \tag{2a.5}$$

As a consequence of (2a.3) we may infer from (2a.5) that

$$\prod_{i=1}^{j} (\mu - \delta_i) \in M_{\phi^*}(G). \tag{2a.6}$$

It follows now from (2a.6) and [2] that $\prod_{i=1}^{j} (\mu - \delta_i) \in M_\phi(G)$ and this establishes (2a.1).

Next, we see that $\prod_{i=1}^{j} (\mu - \delta_i) \in M_\phi(G)$ gives, via part (a) of Theorem C, the result

$$\hat{\mu}_\perp(\Gamma) \subset \mathbb{Z}. \tag{2a.7}$$

Notice that (2a.7) implies that $(\mu - \mu_\perp)\hat{}$ is integer-valued off $-\mathscr{P} \cup S$. Put $\mu - \mu_\perp = \mu_0$. Since μ_0 vanishes at infinity in the direction of ϕ, the set

$$\{\gamma \notin -\mathscr{P} \cup S: |\hat{\mu}_0(\gamma)| \neq 0\} = F$$

must satisfy $\phi(F) \subset [0, M]$ for some $M \in \mathbb{R}^+$.

By (2a.2) and (2a.3) we may conclude that the set

$$L = \{\gamma \in \mathscr{P}^+: |\hat{\omega}(\gamma)\, \hat{\mu}_0(\gamma)| \neq 0\}$$

satisfies $\phi(L) \subset [0, M]$. Applying Theorem 1 of Section 1 to the measure $\omega * \mu_0$ permits the conclusion

$$\bigcup_{i=1}^{k} (\gamma_i + \ker \phi) \supset L \qquad (\gamma_i > 0), \text{ for some } \langle \gamma_i \rangle_1^k \subset \Gamma. \tag{2a.8}$$

Inasmuch as $\hat{\omega}(\Gamma \backslash S^+) > 1$ it follows from (2a.8) that

$$\bigcup_{i=1}^{k} (\gamma_i + \ker \phi) \supset F. \tag{2a.9}$$

Put $\hat{\rho}_i = \hat{\mu}_0|_{\gamma_i + \ker \phi}$ $\quad (i = 1, 2,..., k)$, $\hat{\rho}_i = 0$ on $\Gamma \backslash S_i$ where

$$S_i = S \cap (\gamma_i + \ker \phi).$$

Notice that $\hat{\rho}_i$ is integer-valued off S_i and that S_i is a Sidon set. Since S_i is a weak Rajchman set in Γ (see [10]) it follows that $\hat{\rho}_i$ can be interpolated by an integer-valued transform ξ_i off S_i.

Put

$$\xi = \mu_\perp + \sum \xi_i.$$

Then ξ is integer-valued on Γ and interpolates $\hat{\mu}$ off $-\mathscr{P} \cup S$. Let $g(z)$ be any polynomial in the complex-variable z which fixes the set $\{N_1,..., N_j\}$ and maps every integer in the interval $[-\|\xi\|, \|\xi\|]$ into $\{N_1,..., N_j\}$. Then for the ν of our theorem take $\nu = g \circ \xi$. This proves part (a).

It follows from our proof that if $S = \phi$, then since $\|\mu_0\| \leqslant \|\mu\|$,

$$\bigcup_{i=1}^{N} (\gamma_i + \ker \phi) \supset F \qquad (\gamma_i > 0),$$

where N is as in Theorem 1 of Section 1. Thus when $S = \phi$ we obtain a bound on $\|\nu\|$ which depends only on $\|\mu\|$. If ϕ is an isomorphism then the estimate on the norm of ω([8, p. 42]) establishes (c). This completes the proof.

By the positive octant in $\mathbb{Z}^n$ we mean the set $Q = \{(\gamma_1, ..., \gamma_n); \gamma_i \geqslant 0, \forall\, i\}$. We shall conclude this section with a result concerning idempotents on the positive octant of $\mathbb{Z}^n$. Let Φ be a family of nontrivial homomorphisms of Γ into $\mathbb{R}$. We say $\hat{\mu}$ vanishes at ∞ in the direction of Φ if whenever $\phi(\gamma_n) \to +\infty$ for all $\phi \in \Phi$ then $\hat{\mu}(\gamma_n) \to 0$. As usual

$$M_\Phi^\perp(G) = \{\rho \in M(G): \rho \perp \tau \text{ for each } \tau \in M_\Phi(G)\},$$

where $M_\Phi(G)$ is the space of all transforms vanishing at infinity in the direction of Φ. The next theorem can be found in [10].

THEOREM D. *Suppose $\mu \in M(G)$ satisfies*

$$\prod_{i=1}^{n} (\mu - \delta_i) \in M_\Phi(G),$$

where $N_1, ..., N_n$ are given integers. Then

$$\hat{\mu}_\perp(\Gamma) \subset \mathbb{Z}, \qquad \textit{where} \quad \mu \in M_\Phi^\perp(G).$$

THEOREM 2b. *Let $\mu \in F(N_1, ..., N_j; Q\backslash S)$, where S is Sidon in $\mathbb{Z}^n$. Then there is a $\nu \in F(N_1, ..., N_j; \mathbb{Z}^n)$ such that*

$$\hat{\mu}(\gamma) = \hat{\nu}(\gamma), \qquad \gamma \in Q\backslash S.$$

Proof. Since $\mu \in F(N_1, ..., N_j; Q\backslash S)$ the same technique as that in Theorem 2a shows that

$$\prod_{i=1}^{j} (\mu - \delta_i) \in M_\Phi(\mathbb{T}^n), \tag{2b.1}$$

where Φ is the family of coordinate projections ϕ_i $(i = 1, 2, ..., n)$. Thus Theorem D and (2b.1) imply that $\hat{\mu}_\perp(\Gamma) \subset \mathbb{Z}$. We leave the rest of the proof to the reader.

3. MEASURES WITH CERTAIN CONTINUITY PROPERTIES

Recall that $\mathscr{H}$ denotes the family of closed subgroups of G with infinite index in G. Put $\{H_\alpha\}$ equal to the set of all cosets of $H \in \mathscr{H}$. For any Borel set $E \subset G$ we have defined

$$\mu_H(E) = \sum_{\alpha} \mu(E \cap H_\alpha).$$

The measure μ_H is called the part of μ carried by the cosets of H. The following result relating $\hat{\mu}$ and $\hat{\mu}_H$ is due to Glicksberg and Wik [3]:

THEOREM E. *Let Γ be ordered and suppose $H \in \mathscr{H}$. Then $\hat{\mu}_H(\mathscr{P}) \subset \hat{\mu}(\mathscr{P})^-$. If Γ is ϕ-ordered we also have $\hat{\mu}_H(\mathscr{P}_n) \subset \hat{\mu}(\mathscr{P}_n)^-$, where $\mathscr{P}_n = \phi^{-1}[[n, \infty)]$, $n \in \mathbb{Z}^+$.*

A measure $\mu \in M(G)$ is said to be semi-strongly continuous *if for every $H \in \mathscr{H}$, $\hat{\mu}_H(\gamma) = 0$ for all $\gamma \in \mathscr{P}^+$. If Γ is ϕ-ordered we put $\mathscr{K}$ equal to the set of $K \in \mathscr{H}$ such that $K \not\supset (\ker \phi)^\perp$. Then μ is* continuous in the direction of ϕ *(or simply ϕ-continuous) if $\mu_K = 0$ for all $K \in \mathscr{K}$.*

THEOREM 3a. *Let Γ be ϕ-ordered and suppose $\mu \in M(G)$. Then*

(i) *if $\mu \in M_\phi(G)$ then $\mu_H \in M_\phi(G)$ for all $H \in \mathscr{H}$;*

(ii) *If $\mu_K \in M_\phi(G)$ for $K \in \mathscr{K}$ then $\mu_K = 0$.*

Proof. By Theorem E we have $\hat{\mu}_H(\mathscr{P}_n) \subset \hat{\mu}(\mathscr{P}_n)^-$ for all natural numbers n. Since $\mu \in M_\phi(G)$ we gather that $\mu_H \in M_\phi(G)$ and this confirms (i). We must now establish (ii).

Let $\Psi: M(G) \to M(G|K)$ where Ψ is the usual mapping induced by the natural homomorphism of $G \to G|K$. Fix $\beta \in \Gamma$. We must show that $\{\Psi(\beta\mu)\}_d = 0$. Here $\{\Psi(\beta\mu)\}_d$ denotes the discrete part of $\Psi(\beta\mu)$.

Let $\{\gamma_j\}$ be a sequence in $K^\perp$ such that $\phi(\gamma_j) \geqslant j$, $j \in \mathbb{Z}^+$. Since $\{\Psi(\beta\mu)\}_d\hat{\ }(\gamma) = \hat{\mu}_K(\gamma - \beta)$ is almost periodic on $K^\perp$, the sequence $\hat{\mu}_K(\gamma - \beta + \gamma_j)$ has a uniformly convergent subsequence. Denote this subsequence by $\hat{\mu}_K(\gamma - \beta + \gamma_k)$. Since $\hat{\mu}_K(\gamma - \beta + \gamma_k) \to 0$ pointwise for $\gamma \in K^\perp$ we have that $\hat{\mu}_k(\gamma - \beta + \gamma_k) \to 0$ uniformly in γ. Put $\gamma = \gamma - \gamma_k$. Given $\varepsilon > 0$ choose k such that $|\hat{\mu}_K(\gamma - \gamma_k - \beta + \gamma_k)| < \varepsilon$. Thus $\hat{\mu}_K(\gamma - \beta) = 0$ for all $\gamma \in K^\perp$. This concludes the proof.

We shall now state some corollaries of Theorem 3a. Corollary 2 will be important in the next section.

COROLLARY 1. *If μ is semi-strongly continuous then μ is continuous.*

COROLLARY 2. *Suppose $\prod_{i=1}^n (\mu - \delta_i) \in M_\phi(G)$ where $N_i \in \mathbb{Z}$ and $\delta_i = N_i \delta_0$. Then for all $K \in \mathscr{K}$* we have

$$\hat{\mu}_K(\Gamma) \subset \{N_1, \ldots, N_n\}.$$

COROLLARY 3. *If Γ is ϕ-ordered, then μ is semi-strongly continuous $\Rightarrow \mu$*

is ϕ-continuous. If Γ is Archimedean ordered then μ is strongly continuous if and only if μ is semi-strongly continuous.

The following result characterizes semi-strongly continuous measures in terms of strongly continuous measures.

THEOREM 3b. *Let Γ be ordered and $\mu \in M(G)$. The following statements are equivalent:*

(i) *μ is semi-strongly continuous;*

(ii) *there is a strongly continuous $\nu \in M(G)$ such that $\hat{\mu}(\gamma) = \hat{\nu}(\gamma)$ for all $\gamma \in \mathscr{P}^+$.*

Proof. Suppose $\hat{\mu}(\gamma) = \hat{\nu}(\gamma)$ for all $\gamma \in \mathscr{P}^+$ and ν is strongly continuous. By Theorem E we may conclude that $\hat{\mu}_H(\gamma) = \hat{\nu}_H(\gamma)$ for all $\gamma \in \mathscr{P}^+$ and all $H \in \mathscr{H}$. Thus (ii) $\Rightarrow$ (i).

Next, let μ be semi-strongly continuous. Suppose $\exists\, H_0 \in \mathscr{H}$ such that $\|\mu_{H_0}\| \geqslant 1$. Let $\mu - \mu_{H_0} = {}_1\mu$. Suppose there is an $H_1 \in \mathscr{H}$ satisfying $\|{}_1\mu_{H_1}\| \geqslant 1$. Let ${}_2\mu = {}_1\mu - {}_1\mu_{H_1}$. After at most $\|\mu\|$ steps, we have measures ${}_{q_1}\mu$ and ${}_1\nu = \mu_{H_0} + \cdots + {}_{q_1-1}\mu_{H_{q_1-1}}$, where $\|{}_{q_1}\mu_H\| < 1$ for all $H \in \mathscr{H}$.

Suppose we can find $H_{q_1} \in \mathscr{H}$ such that $\|{}_{q_1}\mu_{H_{q_1}}\| \geqslant \frac{1}{2}$. Let ${}_{q_1+1}\mu = {}_{q_1}\mu - {}_{q_1}\mu_{H_{q_1}}$. Suppose $\exists\, H_{q_1+1} \in \mathscr{H}$ such that $\|{}_{q_1+1}\mu_{H_{q_1+1}}\| \geqslant \frac{1}{2}$. Put ${}_{q_1+2}\mu = {}_{q_1+1}\mu - {}_{q_1+1}\mu_{H_{q_1+1}}$.

In at most $2\|{}_{q_1}\mu\|$ steps we have measures ${}_2\nu = {}_{q_1}\mu_{H_{q_1}} + \cdots + {}_{q_2-1}\mu_{H_{q_2-1}}$ and ${}_{q_2}\mu$, where $\|{}_{q_2}\mu_H\| < \frac{1}{2}$ for all $H \in \mathscr{H}$.

Suppose there exists $H_{q_2} \in \mathscr{H}$ such that $\|{}_{q_2}\mu_{H_{q_2}}\| \geqslant \frac{1}{3}$. Repeating the process we eventually arrive at measures ${}_1\nu$, ${}_2\nu$, ${}_3\nu$,.... Let $\mu_{\mathscr{H}}$ be the norm limit of $\sum_{i=1}^n {}_i\nu$ in $M(G)$.

Observe that $\hat{\mu}_{\mathscr{H}}(\gamma) = 0\ \forall \gamma \in \mathscr{P}^+$. Put $\nu = \mu - \mu_{\mathscr{H}}$. Then by construction $\nu_H = 0$ for all $H \in \mathscr{H}$. Furthermore, the interpolating measure ν satisfies $\|\nu\| \leqslant \|\mu\|$. This concludes the proof.

Given $\mu \in M(G)$ choose $r \in \mathbb{Z}^+ (r \geqslant 31)$ such that $\|\mu\| < r^{1/2}/4$. Then choose N to satisfy $r \leqslant \{\log N/(4 \log \log N)\}^{1/2}$. For $\mu \in M(G)$ put

$$\mathscr{B}(\mu) = \{\gamma \in \Gamma : |\hat{\mu}(\gamma)| \geqslant 1\} = \mathscr{B}$$

and

$$\mathscr{S}(\mu) = \{\gamma \in \Gamma : |\hat{\mu}(\gamma)| \leqslant e^{-r}\} = \mathscr{S}.$$

We next state and prove our main result. The proof uses Theorem 1 of Section 1 and a variant on the argument of Ramsey and Wells [13].

THEOREM 3c. *Let Γ be ϕ-ordered and let $\mu \in M(G)$ be ϕ-continuous.*

Suppose $\mathscr{S}^+ \subset \mathscr{B} \cup \mathscr{S}$. *Then there exists* $\gamma_1, \ldots, \gamma_N$ *such that*

$$\mathscr{B} \cap \mathscr{P}^+ \subset \bigcup_{i=1}^{N} (\gamma_i + \ker \phi) \qquad (\gamma_i > 0).$$

Proof. By Theorems A and B of Section 1 it suffices to confirm that $\phi(\mathscr{B} \cap \mathscr{P}^+)$ is finite. We shall suppose $\phi(\mathscr{B} \cap \mathscr{P}^+)$ is infinite and force a contradiction.

It follows from Theorem 1 of Section 1 that if $\phi(\mathscr{B} \cap \mathscr{P}^+)$ is infinite then the set $\phi(\mathscr{B} \cap \mathscr{P}^+)$ is not bounded above. We shall see that this last assumption leads to the contradiction $\|\mu\| \geqslant r^{1/2}/4$. We adapt the method of Ramsey and Wells to estblish this contradiction. The next lemma may be found in [4].

LEMMA. *Let* μ *be a continuous measure on G. Let* γ_α *be a net in* Γ *such that* $\gamma_\alpha \mu$ *converges weak-* to* $\nu \in M(G)$. *Then*

$$\inf\{|\hat{\nu}(\gamma)| : \gamma \in \Gamma\} = 0.$$

For each natural number n let

$$B_n = \{\gamma \in \mathscr{B}: \phi(\gamma) \geqslant n\}$$

and put $C_n = (\bar{B}_n \mu)^{-*}$ (weak$-*$ closure in $M(G)$). Inasmuch as $\phi(\mathscr{B} \cap \mathscr{P}^+)$ is unbounded above it follows that $C_n \neq \phi$ for all $n \in \mathbb{N}$ (the natural numbers). Since the C_n are weak$-*$ compact it follows by the finite intersection property that

$$C_\infty = \bigcap_{n=1}^{\infty} C_n$$

is not empty.

Choose any element $\nu \in C_\infty$ of minimal norm. Notice $\nu \neq 0$ since $\|\nu\| \geqslant 1$. Suppose $\gamma \notin \mathscr{B}(\nu) = \{\gamma \in \Gamma: |\hat{\nu}(\gamma)| \geqslant 1\}$; then it is easy to check that

$$|\hat{\nu}(\gamma)| \leqslant e^{-r}. \tag{3c.1}$$

Thus ν satisfies

$$\Gamma = \mathscr{B}(\nu) \cup \mathscr{S}(\nu).$$

Choose a net $\{\gamma_\alpha : \alpha \in A\} \subset \mathscr{B}(\mu)$ and a subset $\{\alpha_n : n \in \mathbb{N}\}$ of A satisfying

$$\bar{\gamma}_\alpha \mu \to \nu \qquad \text{weak}-*,$$

with $\alpha > \alpha_n \Rightarrow \gamma_\alpha \in B_n$. For all $\lambda \in \Gamma, \bar{\gamma}_\alpha \bar{\lambda}\mu \to \bar{\lambda}\nu$ weak–* and so

$$\hat{\mu}(\lambda + \gamma_\alpha) \to \hat{\nu}(\lambda).$$

So if $\lambda \in \mathscr{B}(\nu)$, then $\lambda + \lambda_\alpha \in \mathscr{B}_n$ eventually, and so

$$\overline{\mathscr{B}(\nu)} \cdot \nu \subseteq C_\infty.$$

Hence $\|\sigma\| = \|\nu\|$ for every measure σ of the weak–* closure Y of $\overline{\mathscr{B}(\nu)}\nu = Y_0$. It follows as in [13] that the weak–* topology and the norm topology coincide on Y.

Thus Y is compact in $M(G)$ and Y_0 is norm dense in $M(G)$. In particular, Y is covered by a finite number of sets

$$U_a = \{\omega \in M(G) \colon \|\omega - \bar{a}\nu\| < 1 - e^{-r}\},$$

with $a \in \mathscr{B}(\nu)$. We gather that

$$Y \subseteq \bigcup_{k=1}^{m} U_{a_k}, \qquad \{a_k\} \subset \mathscr{B}(\nu). \tag{3c.2}$$

We shall use (3c.2) to show that $\mathscr{B}(\nu)$ is a finite union of cosets of some subgroup Λ of Γ. We repeat some details from [13] for the reader's convenience.

We define an equivalence relation on Γ as follows: Define $a \sim b \Leftrightarrow \mathscr{B}(\nu) - a = \mathscr{B}(\nu) - b$. So $\mathscr{B}(\nu)$ is a union of equivalence classes. If $\|\bar{a}\nu - \bar{b}\nu\| < 1 - e^{-r}$, then, in view of (3c.1), $\gamma + a \in \mathscr{B}(\nu)$ if and only if $\gamma + b \in \mathscr{B}(\nu)$. By (3c.2), $\mathscr{B}(\nu)$ is a finite union of equivalence classes. Let F be an equivalence class contained in $\mathscr{B}(\nu)$ and let $a \in F$. It is clear that $0 \in F - a$. To see that $F - a$ is a group it suffices to show that if b, $c \in F - a$, then $b - c \in F - a$. That is, if $b + a \sim c + a$, then $b - c + a \sim a$. Note that

$$\begin{aligned} \mathscr{B}(\nu) - (b - c + a) &= \mathscr{B}(\nu) - (b + a) + (c + a) - a \\ &= \mathscr{B}(\nu) - (c + a) + (c + a) - a \\ &= \mathscr{B}(\nu) - a. \end{aligned}$$

If $a \in F$, then

$$b \in F - a \Leftrightarrow b + a \sim a \Leftrightarrow b \sim 0.$$

The latter condition is independent of F. It follows that every equivalence class F is a coset of the same subgroup Λ of Γ.

Let $a \in \Gamma$. We claim γ_α is eventually out of $\Lambda + a$. Suppose not. Let λ_α be a cofinal subnet of $\{\gamma_\alpha\}$ contained in $\Lambda + a$ such that $\bar{\lambda}_\alpha \mu \to \nu$. In this case $\Lambda \not\subset \ker \phi$ by the definition of γ_α. Hence

$$\Psi_\Lambda(\bar{a}\mu) \in M_c(G|\Lambda^\perp), \tag{3c.3}$$

where $M_c(G|\Lambda^\perp) \subset M(G|\Lambda^\perp)$ is the space of all continuous measures and Ψ_Λ is the canonical map. Notice that

$$\nu = \lim(\bar{\lambda}_\alpha \cdot \mu) = \lim(\overline{\lambda_\alpha - a + a})\,\mu.$$

Observe that $(\overline{\lambda_\alpha - a})\,\Psi_\Lambda(\bar{a}\mu) \to \Psi_\Lambda(\nu)$ weak-* in $M(G|\Lambda^\perp)$ since $(\overline{\lambda_\alpha - a}) \in \Lambda$. It follows via (3c.3) that $(\overline{\lambda_\alpha - a})\,\Psi_\Lambda(\bar{a}\mu) \in M_c(G|\Lambda^\perp)$ and so

$$\inf\{|\hat{\nu}(\lambda)| : \lambda \in \Lambda\} = 0.$$

This contradicts $\Lambda \subset \mathscr{B}(\nu)$.

Thus we have confirmed that $\{\gamma_\alpha\}$ eventually leaves $\Lambda + a$ for every $a \in \Gamma$. We show that this implies the existence of a set

$$\{m_0\} \cup \{m_{ks}\}, \qquad 1 \leqslant k \leqslant r^2,\ 1 \leqslant s \leqslant r,$$

satisfying condition (2) of Theorem A with respect to $\mathscr{P}^+ \cap \mathscr{B}(\mu)$ and such that

$$|\hat{\mu}(\gamma)| \leqslant e^{-r}$$

for $\gamma \in P_{r^2} \backslash \mathscr{B}(\mu)$.

Choose any $m_0 \in \mathscr{B}(\mu) \cap \mathscr{P}^+$. Put $P_0 = \{m_0\}$. Let $1 \leqslant k \leqslant r^2$ and suppose P_{k-1} has been chosen. We inductively choose $\{m_{ks}\}$ in a way such that

$$\phi(m_{ks}) > \phi(m_{kt}) > 0, \qquad 1 \leqslant s < t < r; \tag{3c.4}$$

$$m_{ks} \in \{\gamma_\alpha\} \backslash (P_{k-1} - \mathscr{B}(\nu)), \qquad 1 \leqslant s \leqslant r; \tag{3c.5}$$

$$(P_{k-1} + m_{ks} - m_{kt}) \subset \mathscr{S}(\mu), \qquad 1 \leqslant s < t \leqslant r. \tag{3c.6}$$

We gather that the set $(P_{k-1} - \mathscr{B}(\nu))$ is a finite union of cosets of Λ, so that $\{\gamma_\alpha\}$ eventually leaves it. Thus we may choose m_{kr} consistent with (3c.5). Suppose for $1 \leqslant j < r$ we have selected m_{ki} consistent with (3c.4), (3c.5) and (3c.6) where $j < i \leqslant r$. Choose m_{kj} satisfying (3c.4) and (3c.5) such that

$$|(\bar{m}_{kj}\mu)^\wedge - \hat{\nu}| < 1 - e^{-r} \qquad \text{on} \qquad \bigcup_{i>j} (P_{k-1} - m_{ki}). \tag{3c.7}$$

Let $\gamma = p + m_{kj} - m_{ki}$, and $p \in P_{k-1}$. Then, for $j < i \leqslant r$,

$$|\hat{\mu}(\gamma)| = |(\bar{m}_{kj}\mu)^\wedge\,(p - m_{ki})|,$$

so by (3c.7)

$$|\hat{\mu}(\gamma)| < 1 - e^{-r} + |\hat{\nu}(p - m_{ki})|. \tag{3c.8}$$

Inasmuch as $p - m_{ki} \in \mathcal{S}(\nu)$ we gather from (3c.8) that

$$|\hat{\mu}(\gamma)| < 1. \tag{3c.9}$$

Thus $\gamma = p + m_{kj} - m_{ki} \in \mathcal{P}^+ \cap \mathcal{S}(\mu)$ and so Theorem B of Section 1 implies that $\|\mu\| \geqslant r^{1/2}/4$. This contradiction shows that $\phi(\mathcal{B} \cap \mathcal{P}^+)$ is bounded above. The proof is complete.

Let S be a Sidon set in Γ. Then by Drury's result [8] there is a measure $\sigma \in M(G)$ satisfying

(i) $\hat{\sigma}(S) = 0$;

(ii) $1 \leqslant |\hat{\sigma}(\Gamma \backslash S)| < 2$;

(iii) the norm of σ depends only on the Sidon constant of S.

For $\mu \in M(G)$ put $\mathcal{S}'(\mu) = \{\gamma \in \Gamma : |\hat{\mu}(\gamma)| < \frac{1}{2}e^{-r}\}$.

COROLLARY 3c.1. *Let $\mu \in M(G)$ with μ ϕ-continuous and S a Sidon set. Suppose $\mathcal{P}^+ \backslash S \subset \mathcal{B} \cup \mathcal{S}'$, Then $(\mathcal{B} \backslash S) \cap \mathcal{P}^+$ is contained in a finite number (depending only on the Sidon constant of S and $\|\mu\|$) of cosets of* ker ϕ.

Proof. Consider the measure σ defined by (i), (ii) and (iii). Since $(\sigma * \mu)_K = \sigma_K * \mu_K$ we see that $\sigma * \mu$ is ϕ-continuous since μ is. We apply Theorem 3c to the measure $\sigma * \mu$ to conclude the proof.

COROLLARY 3c.2. *Let $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_n$, where the Γ_i are subgroups of $\mathbb{R}$. Suppose Γ is lexicographically ordered from the left. Let $\mu \in M(G)$ with $\mathcal{P}^+ \subset \mathcal{B} \cup \mathcal{S}$. Put $\hat{G}_i = \{0\} \oplus \cdots \oplus \{0\} \oplus \Gamma_i \oplus \cdots \oplus \Gamma_n$ $(1 < i \leqslant n)$* and suppose *$H^\perp \not\subset \hat{G}_i \Rightarrow \hat{\mu}_H(\mathcal{P}^+) = 0$. Then there exists $\gamma_1, \ldots, \gamma_N$ such that*

$$\mathcal{P}^+ \cap \mathcal{B} \subset \bigcup_{j=1}^{N} \gamma_j + \hat{G}_i \qquad (\gamma_j \geqslant 0).$$

Proof. Repeated application of Theorem 3c shows that $\mathcal{B} \cap \mathcal{P}^+$ is contained in a finite number of cosets of $\hat{G}_i$. Theorems A and B of Section 1 now give the full result.

Corollary 3c.2 will be of use to us in the next section when we prove the semi-idempotent theorem for $\mathbb{Z}^k$.

THEOREM 3d. *Suppose Γ is ϕ-ordered and $\mu \in M(G)$ is semi-strongly continuous. If $\mathcal{P}^+ \subset \mathcal{B} \cup \mathcal{S}$ then* card$(\mathcal{B} \cap \mathcal{P}^+)$ *is finite.*

Proof. Let μ be semi-strongly continuous. Then Corollary 3 shows that μ is ϕ-continuous. Thus Theorem 3c gives

$$\mathscr{B} \cap \mathscr{P}^+ \subset \bigcup_{i=1}^{N} \gamma_i + \ker \phi \qquad (\gamma_i > 0), \tag{3d.1}$$

for some $\gamma_1,..., \gamma_N \in \Gamma$.

Put $H = (\ker \phi)^\perp$ and consider $\rho_i = \Psi_H(\bar{\gamma}_i\mu)$, $i = 1, 2,..., N$. For every closed subgroup G_0 of $G|H$ of infinite index we have

$$(\rho_i)\hat{}_{G_0}(\gamma) = 0 \qquad \text{for all} \quad \gamma \in \ker \phi. \tag{3d.2}$$

In light of (3d.1) and (3d.2) the Ramsey–Wells Theorem applies to give that the cardinality of $\mathscr{B} \cap \mathscr{P}^+$ is finite. A routine appeal to Theorems A and B of Section 1 establishes that

$$\text{card}\{\mathscr{B} \cap \mathscr{P}^+\} < N$$

if Γ is torsion free. This concludes the proof.

COROLLARY 3d. *Suppose Γ is ϕ-ordered and $\mu \in M(G)$ is semi-strongly continuous. Let S be a Sidon subset of Γ such that $\mathscr{P}^+ \backslash S \subset \mathscr{B} \cup \mathscr{S}'$. Then* $\text{card}\{(\mathscr{B} \backslash S) \cap \mathscr{P}^+\}$ *is finite.*

THEOREM 3e. *Suppose Γ is fully ordered and $\mu \in M(G)$ is semi-strongly continuous. If $\mathscr{P}^+ \subset \mathscr{B} \cup \mathscr{S}$ then* $\text{card}\{\mathscr{B} \cap \mathscr{P}^+\} < N$.

Proof. Suppose $\Gamma = \mathbb{Z}^k$ for some $k \in \mathbb{N}$. Then by [7, p. 104]

$$\mathbb{Z}^k \cong \mathbb{Z}^{k_1} \oplus \cdots \oplus \mathbb{Z}^{k_m},$$

where each $\mathbb{Z}^{k_i}$ is Archimedean ordered and the ordering on $\mathbb{Z}^k$ is lexicographic from left to right. The proof of the present theorem is by induction on the number of summands, m.

If $m = 1$ then the order on $\mathbb{Z}^k$ is Archimedean. By Theorem 3c

$$\text{card}\{\mathscr{B} \cap \mathscr{P}^+\} < N.$$

So, suppose $m \neq 1$. Assume the result is true whenever the number of summands is less than m. Let ϕ be the natural projection such that

$$\phi: \mathbb{Z}^k \to \mathbb{Z}^{k_1}.$$

Put $\mathscr{P}_\phi^+ = \{\gamma \in \mathbb{Z}^k : \phi(\gamma) > 0\}$. Since $\mathscr{P}_\phi^+ \subset \mathscr{B} \cup \mathscr{S}$, Theorem 3d gives

$$\text{card}\{\mathscr{B} \cap \mathscr{P}_\phi^+\} \text{ is finite.} \tag{3e.1}$$

Restrict $\hat{\mu}$ to the group $\{0\} \oplus \mathbb{Z}^{k_2} \oplus \cdots \oplus \mathbb{Z}^{k_m} = \mathbb{Z}_2$.

$$\{\Psi_{\mathbb{Z}_2^\perp}(\mu)\}^\wedge = \hat{\mu}|_{\mathbb{Z}_2}. \tag{3e.2}$$

Since $\Psi_{\mathbb{Z}_2^\perp}(\mu)$ is a semi-strongly continuous measure belonging to $M(\mathbb{Z}^k/\mathbb{Z}_2^\perp)$ we may apply the inductive assumption to conclude via (3e.1) and (3e.2) that

$$\operatorname{card}\{\mathscr{B} \cap \mathscr{P}^+\} \text{ is finite.}$$

Appeal to Theorems A and B of Section 1 yields $\operatorname{card}\{\mathscr{B} \cap \mathscr{P}^+\} < N$ and this concludes the proof for $\mathbb{Z}^k$.

Now suppose Γ is fully ordered. Suppose μ satisfies $\mathscr{P}^+ \subset \mathscr{B} \cup \mathscr{S}$ and $\operatorname{card}\{\mathscr{B} \cap \mathscr{P}^+\} \geqslant N$. Pick N distinct elements in $\mathscr{B} \cap \mathscr{P}^+$ and consider the subgroup $\mathbb{Z}^k$ generated by these characters. Put $(\mathbb{Z}^k)^\perp = G_0$. Clearly,

$$\|\Psi_{G_0}(\mu)\| \leqslant \|\mu\|$$

and $\Psi_{G_0}(\mu)$ is semi-strongly continuous with respect to the induced ordering on $\mathbb{Z}^k$. Put $\mathscr{B}^1 = \mathscr{B}(\Psi_{G_0}(\mu))$. Then by our result for $\mathbb{Z}^k$ we have $\operatorname{card}(\mathscr{P}^+ \cap \mathscr{B}^1) < N$. This contradicts $\operatorname{card}(\mathscr{P}^+ \cap \mathscr{B}^1) \geqslant N$. Our proof is complete.

COROLLARY 3e. *Suppose Γ is fully ordered and $\mu \in M(G)$ is semi-strongly continuous. Let S be a Sidon subset of Γ such that $\mathscr{P}^+ \backslash S \subset \mathscr{B} \cup \mathscr{S}'$. Then* $\operatorname{card}\{\mathscr{B}\backslash S) \cap \mathscr{P}^+\}$ *is finite and depends only on the Sidon constant of S and $\|\mu\|$.*

4. SEMI-IDEMPOTENTS AND ϕ-CONTINUOUS MEASURES

In this section we exhibit a connection between semi-idempotents on ϕ-ordered groups and ϕ-continuous measures. We first re-prove the semi-idempotent theorem of Section 2 for ϕ-ordered groups since the technique involved may be of some interest. The section concludes with a proof of the semi-idempotent theorem for fully ordered groups. As a special case of our semi-idempotent theorem we obtain the result announced by Kessler in [6].

THEOREM 4a. *Let Γ be ϕ-ordered and suppose $\mu \in F(N_1, ..., N_j; \mathscr{P}^+ \backslash S)$ with S a Sidon set in Γ. Then there exists $\nu \in M(G)$ such that $\nu \in F(N_1, ..., N_j; \Gamma)$ satisfying*

(a) $\hat{\nu} = \hat{\mu}$ on $\mathscr{P}^+ \backslash S$;

(b) $\|\nu\|$ *is bounded by a constant depending only on* $\|\mu\|$ *if S is empty*;

(c) *If ϕ is an isomorphism, or Γ is torsion free, then $\|\nu\|$ is bounded by a constant depending only on $\|\mu\|$ and the Sidon constant of S.*

Proof. Let μ satisfy the hypothesis of the present theorem. For simplicity assume $S = \phi$. Then

$$\prod_{i=1}^{j} (\mu - \delta_i) \in M_\phi(G), \tag{4a.1}$$

where $\delta_i = N_i \delta_0$. Thus, we gather from (4a.1) and Corollary 2 of Section 3 that

$$\prod_{i=1}^{j} (\mu_K - \delta_i) = 0$$

for all $K \in \mathscr{K}$.

Suppose $\exists K_1 \in \mathscr{K}$ such that $\mu_1 = \mu_{K_1} \neq 0$. Inasmuch as

$$\|\mu - \mu_1\| \leqslant \|\mu\| - 1, \tag{4a.2}$$

and

$$\prod_{i=1}^{m} (\mu - \mu_1 - \rho_i) \in M_\phi(G),$$

where $\rho_i = M_i \delta_0, M_i \in \mathbb{Z}$, we can repeat the argument for $\prod_{i=1}^{m} (\mu - \mu_1 - \rho_i)$.

As a consequence of (4a.2) this finite descent argument ends in a number of steps $\leqslant \|\mu\|$ with

$$\mu = \mu_1 + \cdots + \mu_n + \nu, \tag{4a.3}$$

where ν is ϕ-continuous and each $\hat{\mu}_i$ is integer-valued. Applying the main result of the previous section to ν we gather that for some γ_i, $i = 1, 2,..., N$,

$$\bigcup_{i=1}^{N} (\gamma_i + \ker \phi) \supset \{\gamma \in \mathscr{P}^+ : |\hat{\nu}(\gamma)| \neq 0\} \qquad (\gamma_i > 0). \tag{4a.4}$$

Here N has the same relation to $\|\mu\|$ as in the main result of the preceding section.

It now follows from (4a.4) that we may interpolate $\hat{\nu}$ on $\mathscr{P}^+$ by the sum of the restrictions of $\hat{\nu}$ to the cosets $\gamma_i + \ker \phi$ in (4a.4). Composing the integer-valued transform which interpolates $\hat{\mu}$ on $\mathscr{P}^+$ with the appropriate polynomial now proves the theorem if $S = \phi$. If $S \neq \phi$, we use Corollary 3c.1 to obtain the full theorem.

In order to prove the semi-idempotent theorem for fully ordered groups we shall need the following two propositions.

PROPOSITION 4b. *Let Γ be fully ordered and suppose $\mu \in F(N_1,..., N_j; \mathscr{P}^+ \backslash S)$ with S a Sidon set in Γ. Then for every $H \in \mathscr{H}, \mu_H \in F(N_1,..., N_j; \mathscr{P}^+)$.*

Proof. Suppose $\mu \in F(N_1,..., N_j; \mathscr{P}^+ \backslash S)$ and $H \in \mathscr{H}$. By Drury's result there exists and $\xi^\varepsilon \in M(G)$ such that

$$(\hat{\xi}^\varepsilon)(S) = 0 \qquad (0 < \varepsilon < 1) \tag{4b.1}$$

and

$$\hat{\xi}^\varepsilon(\Gamma \backslash S) \subset (1-\varepsilon, 1+\varepsilon). \tag{4b.2}$$

Recall that for any $\gamma \in \Gamma$, $\gamma \xi_H^\varepsilon$ restricted to $H^\perp$ is an almost periodic function. As a consequence of [8, p. 48] and (4b.2),

$$\hat{\xi}_H^\varepsilon(\Gamma) \subset (1-\varepsilon, 1+\varepsilon). \tag{4b.3}$$

By Theorem E we also know that

$$(\hat{\mu}_H \cdot \hat{\xi}_H^\varepsilon)(\mathscr{P}^+) \subset (\hat{\mu} \cdot \hat{\xi}^\varepsilon)(\mathscr{P}^+)^-.$$

Thus, if $\gamma_0 \in \mathscr{P}^+$ and $\hat{\mu}_H(\gamma_0) \neq 0$ we gather that

$$\hat{\mu}_H(\gamma_0) \cdot (1-\varepsilon, 1+\varepsilon) \subset \bigcup_i N_i(1-\varepsilon, 1+\varepsilon) \qquad (N_i \neq 0). \tag{4b.4}$$

Let $\varepsilon \to 0$. We gather from (4b.4) that $\hat{\mu}_H(\gamma_0) \in \{N_1,..., N_j\} \backslash \{0\}$. This concludes the proof.

PROPOSITION 4c. *Let $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_n$, where Γ_i $(i = 1, 2,..., n)$ is any subgroup of $\mathbb{R}$. Suppose Γ is lexicographically ordered from the left. If $\mu \in F(N_1,..., N_j; \mathscr{P}^+ \backslash S)$ and $H^\perp \not\subseteq \{0\} \oplus \cdots \oplus \{0\} \oplus \Gamma_j \oplus \Gamma_{j+1} \oplus \cdots \oplus \Gamma_n$ for some $1 < j \leqslant n$, then*

$$\hat{\mu}_H \text{ is integer-valued on } \{0\} \oplus \cdots \oplus \{0\} \oplus \Gamma_{j-1} \oplus \Gamma_j \oplus \cdots \oplus \Gamma_n.$$

Proof. From Proposition 4b we know that $\hat{\mu}_H$ restricted to $\mathscr{P}^+$ is integer-valued. Fix $\gamma \in \Gamma$ and consider $\gamma \mu_H$. Our result now follows from the almost periodicity of the function $\hat{\mu}_H(\beta - \gamma)$, $\beta \in H^\perp$.

We now prove the semi-idempotent theorem for fully ordered groups. The proof uses the result of the previous section on semi-strongly continuous measures.

THEOREM 4d. *Let Γ be fully ordered and suppose $\mu \in F(N_1,..., N_j: \mathscr{P}^+ \backslash S)$ with S a Sidon set in Γ. Then there exists $\nu \in M(G)$ such that $\nu \in F(N_1,..., N_j; \Gamma)$ satisfying*

(a) $\hat{\nu} = \hat{\mu} \qquad \mathscr{P}^+ \backslash S$,

(b) $\|\nu\|$ *is bounded by a constant*

depending only on $\|\mu\|$ *and the Sidon constant of* S.

Proof. We first prove our theorem for $\mathbb{Z}^k$. The general case is obtained by a transfinite induction argument which was suggested to the authors by a reading of [6].

Assume

$$\mathbb{Z}^k \cong \Gamma_1 \oplus \cdots \oplus \Gamma_n,$$

where the Γ_i are finitely generated subgroups of $\mathbb{R}$ and the order is lexicographic (from left to right). We know all full orders on $\mathbb{Z}^k$ are obtained this way; see [7, p. 104].

Let $\mu \in M(\mathbb{T}^k)$ such that $\mu \in F(N_1,..., N_j; \mathscr{P}^+ \backslash S)$. We suppose that $n > 1$ or else we are back in the Archimedean ordered case. Put

$$\mathbb{Z}_t = \{0\} \oplus \cdots \oplus \{0\} \oplus \Gamma_t \oplus \Gamma_{t+1} \oplus \cdots \oplus \Gamma_n,$$

where $1 < t \leqslant n$. Denote by $\mathscr{K}_{t-1}$ the family of subgroups K of $\mathbb{T}^k$ satisfying

$$K^\perp \nsubseteq \mathbb{Z}_t. \tag{4d.1}$$

It follows from Proposition 4c that

$$\mu_K \in F(N_1,..., N_j, \Gamma)$$

if $K \in \mathscr{K}_1$. Suppose there exists a $K_1 \in \mathscr{K}_1$ such that $\mu_{K_1} \neq 0$. Put $\mu_{K_1} = \rho_1$ and notice that

$$\|\mu - \rho_1\| \leqslant \|\mu\| - 1. \tag{4d.2}$$

Since $(\mu - \rho_1)\hat{}$ is integer-valued off $-\mathscr{P} \cup S$ we again apply argument to $\mu - \rho_1$ being careful to pick only subgroups which belonging to $\mathscr{K}_1$. This argument ends in a finite number of steps with the result that

$$\mu = \rho_1 + \rho_2 + \cdots + \rho_r + \eta, \tag{4d.3}$$

where each $\hat{\rho}_i$ is integer-valued and η is ϕ_1-continuous. Here

$$\phi_1 : \mathbb{Z}^k \to \Gamma_1$$

is the natural projection of $\mathbb{Z}^k$ into the "ordering coordinate." Put $\mu_1 = \rho_1 + \cdots + \rho_r$. Notice in (4d.3) that $\hat{\eta}$ is integer valued off $-\mathscr{P} \cup S$. By

Proposition 4c, $\hat{\eta}_K$ is integer-valued on $\mathbb{Z}_2$ for all $K \in \mathscr{K}_2$. Suppose $\hat{\eta}_{K_2}(\gamma_0) \neq 0$ for some $\gamma_0 \in \mathscr{P}^+$ and $K_2 \in \mathscr{K}_2$. Put $\eta_{K_2} = \eta_1$. Then

$$\|\eta - \eta_1\| \leqslant \|\eta\| - 1. \tag{4d.4}$$

Since $(\eta - \eta_1)\hat{}$ is integer-valued off $-\mathscr{P} \cup S$ we again apply argument to $\eta - \eta_1$ being careful to pick only subgroups in $\mathscr{K}_2$. This argument ends in a finite number of steps with the result that

$$\eta = \eta_1 + \eta_2 + \cdots + \eta_m + \xi, \tag{4d.5}$$

each $\hat{\eta}_i$ integer-valued on $\mathbb{Z}_2$. Put $\sum_{i=1}^m \eta_i = \mu_2$. Then by the main result of the previous section we know there are β_i, $i = 1, 2, \ldots, N$, such that

$$\mathscr{B}(\mu_2) \cap \mathscr{P} \subset \bigcup_{i=1}^N \beta_i + \mathbb{Z}_2, \tag{4d.6}$$

since $(\mu_2)_K = 0$ all $K \in \mathscr{K}_1$. Moreover, $\hat{\mu}_2$ is integer-valued on $\mathbb{Z}_2$. We gather that there exists a $\nu_2 \in M(\mathbb{T}^k)$ such that $\hat{\nu}_2 = \hat{\mu}_2$ on $\mathscr{P}^+$ satisfying

(i) $\hat{\nu}_2$ is integer-valued on $\mathbb{Z}^k$;

(ii) $\|\nu_2\| \leqslant N\|\mu_2\|$.

In (4d.5) recall that $\hat{\xi}_K(\mathscr{P}^+) = 0$ for all $K \in \mathscr{K}_2$ and that $\hat{\xi}$ is integer-valued off $-\mathscr{P} \cup S$. Suppose there exists a $K_3 \in \mathscr{K}_3$ such that $\hat{\xi}_{K_3}(\gamma_0) \neq 0$ for some $\gamma_0 \in \mathscr{P}^+$. Put $\xi_{K_3} = \xi_1$ and notice that

$$\|\xi - \xi_1\| \leqslant \|\xi\| - 1. \tag{4d.7}$$

Since $(\xi - \xi_1)\hat{}$ is integer-valued off $-\mathscr{P} \cup S$ we apply argument to $\xi - \xi_1$ being careful to pick only subgroups in $\mathscr{K}_3$. The argument ends in a finite number of steps with the result that

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_l + \rho, \tag{4d.8}$$

where each $\hat{\xi}_i$ is integer-valued on $\mathbb{Z}_3$. Put $\sum_1^l \xi_i = \mu_3$. Corollary 3c.2 of the previous section now shows that for some γ_i, $i = 1, 2, \ldots, N$,

$$\mathscr{B}(\mu_3) \cap \mathscr{P} \subset \bigcup_{i=1}^N \gamma_i + \mathbb{Z}_3, \tag{4d.9}$$

because $(\mu_3)\hat{}_K(\mathscr{P}^+) = 0$ $\forall\, K \in \mathscr{K}_2$. Thus we interpolate $\hat{\mu}_3$ on $\mathscr{P}^+$ by an integer-valued transform $\hat{\nu}_3$ such that

$$\|\nu_3\| \leqslant N\|\mu_3\| \leqslant N\|\mu\|.$$

This finite descent argument ends by showing that

$$\mu = \mu_1 + \mu_2 + \cdots + \mu_s + \tau,$$

with $\hat{\nu}_i = \hat{\mu}_i$ on $\mathscr{P}^+$ $(i = 2, 3, \ldots, s)$, $s \leqslant \|\mu\|$. It is also clear that τ is semi-strongly continuous. By the corollary to Theorem 3e, we see that $\hat{\tau}$ coincides with the transform of a trigonometric polynomial $\hat{t}$ on $\mathscr{P}^+ \backslash S$. The norm of this trigonometric polynomial depends only on the Sidon constant of S and $\|\mu\|$. Put $\mu_1 = \nu_1$. Then for the ν of our theorem take $\sum_{i=1}^{S} \nu_i + t$ composed with an appropriate polynomial. Since $\|\nu_i\| \leqslant N\|\mu\|$ $(i = 1, 2, \ldots, s)$, $\sum_{i=1}^{S} \|\nu_i\| \leqslant N\|\mu\|^2$. This proves the theorem for $\Gamma = \mathbb{Z}^k$.

Next, let Γ be countable with

$$\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i,$$

where $\Gamma_{i+1} \supset \Gamma_i$ and each Γ_i is finitely generated. Let $\mu \in M(G)$ with $\mu \in F(N_1, \ldots, N_j; \mathscr{P}^+ \cap S)$. Then if we restrict $\hat{\mu}$ to Γ_i by defining

$$\hat{\omega}_i(\gamma) = \hat{\mu}(\gamma), \qquad \gamma \in \Gamma_i,$$

there is a $\sigma_i \in M(G/\Gamma_i^{\perp})$ such that

$$\hat{\sigma}_i(\gamma) = \hat{\omega}_i(\gamma), \qquad \gamma \in \Gamma_i \backslash (-\mathscr{P} \cup S),$$

and

$$\sigma_i \in F(N_1, \ldots, N_j; \Gamma_i).$$

Put

$$\zeta_i(\gamma) = \hat{\sigma}_i(\gamma), \qquad \gamma \in \Gamma_i,$$

and

$$\zeta_i(\gamma) = 0 \qquad \text{if} \quad \gamma \in \Gamma \backslash \Gamma_i.$$

We need to show $\|\zeta_i\| = \|\sigma_i\|$. Consider the continuous linear functional on $C(G)$ defined by

$$T(f) = \int_{G/\Gamma_i^{\perp}} \int_G \overline{f(xy)}\, dm_G(x)\, d\sigma_i(y),$$

where m_G is the Haar measure on G. $\|T\| = \|\sigma_i\|$. For $\gamma \in \Gamma_i$,

$$T(\gamma) = \int_{G/\Gamma_i^\perp} \int_G \bar{\gamma}(xy)\, dm_G(x)\, d\sigma_i(y)$$

$$= \int_{G/\Gamma_i^\perp} \int_G \bar{\gamma}(y)\, dm_G(x)\, d\sigma_i(y)$$

$$= \int_{G/\Gamma_i^\perp} \bar{\gamma}(y)\ \ d\sigma_i(y) = \hat{\sigma}_i(\gamma).$$

Now, for $\gamma \in \Gamma \backslash \Gamma_i$,

$$\int_{G/\Gamma_i^\perp} \int_G \bar{\gamma}(xy)\, dm_G(x)\, d\sigma_i(y) = 0.$$

Thus ζ_i is the measure corresponding to T by the Riesz Representation Theorem.

Inasmuch as $\|\sigma_i\|$ depends only on $\|\mu\|$ and the Sidon constant of S, if follows that any weak$-*$ cluster point ν of $\langle \sigma_i \rangle$ interpolates $\hat{\mu}$ off $-\mathscr{P} \cup S$ with the required bound on $\|\nu\|$.

The proof for the general case is obtained by a transfinite induction argument on the cardinality of Γ. Let y_0 be the smallest ordinal number such that the set Y of predecessor's of y_0 has the cardinality of Γ. Let γ_y denote the 1–1 correspondence between Y and $\Gamma \backslash \{0\}$.

For each $y \in Y$ put Γ_y equal to the group generated by $\{\gamma_x : x \leqslant y\}$. Then $\Gamma = \bigcup_{y \in Y} \Gamma_y$ and if $y_1 < y_2$ then $\Gamma_{y_1} \subset \Gamma_{y_2}$. Note also that $\operatorname{card} \Gamma_y < \operatorname{card} \Gamma$ for all $y \in Y$ and $\operatorname{card} \Gamma_y$ is infinite.

Suppose $\mu \in M(G)$ and $\mu \in F(N_1, \ldots, N_j; \mathscr{P}^+ \backslash S)$. Then if we restrict $\hat{\mu}$ to Γ_y we can find a net $\lambda_y \in M(G)$ with $\hat{\lambda}_y = \hat{\mu}$ for all $\gamma \in \Gamma_y \backslash (\mathscr{P}^+ \backslash S)$ satisfying

(a) $\hat{\lambda}_y = 0$ off Γ_y;

(b) $\lambda_y \in F(N_1, \ldots, N_j; \Gamma)$;

(c) $\|\lambda_y\| \leqslant M$.

Thus any weak$-*$ cluster point of the net λ_y is the required ν which interpolates $\hat{\mu}$ off$-\mathscr{P} \cup S$. This concludes our proof.

For some related work the reader is referred to [11].

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