Stochastic Processes and their Applications 9 (1979) 137–145

LOCAL MAXIMA OF THE SAMPLE FUNCTIONS OF THE *N*-PARAMETER BESSEL PROCESS

Madan L. PURI* and Lanh Tat TRAN**

University of Washington, Seattle, U.S.A., and Indiana University, Bloomington, U.S.A.

Received 25 August 1978
Revised 14 March 1979

In this paper we show that almost every sample function of the N-parameter Bessel process associated with the N-parameter Wiener process has a local maximum. In addition some properties related to the local maxima are investigated.

sample functions, Wiener process, Bessel process, local maxima

1. Introduction and preliminaries

Let $W^{(N)}$ be the N-parameter Wiener process, that is a real-valued separable Gaussian process with zero means and covariance $\prod_{i=1}^{N}(s_i \wedge t_i)$ where $s = \langle s_i \rangle$, $t = \langle t_i \rangle$, $s_i \geq 0$, $t_i \geq 0$, $i = 1, \ldots, N$. Then $W^{(N,d)}$ is to be the process with values in the d-dimensional Euclidean space R^d such that each component is an N-parameter Wiener process, the components being independent. Write $W = W^{(N,d)}$ for simplicity, and denote the ith component of W by W^i. Define the N-parameter Bessel process associated with W by

$$B_t = \left[\sum_{i=1}^{d} (W_t^i)^2\right]^{1/2}. \tag{1}$$

It is shown that almost every sample function of B_t has a local maximum. Furthermore, some properties related to the local maxima of B_t are investigated.

As in Orey and Pruitt [5][1], our parameter space is R_+^N, that is the set of $t \in R^N$ with all components nonnegative. When dealing with a point t in the parameter space we sometimes write $t = \langle t_1, \ldots, t_N \rangle$ or simple $\langle t_i \rangle$. In case all $t_i = 0$, we write $t = \langle 0 \rangle$. For $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ with $s_i \leq t_i$, the interval $\mathbf{X}_{i=1}^{N}[s_i, t_i]$ is denoted by $\Delta(s, t)$, and by

* Research supported by the Air Force Office of Scientific Research AFSC, USAF, under Grant No. AFOSR76-2927. Some of this research was done at the University of Washington during my leave of absence from Indiana University. I wish to express my sincere thanks to Professor Donovan Thompson and his colleagues for their kind hospitality at the Department of Biostatistics in the University of Washington, Seattle.

** Research supported by National Science Foundation Grant No. MCS76-00951.

[1] Wherever possible, we shall use the notation of Orey and Pruitt [5].

$\Delta(t)$ in case $s = \langle 0\rangle$. Denote by $S(s, t)$, the symmetric difference of $\Delta(s)$ and $\Delta(t)$. Then it is easy to check that if $s, t \in R_+^N$, the variance of $W^i(t) - W^i(s)$ is $|S(s, t)|$ where $|\cdot|$ denotes the N-dimensional Lebesgue measure. Furthermore, W has continuous sample functions and independent increments. We denote the increment of W over $\Delta(s, t)$ by $W(\Delta(s, t))$. For further information on W, the reader is referred to [1, 4–13].

Local maxima of the sample functions of the two-parameter Wiener process have been studied by Tran [8]. However, due to the complex sample function behavior of the N-parameter Bessel process and the complicated geometrical structures in higher dimensions, Tran's method does not provide a simple generalization to the problem considered in this paper. A much more involved argument is needed here.

Throughout the paper, we will assume that a set of probability zero has been deleted from the probability space so that all sample functions of B_t are continuous.

Definition 1. The sample function $B(\cdot, \omega)$ has a local maximum at s if there exists an open set 0 containing s such that $0 \subset R_+^N$ and $B(t, \omega) \leq B(s, \omega)$ for all $t \in 0$.

We shall need the Orey–Pruitt analogue of the familiar zero-one law. Let $\mathscr{C}_n$ be the class of time intervals in R_+^N with vertices of the form $\langle k_i 2^{-n}\rangle$, k_i nonnegative integers, and having all sides of equal length, and for $n > 0$ each member of $\mathscr{C}_n$ is to be a subcube of one in $\mathscr{C}_0$. Let $\mathscr{C}_\infty = \bigcup_{n=0}^\infty \mathscr{C}_n$, and $\mathscr{F}_n = \mathscr{B}(W(\Delta), \Delta \in \mathscr{C}_n)$, $\mathscr{F}_\infty = \bigvee_{n=0}^\infty \mathscr{F}_n$. Thus $\mathscr{F}_n$ is the Borel field generated by the indicated class of random variables and $\mathscr{F}_\infty$ is the smallest Borel field including all $\mathscr{F}_n$. For a subset D of R_+^N, we put $\mathscr{C}_n(D) = \{\Delta \in \mathscr{C}_n : \Delta \subseteq D\}$, $\mathscr{F}_n(D) = \mathscr{B}\{W(\Delta) : \Delta \in \mathscr{C}_n(D)\}$, $\mathscr{F}_\infty(D) = \bigvee_{n=0}^\infty \mathscr{F}_n(D)$ then we have the following lemma.

Lemma 1 ([5]). *Let $D_m \subseteq R_+^N$, $m = 1, 2, \ldots$, with $D_m \downarrow \emptyset$. If $A \in \mathscr{F}_\infty(D_m)$ for every m, then $P(A) = \{0, 1\}$.*

Lemma 2. *Let ϕ be a nonnegative, nondecreasing, continuous function defined for large arguments. Then for almost all ω there is an $\varepsilon(\omega)$ such that for all intervals $\Delta(s, t)$ with $\Delta(s, t) \subset \Delta(\langle 1\rangle)$ and $|\Delta(s, t)| < \varepsilon(\omega)$,*

$$|W(\Delta(s, t))| < |\Delta(s, t)|^{1/2} \phi(|\Delta(s, t)|^{-1})$$

if and only if

$$\int_0^\infty (\log \xi)^{3N + d/2 - 2} \, e^{-\phi^2(\xi)/2} \, d\xi$$

converges.

For the proof of Lemma 2, see [5, p. 147].

2. Local maxima

In this section we prove the main theorem dealing with the existence of the local maxima of the sample functions of the Bessel process B_t.

Theorem 1. *For almost all sample functions of the Bessel process B_t defined in* (1), *there exists a local maximum.*

Proof. Let s be the center of the unit interval U, and let $C_n \subset U$ be a cube with center at s, sides parallel to the coordinate axes and equal to a_n. Let u^n and v^n be the smallest and the largest vertex of C_n, i.e. closest and farthest from the origin $\langle 0 \rangle$. Pick C_n with $\min(u_1^n, \ldots, u_N^n) > \frac{1}{4}$.

Consider two points s^{nk} and v^{nk} of R_+^N determined by $s_k^{nk} = \frac{1}{2}$, $v_k^{nk} = v_n^k$, $s_j^{nk} = v_j^{nk} = u_j^n$ for $j \neq k$ where $1 \leq j \leq N$.

Define

$$A_{ni} = \bigcap_{k=1}^{N} [W^i(s^{nk}) - W^i(u) > 2a_n^{1/2},\ W^i(s^{nk}) - W^i(v^{nk}) > 2a_n^{1/2}],$$

$$B_{ni} = \bigcap_{k=1}^{N} [W^i(s^{nk}) - W^i(u) < -2a_n^{1/2},\ W^i(s^{nk}) - W^i(v^{nk}) < -2a_n^{1/2}],$$

$$C_{ni} = [\inf_{t \in C_n} W_t^i \geq 0], \qquad E_{ni} = [\sup_{t \in C_n} W_t^i \leq 0],$$

$$F_{ni} = [\sup_{s,t \in U} |W^i(\Delta(s,t))| < (2^{N-1}-1)^{-1} a_n^{1/2} : |s_i - t_i| \leq a_n,$$
$$|s_j - t_j| \leq a_n \text{ for some } i, j \text{ with } i \neq j].$$

The variables $W^i(s^{nk}) - W^i(u)$, $W^i(s^{nk}) - W^i(v^{nk})$ are normally distributed with mean 0 and variances greater than $4^{-N+1/2} a_n$. Thus $P(A_{ni}) > \beta$ for some constant β.

Let $\{a_n\}$ be a sequence of positive numbers with $a_n \downarrow 0$ and let D_n be the interior of $S(u^n, v^n)$. Clearly $D_n \downarrow \emptyset$ as $a_n \downarrow \emptyset$. Observe that the event $[A_{ni}$ infinitely often$] \in \mathscr{F}_\infty(D_n)$. Thus, from Lemma 1, it follows that

$$\mathbf{P}[A_{ni} \text{ infinitely often}] = 1.$$

Analogously,

$$\mathbf{P}[B_{ni} \text{ infinitely often}] = 1.$$

Let $\eta > 0$. Then by Lemma 2, or by the continuity of the sample functions of W,

$$\mathbf{P}\left[\bigcap_{n=n_0}^{\infty} C_{n_i} \mid W_s^i > \eta\right] \to 1 \quad \text{as } n_0 \to \infty.$$

Furthermore,

$$\mathbf{P}[A_{ni}C_{ni} \text{ infinitely often}] \geqslant \mathbf{P}\left[(A_{ni} \text{ infinitely often}) \bigcap_{n=n_0}^{\infty} C_{ni}\right]$$

$$\geqslant \mathbf{P}\left[\bigcap_{n=n_0}^{\infty} C_{ni}\right] \geqslant \mathbf{P}\left[\bigcap_{n=n_0}^{\infty} C_{ni}[W_s^i > \eta]\right].$$

Since $\mathbf{P}[W_s^i > \eta]$ converges to $\frac{1}{2}$ as $\eta \to 0$, by picking η small enough and then n_0 large enough, the probability of the last event can be made as close to $\frac{1}{2}$ as desired.

It is now clear that

$$\mathbf{P}[A_{ni}C_{ni} \cup B_{ni}E_{ni}) \text{ i.o.}] = 1.$$

Also, by Lemma 2 and by the independence of the components of W

$$\mathbf{P}\left[\bigcap_{i=1}^{d} (A_{ni}C_{ni}F_{ni} \cup B_{ni}E_{ni}F_{ni}) \text{ i.o.}\right] = 1. \tag{2}$$

We claim that

$$\left[\bigcap_{i=1}^{d} (A_{ni}C_{ni}F_{ni} \cup B_{ni}E_{ni}F_{ni})\right] \subset [\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]. \tag{3}$$

Observe that $[\bigcap_{i=1}^{d} (A_{ni}C_{ni}F_{ni} \cup B_{ni}E_{ni}F_{ni})]$ can be written as the union of 2^d events. Each of these events is the intersection of d sets with the ith set $(1 \leqslant i \leqslant d)$ being either $A_{ni}C_{ni}F_{ni}$ of $B_{ni}E_{ni}F_{ni}$. To complete the proof of (3), it is sufficient to show that every one of these 2^d events is a subset of $[\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]$. Without loss of generality we will show

$$\left[\bigcap_{i=1}^{d} A_{ni}C_{ni}F_{ni}\right] \subset [\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]. \tag{4}$$

A slight variation of the proof along the same lines can be applied to show that the remaining $2^d - 1$ events are also subsets of $[\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]$.

Let $t \in \partial C_n$ where ∂C_n is the boundary of C_n, and let

$$p^t = \langle \sigma(t_1), \ldots, \sigma(t_N) \rangle$$

where

$$\sigma(t_i) = \begin{cases} t_i, & \text{if } u_i^n < t_i < v_i^n, \\ s_i, & \text{otherwise.} \end{cases}$$

Observe that p^t lies in the interior of C_n.

Let

$$H_n = \bigcap_{t \in \partial C_n} \left\{ \bigcap_{i=1}^{d} [W^i(p^t) - W^i(t) > a_n^{1/2}] \right\}.$$

We will complete the proof of (4) by showing

$$\left[\bigcap_{i=1}^{d} A_{ni}C_{ni}F_{ni}\right] \subset \left[H_n \bigcap_{i=1}^{d} C_{ni}\right], \tag{5}$$

$$\left[H_n \bigcap_{i=1}^{d} C_{ni}\right] \subset [\sup_{t\in C_n^0} B_t > \sup_{t\in\partial C_n} B_t]. \tag{6}$$

Observe that for any $1 \leq i \leq d$,

$$\begin{aligned} W^i(p^t) - W^i(t) &= W^i(\langle\sigma(t_1), t_2, \ldots, t_N\rangle) - W^i(\langle t_1, t_2, \ldots, t_N\rangle) \\ &+ W^i(\langle\sigma(t_1), \sigma(t_2), t_3, \ldots, t_N\rangle) - W^i(\langle t_1), t_2, \ldots, t_N\rangle) \\ &+ \cdots + W^i(\langle\sigma(t_1), \sigma(t_2), \ldots, \sigma(t_n)\rangle) - W^i(\langle\sigma(t_1), \ldots, \sigma(t_{N-1}), t_N\rangle). \end{aligned} \tag{7}$$

Consider the random variable

$$\begin{aligned} &W^i(\langle\sigma(t_1), \ldots, \sigma(t_{j-1}), \sigma(t_j), t_{j+1}, \ldots, t_N\rangle) \\ &\quad - W^i(\langle\sigma(t_1), \ldots, \sigma(t_{j-1}), t_j, t_{j+1}, \ldots, t_N\rangle). \end{aligned} \tag{8}$$

The variance of this variable is equal to

$$(u_1^n + e_1^n) \cdots (u_{j-1}^n + e_{j-1}^n)\,|\sigma(t_j) - t_j|\,(u_{j+1}^n + e_{j+1}^n) \cdots (u_N^n + e_N^n) \tag{9}$$

where $0 \leq e_i^n \leq a_n$. Observe that (9) can be expanded into a sum of 2^{N-1} terms. At this point, it is easy to see that (8) is equal to

$$\begin{aligned} &W^i(\langle u_1^n, \ldots, u_{j-1}^n, \sigma(t_j), u_{j+1}^n, \ldots, u_N^n\rangle) \\ &\quad - W^i(\langle u_1^n, \ldots, u_{j-1}^n, t_j, u_{j+1}^n, \ldots, u_N^n\rangle) + L^i \end{aligned}$$

where L^i can be decomposed into no more than $2^{N-1} - 1$ random variables, such that each random variable is the increment of W^i over an interval in U with at least two sides smaller than or equal to a_n.

The random variables in (8) is zero if $\sigma(t_j) = t_j$. Assume $\sigma(t_j) \neq t_j$ and let $\omega \in [\bigcap_{i=1}^{d} A_{ni}C_{ni}F_{ni}]$. Then $\omega \in F_{ni}$ for each i. Hence

$$|L^i(\omega)| < (2^{N-1} - 1)(2^{N-1} - 1)^{-1} a_n^{1/2} = a_n^{1/2}.$$

Now, $\sigma(t_j) = \frac{1}{2}$ and $t_j = u_j^n$ or v_j^n since $\sigma(t_j) \neq t_j$. Therefore

$$\begin{aligned} &W^i(\langle u_1^n, \ldots, u_{j-1}^n, \sigma(t_j), u_{j+1}^n, \ldots, u_N^n\rangle, \omega) \\ &\quad - W^i(\langle u_1^n, \ldots, u_{j-1}^n, t_j, u_{j+1}^n, \ldots, u_N^n\rangle, \omega) \\ &= \begin{cases} W^i(s^{nj}, \omega) - W^i(u, \omega), & \text{if } t_j = u_j^n, \\ W^i(s^{nj}, \omega) - W^i(v^{nj}, \omega), & \text{if } t_j = v_j^n, \end{cases} \end{aligned}$$

which is greater than $2a_n^{1/2}$ since $\omega \in A_{ni}$. We obtain

$$\begin{aligned} &W^i(\langle\sigma(t_1), \ldots, \sigma(t_{j-1}), \sigma(t_j), t_{j+1}, \ldots, t_N\rangle, \omega) \\ &\quad - W^i(\langle\sigma(t_1), \ldots, \sigma(t_{j-1}), t_j, t_{j+1}, \ldots, t_N\rangle, \omega) \\ &\quad > 2a_n^{1/2} - a_n^{1/2} = a_n^{1/2}. \end{aligned} \tag{10}$$

By (7) and (10) we have

$$W^i(p^t, \omega) - W^i(t, \omega) > [\# \text{ of } j\text{'s}, 1 \leqslant j \leqslant N, \text{ with } \sigma(t_j) \neq t_j] a_n^{1/2}.$$

If $t \in \partial C_n$, t_j is equal to u_j^n or v_j^n and $\sigma(t_j) = \frac{1}{2}$ for some $1 \leqslant j \leqslant N$. Therefore $\sigma(t_j) \neq t_j$ for some j and $[\#$ of j's, $1 \leqslant j \leqslant N$, with $\sigma(t_j) \neq t_j] \geqslant 1$. Hence for $\omega \in A_{ni}$, $W^i(p^t, \omega) - W^i(t, \omega) > a_n^{1/2}$ for all $t \in \partial C_n$ and for all i.

We have shown $\omega \in H_n$. Furthermore $\omega \in \bigcap_{i=1}^d C_{ni}$ since $\omega \in [\bigcap_{i=1}^d A_{ni} C_{ni} F_{ni}]$. The proof of (5) is completed.

Let us turn to the proof of (6). We argue by contradiction. Let $\omega \in [H_n \bigcap_{i=1}^d C_{ni}]$ and assume $\omega \notin [\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]$. Then $\sup_{t \in \partial C_n} B(t, \omega) \geqslant \sup_{t \in C_n^0} B(t, \omega)$. Now, $B(t, \omega)$ is a continuous function of t on ∂C_n. Hence for some $l \in \partial C_n$,

$$B(l, \omega) = \sup_{t \in \partial C_n} B(t, \omega) \geqslant \sup_{t \in C_n^0} B(t, \omega) \geqslant B(p^l, \omega)$$

since p^l is an interior point of C_n.

We obtain $(W^i(l, \omega))^2 \geqslant (W^i(p^l, \omega))^2$ for some $1 \leqslant i \leqslant d$ since

$$\left[\sum_{i=1}^d (W^i(l, \omega))^2\right]^{1/2} \geqslant \left[\sum_{i=1}^d (W^i(p^l, \omega))^2\right]^{1/2}.$$

Since $\omega \in [\inf W_t^i \geqslant 0]$, both $W^i(l, \omega)$ and $W^i(p^l, \omega)$ are nonnegative. Therefore $W^i(l, \omega) \geqslant W^i(p^l, \omega)$ for some $1 \leqslant i \leqslant d$, contradicting the fact that $W^i(p^l, \omega) - W^i(l, \omega) > a_n^{1/2}$.

The proof of (6) is completed. From (2) and (3), we conclude that with probability one, the event $[\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]$ occurs infinitely often.

Pick $\omega \in [\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]$. Since $B(t, \omega)$ is a continuous function of t on C_n, there exists an $l \in C_n$ such that $B(l, \omega) = \sup_{t \in C_n} B(t, \omega)$. If $l \in \partial C_n$, then

$$\sup_{t \in \partial C_n} B(t, \omega) = B(l, \omega) = \sup_{t \in C_n} B(t, \omega) \geqslant \sup_{t \in C_n^0} B(t, \omega),$$

contradicting the fact that $\omega \in [\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t]$. Finally, C_n^0 can be chosen to be the open set $0 \subset R_+^N$ mentioned in Definition 1.

The proof is now completed. Recall that s was picked to be the center of U. Actually, s can be chosen to be any point in U^0. Therefore, for almost all sample functions of B_t, the set of local maxima is dense in R_+^N.

We shall now investigate some properties of the local maxima of B_t.

Definition 2. The sample function $B(\cdot, \omega)$ has a strict local maximum at s if there exists an open set 0 containing s such that $0 \subset R_+^N$ and $B(t, \omega) < B(s, \omega)$ for all $t \in 0$.

We have the following theorem.

Theorem 2. *For almost every sample function of* $\{B_t, t \in R_+^N\}$, *all the local maxima are strict and the set of local maxima is countable.*

Proof. Let I and J be two disjoint, closed intervals in the interior of R_+^N. We claim that

$$\mathbf{P}\{\sup_{t\in I} B_t = \sup_{t\in J} B_t\} = 0. \tag{11}$$

Let $I = \Delta(u, v)$, $J = \Delta(s, t)$. Denote the complements of $\Delta(t)$ and $\Delta(v)$ by $[\Delta(t)]'$ and $[\Delta(v)]'$. Since I and J are disjoint intervals, it is clear that

$$[\Delta(u) \cap [\Delta(t)]'] \cup [\Delta(s) \cap [\Delta(v)]']$$

contains a nondegenerate interval, i.e., an interval with a positive N-dimensional Lebesgue measure. Let I' be any such interval, and without loss of generality assume that

$$I' \subset \Delta(u) \cap [\Delta(t)]'.$$

Consider now

$$\mathbf{P}[\sup_{t\in I} B_t = \sup_{t\in J} B_t] = \mathbf{P}\left[\sup_{t\in I}\left[(W_t^1)^2 + \sum_{i=2}^{d} (W_t^i)^2\right]^{1/2} = \sup_{t\in J}\left[\sum_{i=1}^{d} (W_t^i)^2\right]^{1/2}\right]. \tag{12}$$

Let $W^1(I')$ be the increment of W^1 over I'. Since W^1 has independent increments, for $t \in I$, we can write

$$W_t^1 = W_t^1 - W^1(I') + W^1(I')$$

such that $W^1(I')$ is independent of $W_t^1 - W^1(I')$ for all $t \in I$. Also, $W^1(I')$ is independent of $\sup_{t\in J}[\sum_{i=1}^{d} (W_t^i)^2]$ since $I' \subset [\Delta(t)]'$.

Let $X = W^1(I')$, $Y_t = W_t^1 - W^1(I')$. Then the σ-fields generated by X and $(Y_t, W_t^2, \ldots, W_t^d, t \in I, W_t^1, \ldots, W_t^d, t \in J)$ are independent. Let $(\Omega, \mathscr{F}, P)$ be the original probability space and let $(\Omega_1, \mathscr{F}_1, P_1)$ and $(\Omega_2, \mathscr{F}_2, P_2)$ be two identical copies of $(\Omega, \mathscr{F}, P)$. Define

$$S = \left[(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \sup_{t\in I}\left\{(X(\omega_1) + Y_t(\omega_2))^2 + \sum_{i=2}^{d} (W_t^i(\omega_2))^2\right\}^{1/2} = \sup_{t\in J}\left\{\sum_{i=1}^{d} (W_t^i(\omega_2))^2\right\}^{1/2}\right].$$

Let χ_S denote the indicator function of S. Using independence and Fubini's theorem and following an argument similar to that of [3, p. 37], we obtain that (12) is equal to

$$\int_{\Omega_2}\int_{\Omega_1} \chi_S(\omega_1, \omega_2) P_1(d\omega_1) P_2(d\omega_2) =$$

$$= \int_{\Omega_2} P_1\Big[\sup_{t\in I}\Big\{(X+Y_t(\omega_2))^2 + \sum_{i=2}^{d}(W_t^i(\omega_2))^2\Big\}^{1/2}$$

$$= \sup_{t\in J}\Big\{\sum_{i=1}^{d}(W_t^i(\omega_2))^2\Big\}^{1/2}\Big] P_2(d\omega_2).$$

For an arbitrary fixed ω_2, consider

$$P_1\Big[\sup_{t\in I}\Big\{(X+Y_t(\omega_2))^2 + \sum_{i=2}^{d}(W_t^i(\omega_2))^2\Big\}^{1/2} = \sup_{t\in J}\Big\{\sum_{i=1}^{d}(W_t^i(\omega_2))^2\Big\}^{1/2}\Big]. \tag{13}$$

We shall now show that (13) equals zero for a fixed ω_2. Consider the function $f(x)$, defined by

$$f(x) = \sup_{t\in I}\Big\{(x+Y_t(\omega_2))^2 + \sum_{i=2}^{d}(W_t^i(\omega_2))^2\Big\}^{1/2}.$$

Note that $f(x)$ equals the supremum of the distance from the origin of the set D_x in R^d defined by

$$D_x = \{z \in R^d : z_1 = Y_t(\omega_2) + x,\ z_2 = W_t^2(\omega_2), \ldots, z_d = W_t^d(\omega_2) \text{ for some } t \in I\}.$$

It is now easy to see that as x varies from $-\infty$ to $+\infty$, the set D_x is translated along a vector parallel to a coordinate axis and so $f(x)$ decreases and then increases as x goes from $-\infty$ to $+\infty$. For a fixed ω_2, (13) equals

$$P_1\Big[X = f^{-1}\Big(\sup_{t\in J}\sum_{i=1}^{d}(W_t^i(\omega_2))^2\Big)\Big] \tag{14}$$

where f^{-1} is the inverse of f. It is clear that there are at most 2 values of $f^{-1}(\sup_{t\in J}\sum_{i=1}^{d}(W_t^i(\omega_2))^2)$, and since X is normal random variable, (14) equals 0. Thus, for each fixed ω_2, (13) equals zero. The proof of (11) follows by integrating (13) over the probability space.

Consider the set $\bigcap[\sup_{t\in I} B_t \neq \sup_{t\in J} B_t]$ where the intersection is taken over all intervals I and J with rational least and largest vertices, i.e., the coordinates of u, v, s, t are all rational.

We claim that $\bigcap[\sup_{t\in I} B_t \neq \sup_{t\in J} B_t]$ is contained in the set of ω such that all local maxima of $B(\cdot, \omega)$ are strict. Let $\omega \in \bigcap[\sup_{t\in I} B_t \neq \sup_{t\in J} B_t]$ and suppose $B(\cdot, \omega)$ has a local maximum at s. Then there exists an open set $0 \subset R_+^N$ with $s \in 0$ and $B(t, \omega) \leq B(s, \omega)$ for all $t \in 0$. If this local maximum is not strict, then there exists an $l \in 0$ with $l \neq s$ such that $B(l, \omega) = B(s, \omega)$. Let I, J be two disjoint, closed intervals with rational least and largest vertices such that $l \in I \subset 0$ and $s \in J \subset 0$. Then

$$\sup_{t\in I} B(t, \omega) = B(l, \omega) = B(s, \omega) = \sup_{t\in J} B(t, \omega),$$

contradicting the fact that $\omega \in \bigcap[\sup_{t\in I} B_t \neq \sup_{t\in J} B_t]$. Clearly $\mathbf{P}[\bigcap(\sup_{t\in I} B_t \neq \sup_{t\in J} B_t)] = 1$.

Countability of the set of local maxima is a consequence of the following lemma.

Lemma 3. *Let f be a continuous, real valued function of R_+^N with all local maxima strict. Then f has countably many local maxima.*

This lemma is a straightforward generalization of the univariate case, the proof of which can be found in [3, p. 38].

Acknowledgment

The authors would like to express their sincere thanks to Professor J. Gani and the referee for the critical examination of the original draft. Their detailed comments and suggestions for improvements are gratefully acknowledged.

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