Baer sums in homological categories

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Abstract

We give a unified treatment of the Baer sums in the context of efficiently homological categories which, on the one hand, contains any category of groups with multiple operators and more generally any semi-abelian variety and, on the other hand, the category of Hausdorff groups and more generally any category of semi-abelian Hausdorff algebras. This gives rise to a generalized “Euclid’s Postulate” and a five terms exact sequence.

Keywords: Baer sum; Protomodular, additive and homological categories; Topological group; Topological semi-abelian algebra

Introduction

On the one hand, the notion of homological (i.e. pointed protomodular and regular) category is a context in which it is possible to deal, in full generality, with the notion of exact sequence, and it was shown in [2,9] that any homological category \( \mathcal{C} \) satisfies the “passive” homological machinery: namely when a diagram satisfies some homological hypothesis, it satisfies also the homological conclusions; see, for instance, the Noether isomorphisms, the short five, \( 3 \times 3 \) and “snake” lemmas. But this notion is unable in general to produce the “active” part of homology, namely to produce new exact sequences from given ones, as it is the case, for instance, in the category \( \mathcal{Gp} \) of groups with the calculation of the Baer sum of two exact sequences having same abelian kernel and producing the same action on this kernel.
On the other hand, there was described in [7], for any finitely complete Barr exact category $\mathcal{C}$, a very general description of the Baer sum process concerning the objects $X$ having a global support in $\mathcal{C}$ and endowed with an autonomous Mal’cev operation $p : X \times X \times X \to X$: actually, with any object of this kind, there is associated an abelian group object $d(X)$ in $\mathcal{C}$ which defines a “direction” functor $d : \text{Mal}_G\mathcal{C} \to \text{Ab}_\mathcal{C}$, in the same way as with any nonempty $K$-affine space is associated a $K$-vector space. This direction functor $d$ is a cofibration whose any fibre is a groupoid canonically endowed with a closed symmetric tensor product which “is” the Baer sum. Indeed, if you consider a group $G$ and the slice category $\mathcal{C} = Gp/C$ of groups above $C$ (which is a Barr exact category), an object $g : G \to C$ is abelian in $\mathcal{C}$ if and only if it is a group homomorphism with abelian kernel [2]. It has a global support if and only if it is surjective. The “direction” of such an extension $g$ with abelian kernel $A$ is then nothing but the projection $C \ltimes \phi A \to C$ whose domain is the semidirect product given by the classical group action $\phi : C \to \text{Aut} A$ associated with this extension (in other words, its direction is its associated $C$-module). In this context, the symmetric tensor product on such extensions is nothing but the classical Baer sum, and the abelian group structure of the isomorphism classes of such extensions is nothing but $\text{Opext}(C, A, \phi)$ as in [26] for instance.

More generally, when a category $\mathcal{C}$ is protomodular [6] (as this is still the case for $\mathcal{C} = Gp/C$), any object $X$ has at most one Mal’cev operation which is necessarily autonomous; so that the existence of such an operation becomes a property and is no longer a further structure. In those circumstances, the object $X$ in question is said to be abelian in $\mathcal{C}$. Accordingly, if the category $\mathcal{C}$ is both protomodular and Barr exact, the subcategory (actually subgroupoid) of abelian objects with global support and direction $A$ inherits certainly a closed symmetric monoidal structure (= Baer sum), and the set of isomorphism classes of such objects inherits an abelian group structure, which (via Corollary 4 in [7]) is nothing but the first cohomology group $H^1_C(A)$ of $\mathcal{C}$ with coefficients in $A$ (in the sense of [1] for instance).

The first aim of this work, as the previous remark about the category $\mathcal{C} = Gp/C$ makes us hope, is to show that the description, recalled in the second paragraph above, of the abstract Baer sum defined in any Barr exact context can be greatly simplified when the ground category $\mathcal{C}$ in question is moreover protomodular, and can be reduced to a classical scheme concerning the exact sequences. This produces a unified treatment of the Baer theory in the context of any semi-abelian variety [13] and in particular of any category of groups with multiple operators [21]. This is the root of the classically known (but unexplained up to now) parallelism of treatment of homology theory for groups and Lie $R$-algebras (see, for instance, [17,22,25] for some classical illustrations of this parallelism, and [20] for a less classical one). More generally, these processes lead to a generalized “Euclide’s Postulate” and, on the model of the homology of groups or Lie $R$-algebras, to a five terms exact sequence.

Actually this simplified description does hold, with no extra charge, in the wider context of efficiently homological categories (see Definition 1.1) which allows us to enlarge the class of examples dealing with Baer theory, including the categories $\text{AbTop}$, $\text{AbHaus}$, $\text{GpTop}$ and $\text{GpHaus}$ of topological and Hausdorff (abelian) groups and more generally any category of semi-abelian topological and Hausdorff algebras in the sense of [3].

Certainly, the pioneering work in aiming a unified treatment of the Baer sums goes back to Gerstenhaber [18]. But there were no detailed proofs (for instance, on p. 63, of the main fact that $i \lor -j$ is a kernel map), and, more importantly, the context was much more restricted, see [8,27] for a precise comparison of that context with the protomodular one. On the other hand, our present work completes the attempt of [14].
1. Normal subobjects and connectors

Recall that a category $\mathcal{C}$ is protomodular (see [2,6] where the fundamentals on this notion are collected) when it is finitely complete and such that, given any split epimorphism $(g, s)$ and any pullback diagram:

$$
\begin{array}{ccc}
U & \rightarrow & G \\
\downarrow^{\tilde{h}} & & \downarrow^{s} \\
V & \rightarrow & C \\
\downarrow^{h} & & \\
\end{array}
$$

the pair $(s, \tilde{h})$ is jointly strongly epic. When the category $\mathcal{C}$ is pointed (i.e. finitely complete with a zero object [26]), the previous condition can be reduced to the only pullbacks:

$$
\begin{array}{ccc}
\text{Ker } g & \rightarrow & G \\
\downarrow & & \downarrow^{s} \\
1 & \rightarrow & C \\
\downarrow^{\alpha_C} & & \\
\end{array}
$$

Any finitely complete additive category is therefore pointed protomodular. The category $\text{Gp}$ of groups is clearly protomodular as, for any object $x$ in the group $G$, the equality $x = (x.sg(x^{-1})).sg(x)$ shows it immediately, the element $x.sg(x^{-1})$ being in Ker $g$. Among other examples, there are the categories of rings, commutative rings, Lie algebras on a ring $R$, topological groups, Hausdorff groups [2] and $C^*$-algebras [19], the dual of the category $\text{Set}_*$ of pointed sets, and more generally the dual of the category $\mathcal{E}_*$ of pointed objects when $\mathcal{E}$ is a topos. The pointed protomodular varieties are characterized in [13]. More generally, when $\mathcal{E}$ is a finitely complete category, the category $\text{Gp}\mathcal{E}$ of internal groups in $\mathcal{E}$ is pointed protomodular. In particular, any category of presheaves or sheaves of groups on a topological space is pointed protomodular. Let us recall that a protomodular category is necessarily a Mal’cev category [2], i.e. a finitely complete category such that any reflexive relation is an equivalence relation [15,16].

1.1. Connectors

A connector [11] between two equivalence relations $R$ and $T$ on the same object $X$ is a morphism

$$p : R \times_X T \rightarrow X, \quad (xRySz) \mapsto p(x, y, z),$$

which, internally speaking, satisfies the Mal’cev identities: $p(x, y, y) = x$ and $p(y, y, z) = z$. In a protomodular category $\mathcal{C}$, a connector is necessarily unique when it exists, and thus the existence of such a connector becomes a property. We say then that $R$ and $T$ are connected and denote this situation by $[R, T] = 0$, since in any protomodular variety the existence of such a connector is equivalent to saying that the commutator $[R, T]$ in the sense of J.D.H. Smith [28] is trivial. On the classical model, an equivalence relation is said to be abelian when $[R, R] = 0$ and central when $[R, \nabla_X] = 0$, where $\nabla_X$ is the coarse relation on $X$. In that sense, an object
$X$ is abelian in $\mathbb{C}$ if and only if $\nabla_X$ is central. More importantly, in the protomodular context, a connector produces a double equivalence relation whose underlying diagram is the following:

\begin{align*}
R \times_X T & \xrightarrow{p_2} T \\
p_0 \downarrow & \quad \downarrow (p_0, p) \quad \downarrow t_1 \\
R \quad & \xrightarrow{r_0} X
\end{align*}

In the set theoretical context, this would mean that any diagram as on the left-hand side below can be completed into a diagram as on the right-hand side:

\begin{align*}
x & \xrightarrow{R} y \\
R & \xrightarrow{T} x \quad p(x, y, z) \\
y & \xrightarrow{T} z
\end{align*}

On the other hand, let us recall that in a protomodular category there is an intrinsic notion of normal subobject. A map $m : I \to X$ in $\mathbb{C}$ is normal to an equivalence relation $(r_0, r_1) : R \rightrightarrows X$ when $m^{-1}(R)$ is the coarse relation $\nabla_I$ on $I$ and the induced map $\nabla_I \to R$ in the category $\text{Rel} \mathbb{C}$ of equivalence relations in $\mathbb{C}$ is fibrant. This means that any of the following commutative squares is a pullback:

\begin{align*}
I \times I & \xrightarrow{\tilde{m}} R \\
p_0 \downarrow & \quad \downarrow p_1 \quad \downarrow r_0 \quad \downarrow r_1 \\
I & \xrightarrow{m} X
\end{align*}

This implies that the map $m$ is necessarily a monomorphism. This definition gives an intrinsic way to express that $I$ is an equivalence class of $R$. When the category $\mathbb{C}$ is protomodular, the map $m$ is normal to at most one equivalence relation, and consequently the fact to be normal in such a category becomes a property. When, moreover, $\mathbb{C}$ is pointed, there is a bijection between the set of isomorphism classes of equivalence relations on an object $X$ and the set of isomorphism classes of normal subobjects of $X$. Indeed the normal subobject associated with the equivalence relation $R$ (its normalization) is given by the map $i_R = r_1.k_0 : I_R = K \rightrightarrows X$ in the following diagram in which, for sake of simplicity, we set $\text{Ker} r_0 = K$:
The normalization of the kernel equivalence relation of a map \( g : G \to C \):

\[
\begin{array}{ccc}
R[g] & \xrightarrow{g_1} & G \\
\downarrow{g_0} & & \downarrow{g} \\
& C &
\end{array}
\]

is nothing but its kernel map \( K \hookrightarrow G \). Of course, in a pointed protomodular category \( \mathbb{C} \), any normal subobject is not in general a kernel map; this is the case if and only if the equivalence relation \( R \) to which it is normal is effective (i.e. is the kernel equivalence relation of some map). For instance, in the additive category \( \text{AbHaus} \) of Hausdorff abelian groups, any subobject is normal, and it is a kernel if and only if the subgroup in question is endowed with the induced topology.

Let us recall the following result [11, Theorem 5.2] which connects the property \([R, T] = 0\) to a normality condition:

**Proposition 1.1.** Let \( \mathbb{C} \) be a pointed protomodular category, and \( (R, T) \) a pair of equivalence relations on an object \( X \). Then \([R, T] = 0\) if and only if the monomorphism \( i_T : X \hookrightarrow R \) is normal, where \( i_T \) is the normalization of \( T \), and \( s_0 \) denotes (in a simplicial way) the inclusion corresponding to the reflexivity of \( R \). In particular \( R \) is central if and only if \( s_0 : X \hookrightarrow R \) is normal.

We shall suppose from now on \( \mathbb{C} \) is pointed protomodular. We shall be mainly interested in maps \( g : G \to C \) with abelian (respectively central) kernel equivalence relation \( R[g] \). The slice category \( \mathbb{C}/C \) is no more pointed, but still protomodular. Recall that the map \( g \) has an abelian equivalence relation in \( \mathbb{C} \) if and only if, seen as an object in the slice category \( \mathbb{C}/C \), this object \( g \) is abelian, see [2].

The pullback functor along any map being left exact, then certainly the kernel \( \text{Ker} g \) of \( g \) is an abelian object in \( \mathbb{C} \). The fact that the kernel \( \text{Ker} g \) of \( g \) is abelian does not guarantee in general that its kernel relation \( R[g] \) is abelian, see [8] for a counterexample. But according to the previous proposition, the map \( g \) has an abelian (respectively central) kernel equivalence relation if and only if \( \text{Ker} g : G \hookrightarrow R[g] \) (respectively \( G \hookrightarrow R[g] \)) is normal.

**1.2. Homological categories**

A category \( \mathbb{C} \) is homological [2] when it is pointed, protomodular and regular in the sense of [1]. It is exact homological when, moreover, any equivalence relation is effective, or, in other words when any normal monomorphism is a kernel map. In a homological category all the classical homological lemmas (short five, \( 3 \times 3 \) and snake lemmas, Noether isomorphisms) do hold, see [2,9]. An abelian category is exact homological. The category \( \text{AbHaus} \) of Hausdorff abelian groups is homological but not exact (and consequently not abelian) since any subobject is not a kernel. Any example of protomodular category given above is exact homological, except the ones of topological and Hausdorff groups which are only homological, and the ones of the form \( Gp \Rightarrow E \) when \( E \) is not assumed to be exact. Any category of presheaves or sheaves of groups on a topological space is exact homological. Here, we shall introduce a halfway notion which, precisely, will allow us to include also the case of topological or Hausdorff groups and abelian groups.
**Definition 1.1.** A regular category $C$ is said to be efficiently regular when any equivalence relation $T$ on an object $X$ which is a subobject $j : T \rightarrow R$ of an effective equivalence relation on $X$ by an effective monomorphism in $C$ (which means that $j$ is the equalizer of some pair of maps in $C$), is itself effective. A homological category $C$ is said to be efficiently homological when it is efficiently regular.

**Examples.**

1. The additive categories $AbTop$ and $AbHaus$ of topological and Hausdorff abelian groups, as well as the homological categories $GpTop$ and $GpHaus$ of topological and Hausdorff groups are efficiently homological. Indeed, in any of these categories, an internal equivalence relation $R$ on a topological group $X$ is effective if and only $R$ is endowed with the topology induced by the product topology: $R \rightarrow X \times X$. Now if $j : T \rightarrow R$ is a subobject among the equivalence relation on $X$, and $j$ is effective, $T$ is endowed with the topology induced by the inclusion: $T \rightarrow R \rightarrow X \times X$ and is effective. Actually, in the category $GpTop$, effective monomorphisms coincide with strong monomorphisms; and the category $GpTop$ satisfies the stronger property of being “quasi-effective regular,” i.e. of being such that strong equivalence relations coincide with effective equivalence relations, see [4,24].

2. The same arguments and result apply to the homological categories $Top^T$ and $Haus^T$ of topological and Hausdorff semi-abelian algebras in the sense of [3], i.e. of topological and Hausdorff models of a semi-abelian theory $T$. A theory $T$ is said to be semi-abelian when the variety $\mathcal{V} = Set^T$ of its models is pointed protomodular, see [13,23].

3. When $E$ is an efficiently regular category, then the homological category $GpE$ of internal groups in $E$ is efficiently homological. This comes from the fact that, $E$ being regular and the regular epimorphisms being stable by products, any equivalence relation $R$ in $GpE$ whose underlying relation is effective in $E$ is effective in $GpE$.

4. The same arguments and result apply to the homological category $E^T$ of internal $T$-models in $E$ for any semi-abelian theory $T$, when $E$ is efficiently regular.

5. It is a simple exercise to check that a finitely complete additive category $\mathcal{A}$ is efficiently homological if and only if the kernel maps are stable under composition.

Here is our first observation.

**Proposition 1.2.** Suppose $C$ is efficiently regular. Let be given an equivalence relation $R$ on an object $U$ which is fibrant above an effective equivalence relation $R[q]$ on $V$:

$$
\begin{array}{c}
R & \xleftarrow{r_1} & U \\
\downarrow{h} & & \downarrow{q_1} \\
R[q] & \xrightarrow{q_0} & V & \xrightarrow{q} & W
\end{array}
$$

Then $R$ is effective.

**Proof.** The fact that $R$ is fibrant above $R[q]$ means that any of the left-hand side downward squares above are pullbacks. Now consider $R[q,h] = h^{-1}(R[q])$. Then there is a natural inclusion $j : R \rightarrow R[q,h]$. But $R$ is fibrant and $j$ is split, and thus an effective monomorphism:
Accordingly $R$ is effective.  \hfill \Box

We have the following consequence.

**Proposition 1.3.** Suppose $C$ is efficiently homological. Let $g : G \to C$ be a map and $T$ an equivalence relation on $G$ such that $[R[g], T] = 0$. Then the following equivalence relation on $T$ induced by the double relation associated with the connector $p$:

$$R[g] \times_X T \xrightarrow{p_2} T$$

is effective.

**Proof.** This is a particular case of the previous proposition since the equivalence relation in question is fibrant above $R[g]$:

$$R[g] \times_X T \xrightarrow{p_2} T$$

At this stage, we have the following specification which will be the driving force of the definition of the direction and of the construction of the Baer sums. This is the main justification of the introduction of efficiently regular categories.

**Corollary 1.1.** Let $C$ be an efficiently homological category, $g : G \to C$ a map in $C$ with kernel $k : K \to G$ and $T$ an equivalence relation on $G$. When $[R[g], T] = 0$, then the monomorphism $K \xrightarrow{k} G \xrightarrow{s_0} T$ is a kernel map. In particular, when $R[g]$ is abelian (respectively central) then $s_0.k : K \to G \to R[g]$ (respectively $G \to R[g]$) is a kernel map.

**Proof.** The monomorphism $K \xrightarrow{k} G \xrightarrow{s_0} T$ is precisely the normalization of the equivalence relation of the previous proposition, which is effective. Accordingly this monomorphism is a kernel map.  \hfill \Box
2. The direction functor in the categories \(C/C\)

We shall suppose from now on the category \(C\) efficiently homological. Let \(C\) be an object in \(C\). Then an object \(g: G \rightarrow C\) in the slice category \(C/C\) has a global support if and only if it is a regular epimorphism, and it is endowed with an autonomous Mal’cev operation if and only if its kernel equivalence relation is abelian [2]. So both conditions are satisfied if and only if \(g\) is an extension with abelian kernel equivalence relation. Let us introduce more precisely the category \(Abx C\) whose objects are the extensions with codomain \(C\) and a given abelian kernel relation:

\[
R[g] \xrightarrow{g_1} G \xrightarrow{g} C
\]

and whose morphisms are the maps \(\chi: G \rightarrow H\) making commute the following right-hand side square:

\[
\begin{array}{ccc}
R[g] & \xrightarrow{g_1} & G & \xrightarrow{g} & C \\
\downarrow{R(\chi)} & & \downarrow{\chi} & & \downarrow{1_C} \\
R[h] & \xrightarrow{h_1} & H & \xrightarrow{h} & C
\end{array}
\]

Of course it determines a unique factorization \(R(\chi)\) between the kernel relations which necessarily commutes with the connectors. The category \(Abx C\) is not finitely complete since the notion of objects with global support is not pullback stable; however it is essentially affine in the sense of [6]. We shall need this important distinction: the dual of the extension \((g, R[g])\), is defined to be the extension \((g, R^*[g])\), where \(R^*[g]\) is the dual of \(R[g]\), i.e. the relation with switched domain and codomain maps:

\[
R[g] \xrightarrow{g_0} G \xrightarrow{g} C
\]

We shall denote it by \(g^*\) for short. Clearly \(1_G\) determines a morphism between \(g\) and its dual \(g^*\), and \(R(1_C)\) is the twisting isomorphism \(R[g] \rightarrow R[g]\).

We shall denote by \(Sax C\) the category of abelian group objects in \(C/C\) which are nothing but split extensions with codomain \(C\) and abelian kernel relations (we do not need here to specify the kernel relation). We shall heavily use the fact that the category \(Sax C\) is finitely complete and, as any category of internal abelian groups, additive.

Now let \(g\) be an object in \(Abx C\).

**Definition 2.1.** The direction of this extension is defined as the right-hand side split extension given by the following diagram:
where $\alpha : A \rightarrow G$ is the kernel of the map $g$, the diagonal map $s_0.\alpha$ is also a kernel map since $C$ is efficiently homological, and $q_g$ a cokernel of this kernel.

**Remark.** 1. Necessarily the downward right-hand side square is a pullback, and consequently also the upward one. Moreover, the following diagram is necessarily a pushout in $C$, and consequently the object $d_C(G)$ can be described this way:

$$\begin{array}{ccc}
R[g] & \xrightarrow{q_g} & d_C(G) \\
s_0 \downarrow & & \uparrow s_g \\
G & \xrightarrow{g} & C
\end{array}$$

since, in a protomodular category, any pullback with parallel regular edges is a pushout.

2. Actually this diagram must be understood as a cokernel inside the category $Grd\ C$ of internal groupoids in $C$ to give $d_g : d_C(G) \rightarrow C$ its whole structure of abelian group in $C/C$:

$$\begin{array}{ccc}
1 & \xrightarrow{\text{dis}(A)} & R[g] & \xrightarrow{q_g} & d_C(G) & \xrightarrow{\bullet} & 1, \\
& & s_0 & \downarrow & \uparrow s_g \\
& & G & \xrightarrow{g} & C
\end{array}$$

where $\text{dis}(A)$ is the discrete groupoid associated with $A$ and $d_C(G)\bullet$ the groupoid whose object of objects $d_C(G)_0$ is $C$, whose object of morphisms $d_C(G)_1$ is $d_C(G)$ and whose domain and codomain maps are equal to $d_g$.

**Warning.** Of course, the definition of the direction involves a fixed choice of the cokernel $q_g$. We already mentioned that an object $g$ in $Abx\ C$ involved the choice of a given kernel relation $R[g]$. So that the definition of the direction involves the whole diagram introducing it above. Actually, the diagram defining $d_g$ makes the object $g$ of $C/C$ a $d_g$-torsor in the category $C/C$, see [7]. If you consider the dual abelian extension $g^\ast$, its direction is the same via the following diagram:

$$\begin{array}{ccc}
R[g] & \xrightarrow{q_g} & d_C(G) \\
s_0.\alpha \downarrow & & \uparrow s_g \\
A & \xrightarrow{g_0} & G & \xrightarrow{g} & C
\end{array}$$

But we shall see that the isomorphism between $g$ and its dual $g^\ast$ induced by the identity on $G$ does not induce the identity on $d_C(G)$ and, consequently does not belong to the fibre above $d_C(G)$.

More generally, given any extension $h$ with abelian kernel relation and any split abelian extension $(d, s) : D \rightarrowtail C$, we shall say that the later is the direction of the former when there is a fixed map $\bar{h} : R[h] \rightarrow D$ making the following downward square a pullback (or equivalently the upward square a pushout):

$$\begin{array}{ccc}
R[h] & \xrightarrow{q_h} & d_C(D) \\
s_0 \downarrow & & \uparrow s_g \\
A & \xrightarrow{g_0} & G & \xrightarrow{g} & C
\end{array}$$
The $3 \times 3$ lemma applied to the following diagram shows that the unique map $\hat{\alpha} : A \to d_C(G)$ which makes commutes the upper right-hand side square is the kernel of the direction $d_g : d_C(G) \to C$:

We get then: $\hat{\alpha} = \hat{\alpha} \cdot (-1, 1) \cdot (0, 1) = q_g \cdot R(\alpha) \cdot (0, 1) = q_g \cdot (0, \alpha)$.

It is now possible to single out extensions $g$ with central equivalence relation among those with abelian kernel equivalence relation by means of their directions.

**Proposition 2.1.** An extension $g$ with abelian kernel equivalence relation has a central kernel equivalence relation if and only if its direction is the projection $p_C : C \times A \to C$ with section the canonical injection $l_C : C \hookrightarrow C \times A$, i.e. the trivial $C$-module associated with $A$.

**Proof.** Suppose the direction is $p_C : C \times A \to C$, then its section $l_C : C \hookrightarrow C \times A$ is a kernel map; and, the right-hand square being a pullback in the diagram introducing the direction above, the map $s_0 : X \to R[g]$ is still a kernel map. Consequently $R[g]$ is central. Conversely suppose $R[g]$ is central; then $s_0 : X \to R[g]$ is a kernel map, and its direct image along the regular epimorphism $q_g$, which is $s_g$, is thus normal. Then by Theorem 4.9 in [12], $d_C(G)$ is isomorphic to $C \times \ker d_g$. But $\ker d_g$ is $A$. □

Clearly the construction of $d_C(G)$ defines a functor $d_C : \text{Ab}_x C \to \text{Sax}_x C$. We are now going to precise its effect on the morphisms. Let us consider a morphism $\chi$ in $\text{Ab}_x C$, and call $\psi$ the restriction to the kernels:

$$
\begin{array}{cccc}
1 & \longrightarrow & A & \xrightarrow{\alpha} & G & \xrightarrow{g} & C & \longrightarrow & 1 \\
\downarrow{\psi} & & \downarrow{\chi} & & \downarrow{l_C} & & \\
1 & \longrightarrow & B & \xrightarrow{\beta} & H & \xrightarrow{h} & C & \longrightarrow & 1
\end{array}
$$
**Proposition 2.2.** There is a unique map \( d_C(\chi) : d_C(G) \to d_C(H) \) which makes the following diagram commute:

\[
\begin{array}{ccc}
1 & \xrightarrow{\hat{\alpha}} & A & \xrightarrow{s_g} & d_C(G) & \xrightarrow{d_g} & C & \xrightarrow{1_C} & 1 \\
\downarrow{\psi} & & \downarrow{d_C(\chi)} & & \downarrow{1} & & \downarrow{1} \\
1 & \xrightarrow{\hat{\beta}} & B & \xrightarrow{s_h} & d_C(H) & \xrightarrow{d_h} & C & \xrightarrow{1} & 1
\end{array}
\]

Moreover, if \( k : K \to G \) is a kernel of \( \chi \) and \( \tilde{k} : K \to A \) is the induced factorization (which is the kernel of \( \psi \)), then \( \hat{\alpha} \tilde{k} : K \to A \to d_C(G) \) is a kernel of \( d_C(\chi) \).

**Proof.** The unicity is straightforward, the pair \((\hat{\alpha}, s_g)\) being jointly strongly epimorphic. On the other hand, the map \( \chi : G \to H \) extends to a map \( R(\chi) : R[g] \to R[h] \) underlying a morphism of equivalence relation which consequently is such that: \( R(\chi).s_0.\alpha = s_0.\beta.\psi \). So that there is a unique map \( d_C(\chi) \) which completes the following diagram:

\[
\begin{array}{ccc}
1 & \xrightarrow{s_0.\alpha} & A & \xrightarrow{q_g} & R[g] & \xrightarrow{d_C(\chi)} & d_C(G) & \xrightarrow{1} & 1 \\
\downarrow{\psi} & & \downarrow{R(\chi)} & & \downarrow{d_C(\chi)} & & \downarrow{1} \\
1 & \xrightarrow{s_0.\beta} & B & \xrightarrow{q_h} & R[h] & \xrightarrow{1} & d_C(H) & \xrightarrow{1} & 1
\end{array}
\]

Actually, this must be understood as a diagram of internal groupoids, in such a way that \( d_C(\chi) \) is underlying a group homomorphism in \( \mathcal{C}/\mathcal{C} \):

\[
\begin{array}{ccc}
1 & \xrightarrow{\text{dis}(A)} & \xrightarrow{\text{dis}(\psi)} & \xrightarrow{\text{dis}(\chi)} & \xrightarrow{1} \\
& & \xrightarrow{R(\chi)} & & \xrightarrow{d_C(\chi)} \\
1 & \xrightarrow{\text{dis}(B)} & \xrightarrow{\text{dis}(\psi)} & \xrightarrow{\text{dis}(\chi)} & \xrightarrow{1}
\end{array}
\]

The end of the proof is straightforward. \( \square \)

### 2.1. Exactness properties of the direction functor

A map \( \chi \) in \( \text{Abx} \mathcal{C} \) is a monomorphism (respectively regular epimorphism, isomorphism) when it is a monomorphism (respectively regular epimorphism, isomorphism) in \( \mathcal{C} \). But the domain and codomain sequences being exact in the diagram introducing \( \chi \) above, this is the case if and only if the only map \( \psi \) is a monomorphism (respectively regular epimorphism, isomorphism) \([2]\). Accordingly:

**Corollary 2.1.** The direction functor \( d_C : \text{Abx} \mathcal{C} \to \text{Sax} \mathcal{C} \) preserves and reflects monomorphisms (respectively regular epimorphisms, isomorphisms). In particular any map whose image by \( d_C \) is an identity is an isomorphism.

We have also:
Proposition 2.3. The direction functor $d_C: \text{Ab}_X C \to \text{Sax}_X C$ preserves products.

Proof. Since $R[g \times h] = R[g] \times R[h]$ and the exact sequences are stable by products in $C$, we have also $d_{C \times C}(G \times H) = d_C(G) \times d_C(H)$. Now the product $G \times C H$ in $\text{Ab}_X C$ is given by the following pullback:

\[
\begin{array}{ccc}
G \times_C H & \xrightarrow{j} & G \times H \\
\pi & \downarrow & \downarrow g \times h \\
C & \xrightarrow{s_0} & C \times C
\end{array}
\]

which produces the following left-hand side pullback:

\[
\begin{array}{ccc}
R[\pi] & \xrightarrow{R(j)} & R[g] \times R[h] \\
\downarrow d_0 & \downarrow d_1 & \downarrow d_0 \\
G \times_C H & \xrightarrow{j} & G \times H \\
\downarrow g \times h & \downarrow & \downarrow C \times C
\end{array}
\]

But the right-hand side square is also a pullback, and consequently the whole rectangle is a pullback. This rectangle is the same as the following one (where $\bar{j}$ is the factorization induced by $R(j)$):

\[
\begin{array}{ccc}
R[\pi] & \xrightarrow{q_\pi} & d_C(G \times_C H) \\
\downarrow d_0 & \downarrow d_1 & \downarrow d_0 \\
G \times_C H & \xrightarrow{\pi} & C \\
\downarrow & \downarrow & \downarrow s_0 \\
& & C \times C
\end{array}
\]

Now the left-hand side square is a pullback with horizontal regular epimorphisms, and thus the right-hand side is a pullback which means that $d_\pi$ is the desired product in $C/C$. \qed

We shall need some more specific calculations.

Lemma 2.1. Consider a split extension with abelian kernel relation:

\[
1 \xrightarrow{1} A \xrightarrow{\alpha} G \xrightarrow{g} C \xrightarrow{s} 1
\]

Then it is the direction of its underlying nonsplit extension with abelian kernel relation.

Proof. It is the consequence of the fact that in any additive category $A$, the “division” map $d = (-1, 1): A \times A \to A$ satisfies the equations of an internal functor $\nabla_A \to A$ (where $A$ is equipped with its canonical group structure), which is actually a discrete fibration. This means that any of the following downward squares are pullbacks:
In the additive category $Sax C$, they are the following pullbacks:

\[
\begin{array}{ccc}
R[g] & \longrightarrow & G \\
\uparrow g_0 & \quad & \uparrow \gamma \\
G & \longrightarrow & C
\end{array}
\]

This implies that the map $(-1, 1)_C$ is the cokernel of $s_0.\alpha$. □

We show now that the direction of the isomorphism between an extension $g$ and its dual $g^*$ induced by the identity map $1 : G \to G$ is not the identity map $1 : d_C(G) \to d_C(G)$, in other words this isomorphism does not lie in a fibre of the direction functor $d_C$.

**Lemma 2.2.** The direction of the canonical isomorphism between an extension $g$ and its dual $g^*$ is the isomorphism $-1 : d_C(G) \to d_C(G)$.

**Proof.** This is a consequence of the fact that, in any additive category $A$, the map $-1_A : A \to A$ is the unique one making the following diagram commute:

\[
\begin{array}{ccc}
A \times A & \longrightarrow & A \\
\downarrow tw & & \downarrow -1_A \\
A \times A & \longrightarrow & A \quad (-1,1)
\end{array}
\]

where $tw$ is the twisting isomorphism. □

We shall characterize now those maps in $Abx C$ whose images in the additive category $Sax C$ are the zero maps.

**Lemma 2.3.** A map $\chi : g \to h$ in $Abx C$ has a null direction if and only if $\chi$ is a constant map, i.e. if and only if there is a global element $e : C \to H$ such that $\chi = e.g$.

**Proof.** Of course if $\chi = e.g$, then $d_C(\chi) = d_C(e).d_C(g) = s_h.d_g$ which is the zero map $d_C(G) \to d_C(H)$. Conversely if $d_C(\chi) = 0$ in $Sax C$, then the restriction $\psi : A \to B$ is 0 in $C$. Thus $\chi.\alpha = \beta.\psi = 0$, and there is a factorization $e : C \to H$ such that $e.g = \chi$. □

The functor $d_C$ has also the following very strong property.

**Lemma 2.4.** When two parallel maps $\chi$ and $\chi'$ in $Abx C$ have the same direction and are equalized by a map $v$, they are equal.
Proof. Let us consider the following map $\chi$:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\alpha} & G & \xrightarrow{g} & C & \longrightarrow & 1 \\
\psi & & \downarrow & & \chi & & \downarrow & & 1_C \\
1 & \longrightarrow & B & \xrightarrow{\beta} & H & \xrightarrow{h} & C & \longrightarrow & 1
\end{array}
\]

First let us consider the following factorization $\bar{v}$ which makes commute the following diagram of split epimorphisms:

\[
\begin{array}{ccccccc}
F \times_C d_C(G) & \xrightarrow{p\bar{v}} & d_C(G) & \xrightarrow{qg} & d_C(G) \\
\downarrow p_F & & \downarrow p_F & & \downarrow p_F \\
F & \xrightarrow{v} & G & \xrightarrow{g} & C
\end{array}
\]

Provided we know that $d_C(qg) = (-1, 1)_C$ (which will be shown in Corollary 2.2), a straightforward calculation in the additive category $Sax C$ shows that

\[
d_C(d_1.\bar{v}) = (d_C(v), 1)_C : d_C(F) \times_C d_C(G) \rightarrow d_C(G)
\]

which is a split epimorphism. Accordingly the map $d_1.\bar{v}$ is itself a regular epimorphism. On the other hand, setting $\chi.v = v'$, the map $\chi$ induces a commutative square:

\[
\begin{array}{ccccccc}
F \times_C d_C(G) & \xrightarrow{\bar{v}} & R[g] & \xrightarrow{R(\chi)} & d_C(G) \\
\downarrow 1 \times_C d_C(\chi) & & \downarrow & & \downarrow R(\chi) \\
F \times_C d_C(H) & \xrightarrow{\bar{v}'} & R[h]
\end{array}
\]

Now if $d_C(\chi) = d_C(\chi')$, we have $\chi.d_1.\bar{v} = d_1.R(\chi).\bar{v} = d_1.\bar{v}'.(1 \times_C d_C(\chi)) = d_1.\bar{v}'.(1 \times_C d_C(\chi')) = \chi'.d_1.\bar{v}$. But $d_1.\bar{v}$ is a regular epimorphism, so that $\chi = \chi'$. □

Remark. There must be recall that the category $Abx C$ is not finitely complete, and that a pair of parallel arrows, in this category, does not admit in general an equalizer; if there were equalizers, the direction functor, being conservative and satisfying the previous lemma, would be faithful, which is far from being the case.

Let us close this section with the following more general observation. Suppose we have a map $\chi$ in $Abx C$:

\[
\begin{array}{cccccc}
G & \xrightarrow{\chi} & H \\
g & \downarrow & h
\end{array}
\]
Then, since we have an inclusion $j : R[\chi] \hookrightarrow R[g]$, we have $[R[\chi], R[g]] = 0$ and then $[R[\chi], R[\chi]] = 0$. Consequently the map $\chi$ has an abelian kernel relation. If, moreover, $\chi$ is a regular epimorphism, it is an extension with abelian kernel relation. Now the question is: what is its direction? Let $k : K \rightarrow dC(G)$ be the kernel of $dC(\chi)$ in the additive category $Sax C$, and $(d_K, s_K) : K \Rightarrow C$ its domain.

**Proposition 2.4.** Then there is a map $r_\chi$ making the following square a pullback in $C$:

$$
\begin{array}{ccc}
R[\chi] & \xrightarrow{r_\chi} & K \\
\downarrow{\chi_0} & & \downarrow{d_K} \\
G & \xrightarrow{g} & C
\end{array}
$$

Accordingly, the direction of the extension $\chi$ is the group $h^* ((d_K, s_K))$ in $Sax H$.

**Proof.** First consider the whole rectangle given by the following pullbacks:

$$
\begin{array}{ccc}
R[\chi] & \xrightarrow{\chi \cdot \chi_0} & H \\
\downarrow{j} & & \downarrow{s_0} \\
R[g] & \xrightarrow{j} & R[h] \xrightarrow{s_h} dC(H)
\end{array}
$$

It is the same as the following rectangle since the horizontal composites are the same in the two diagrams:

$$
\begin{array}{ccc}
R[\chi] & \xrightarrow{g \cdot \chi_0} & K \\
\downarrow{j} & & \downarrow{k} \\
R[g] & \xrightarrow{qg} & dC(G) \xrightarrow{dC(\chi)} dC(H)
\end{array}
$$

Since the right-hand side square is a pullback by definition of $k$, there is a factorization $r_\chi$ which completes the left-hand square as a pullback. Then consider the following vertical rectangle:

$$
\begin{array}{ccc}
R[\chi] & \xrightarrow{r_\chi} & K \\
\downarrow{j} & & \downarrow{k} \\
R[g] & \xrightarrow{qg} & dC(G) \xrightarrow{dC(\chi)} dC(H)
\end{array}
$$
Clearly it is a pullback. And we have $g_0.j = \chi_0, g_1.j = \chi_1$. Moreover, we have $d_g.k = d_K$, since $s_h.d_g.k = s_h.d_h.d_C(\chi).k = s_h.d_h.s_h.d_K = s_h.d_K$. The conclusion is obtained by considering the following right-hand side pullback:

\[
\begin{array}{c}
R[\chi] \\
\xrightarrow{r_\chi} \\
H \times C \xrightarrow{K} K \\
\xrightarrow{\chi_0} \\
G \\
\xrightarrow{\chi} \\
H \xrightarrow{h} C \\
\xrightarrow{g} \\
\end{array}
\]

which produces the left-hand side one. \qed

2.2. The direction functor is a cofibration

To recover all the properties of the direction functor in the protomodular and efficiently regular context, we must show now that it is a cofibration. Before going any further, let us state the following:

**Proposition 2.5.** Let $g$ be an extension with abelian kernel relation, and $i : A' \hookrightarrow G$ a kernel map such that $A' \subseteq A$. Then the unique factorization $h : G/A' \to C$ of $g$ through the quotient $q : G \to G/A'$ is an extension with abelian kernel relation whose direction is $d_C(G/A') = d_C(G)/A'$.

**Proof.** Let us consider the following diagram, where $i'$ is necessarily a kernel map since kernel maps are stable by pullbacks:

\[
\begin{array}{c}
1 \\
\xrightarrow{i'} \\
A' \\
\xrightarrow{\alpha} \\
A \\
\xrightarrow{q'} \\
A/A' \\
\xrightarrow{1} \\
1 \\
\end{array} \quad \begin{array}{c}
1 \\
\xrightarrow{i} \\
G \\
\xrightarrow{\tilde{q}} \\
G/A' \\
\xrightarrow{1} \\
1 \\
\end{array} \quad \begin{array}{c}
1 \\
\xrightarrow{1} \\
C \\
\xrightarrow{l_C} \\
C \\
\xrightarrow{1} \\
1 \\
\end{array}
\]

The $3 \times 3$ lemma produces a right-hand side exact sequence. Moreover, the regular epimorphism $\tilde{q}$ produces a regular epimorphic factorization $R(\tilde{q}) : R[g] \to R[h]$ which shows that $R[h]$ is the direct image of $R[g]$ along the regular epimorphism $\tilde{q}$. Consequently $R[h]$ is abelian since $R[g]$ is abelian (see [2, Corollary 2.6.16]), and $h$ is an extension with abelian kernel relation. The following $3 \times 3$ diagram determines its direction as $d_C(G)/A'$, where $\beta$ is a kernel of $h$: 
and where the commutation of the upper right-hand side diagram is straightforward. □

Actually the previous proposition implies the following:

**Proposition 2.6.** Any regular epimorphism \( \theta \) in \( \text{Sax} C \):

\[
\begin{align*}
\begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \downarrow \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array} & \begin{array}{c}
\theta \\
\Downarrow \theta \\
\downarrow
\end{array}
\end{array}
\]

**Proof.** Let us denote by \( k : K \to d_C(G) \) the kernel of \( \theta \) in the additive category \( \text{Ab} x C \). Then consider the following diagram where each square is a pullback:

\[
\begin{array}{ccc}
A' & \xrightarrow{\alpha'} & T & \xrightarrow{\gamma} & K & \xrightarrow{d_K} & C \\
\downarrow i' & \downarrow j & \downarrow k & \downarrow t \\
A & \xrightarrow{R[g]} & R[\alpha] & \xrightarrow{q_g} & d_C(G) & \xrightarrow{\theta} & L \\
\downarrow g_0 & \downarrow \alpha & \downarrow d_g & \downarrow \theta \\
G & \xrightarrow{g} & C
\end{array}
\]

Notice that \( d_K = d_g.k \) since \( t.d_g.k = t.l.\theta.k = t.l.t.d_K = t.d_K \). Accordingly \( \gamma.\alpha' \) is the kernel of \( d_K \), and thus \( \alpha.i' \) is the kernel of \( \theta \). On the other hand, the pair \( (t_0, t_1) = j.(g_0, g_1) : T \to G \) determines a reflexive relation, and thus an equivalence relation, since \( C \) is protomodular. Moreover, this equivalence relation is effective, since \( j \) is a pullback of the effective (since split) monomorphism \( t \) and is consequently itself an effective monomorphism. The fact that the upper left-hand side square is a pullback implies that the normalization of the equivalence relation \( T \) is the map \( t_1.\alpha' = g_1.j.\alpha' = g_1.(0, \alpha).i' = \alpha.i' \). This map \( i = \alpha.i' : A' \to G \) is a kernel map, since
$T$ is an effective equivalence relation. Then, according to the previous proposition, its cokernel $\bar{\theta} : G \twoheadrightarrow H$ (which is also the quotient of $T$) determines a factorization $h : H \rightarrow C$ and a map $q_h$ which completes the following $3 \times 3$ diagram:

\[
\begin{array}{cccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
A' & A' \times A' & A' \\
\downarrow i' & \downarrow R(i') & \downarrow \hat{\alpha}.i' \\
A & R[\hat{\alpha}] & dC(G) \\
\downarrow q' & \downarrow R(\tilde{\theta}) & \downarrow \theta \\
A/\hat{\alpha} & R[h] & L \\
\downarrow 1 & \downarrow 1 & \downarrow 1 \\
\end{array}
\]

and proves that $d_C(\bar{\theta}) = \theta$ and $d_C(H) = L$. Now if $\phi$ is a morphism in $Abx C$ between the extensions $g$ and $g'$, such that there is a factorization $u : L \rightarrow d_C(G')$ in $Sax C$ satisfying $u.\theta = d_C(\phi)$, we must show that $\phi$ factorizes in a unique way through $\bar{\theta}$. This is the case if and only if $A' \subseteq \text{Ker} \phi$. But, thanks to Proposition 2.2, this is precisely what implies the factorization $u$ and the equality $u.\theta = d_C(\phi)$.

Let us show now that any regular epimorphism is cocartesian. If $\chi$ is a regular epimorphism in $Abx C$, then $d_C(\chi)$ is a regular epimorphism. Let $\bar{\chi}$ be its associated cocartesian map. Then $d_C(\bar{\chi}) = d_C(\chi)$, and there is a unique factorization $\xi$, such that $\xi.\bar{\chi} = \chi$ and $d_C(\xi) = 1$. Thus $\xi$ is an isomorphism (since $d_C$ is conservative), and $\chi$ is cocartesian. \hfill $\square$

On the other hand, the diagram defining the direction of an extension $g$ with abelian kernel relation is clearly a diagram in $Abx C$ and the map $q_C : R[g] \rightarrow d_C(G)$ is itself in $Abx C$.

**Corollary 2.2.** The direction of the map $q_C : R[g] \rightarrow d_C(G)$ is the “division” map

\[
(-1, 1)_C : d_C(G) \times d_C(G) \rightarrow d_C(G)
\]

in the additive category $Sax C$.

**Proof.** The shortest way to prove it will be to show that $q_g$ is obtained by the construction of the cocartesian map with domain $R[g]$ above $(-1, 1)_C : d_C(G) \times d_C(G) \rightarrow d_C(G)$. The kernel of this last map is the map $(\hat{\alpha}, \hat{\alpha}) : A \rightarrow d_C(G) \times d_C(G)$. Its factorization through the kernel $R(\hat{\alpha}) : A \times A \rightarrow d_C(G) \times d_C(G)$ of the product $d_C(g) \times d_C(g) : d_C(G) \times d_C(G) \rightarrow C$ is clearly $s_0 : A \rightarrow A \times A$. Consequently the cocartesian map with domain $R[g]$ above the map $(-1, 1)_C : d_C(G) \times d_C(G) \rightarrow d_C(G)$ is given by the cokernel of:

\[
R(\alpha).s_0 = s_0.\alpha : A \twoheadrightarrow R[g] \rightarrow d_C(G)
\]

which is the map $q_g$. \hfill $\square$
**Remark.** More technically, let us denote by $\tau : A \rightarrow B = \text{Ker} l$ the unique map making the following diagram commute:

\[
\begin{array}{c}
1 \rightarrow A \xrightarrow{\delta} dC(G) \xrightarrow{s\gamma} C \rightarrow 1 \\
\tau \\
1 \rightarrow B \xrightarrow{\beta} L \xrightarrow{l} C \rightarrow 1
\end{array}
\]

The left-hand side square is a pullback, and as any pullback with regular epimorphic parallel edges, it is also a pushout. From that, we just built an extension with abelian kernel relation:

\[
\begin{array}{c}
1 \rightarrow A \xrightarrow{\alpha} G \xrightarrow{g} C \rightarrow 1 \\
\tau \\
1 \rightarrow B \xrightarrow{\beta} H \xrightarrow{h} C \rightarrow 1
\end{array}
\]

where the left-hand side square is necessarily a pullback and a pushout for the same reasons as above.

On the other hand, in the more restricted context of exact homological categories, we can reinforce the similarity (already noticed in the introduction) between the direction of an extension with abelian kernel relation and the direction of a $K$-affine space by asserting an analogue of the Euclide’s Postulate: given any point $x$ of a $K$-affine space $X$ and any subspace $X' \subset X$ of its direction, there is a unique affine subspace $X'$ with direction $X'$ such that $x \in X'$.

**Corollary 2.3 (Euclide’s Postulate).** Let $C$ be an exact homological category. Let $g : G \rightarrow C$ be any extension with abelian kernel relation, $d' : D' \rightarrow C$ be any subgroup of its direction:

\[
\begin{array}{c}
D' \xrightarrow{i} dC(G) \\
\downarrow_{s'} \quad \downarrow_{d'} \quad \downarrow_{s} \\
C
\end{array}
\]

and $e : C \hookrightarrow G$ be any global element of $g$. Then there is a unique subextension $g'$ having $d' : D' \rightarrow C$ as direction and $e$ as global element.

**Proof.** Since $C$ is Barr exact, then $\text{Sax} C$, which is additive and Barr exact, is abelian. Let us denote by $\theta : dC(G) \rightarrow L$ the quotient in the abelian category $\text{Sax} C$ of the inclusion $i : D' \hookrightarrow dC(G)$. Let $\tilde{\theta} : G \rightarrow H$ be the cocartesian map above $\theta$, and consider the following pullback in $C$:

\[
\begin{array}{c}
G' \xrightarrow{i'} G \\
\downarrow_{e'} \quad \downarrow_{g'} \quad \downarrow_{e} \\
C \xrightarrow{\tilde{\theta}, e} H
\end{array}
\]
It actually lies in $Abx \mathcal{C}$ since $g'$ is necessarily a regular epimorphism. It determines, via $i'$ a subextension $g'$ with global element $e'$, whose direction is given by the following pullback in $Sax \mathcal{C}$:

\[
\begin{array}{ccc}
G' & \xrightarrow{d_C(i')} & G \\
d_C(g') & & \downarrow \theta \\
C & \xrightarrow{t} & L \\
\end{array}
\]

since, according to [7, Proposition 6], the direction functor preserves the pullbacks whenever they exist in $Abx \mathcal{C}$. This means that $d_C(i') : d_C(G') \to d_C(G)$ is (up to isomorphism) the kernel $\iota : D' \to d_C(G)$ of $\theta$ in the abelian category $Sax \mathcal{C}$. Suppose now you have a pair $(G'', e'')$ satisfying the same properties. Then you can check that the following diagram commutes:

\[
\begin{array}{ccc}
G'' & \xrightarrow{i''} & G \\
g'' & & \downarrow \tilde{\theta} \\
C & \xrightarrow{\bar{\theta}, e} & H \\
\end{array}
\]

since its image by the direction functor $d_C$ commutes and since, thanks to Lemma 2.4 and to the existence of the global element $e''$, it is enough to check the commutation by composition with $e''$. Thus there is a factorization $G'' \to G'$ whose image by $d_C$ is an identity map. But the direction functor is conservative, and this map is an isomorphism. □

**Remark.** The same arguments, and consequently this same Euclidean Postulate, apply in the more general context of [7].

We are now in position to assert our main result.

**Theorem 2.1.** Let $\mathcal{C}$ an efficiently homological category. Then the direction functor $d_C : Abx \mathcal{C} \to Sax \mathcal{C}$ is a cofibration whose any map is cocartesian.

**Proof.** So let us consider now an object $g : G \to C$ in $Abx \mathcal{C}$ and any morphism in $Sax \mathcal{C}$:

\[
\begin{array}{ccc}
d_C(G) & \xrightarrow{\theta} & L \\
& & \uparrow t \\
& & \downarrow s_g \\
C & \xrightarrow{1} & C \\
\end{array}
\]

Then the morphism $(\theta, 1) : d_C(G) \times_C L \to L$, in the additive category $Sax \mathcal{C}$, is split, and consequently a regular epimorphism. Let us consider its associated cocartesian map $\overline{(\theta, 1)} : G \times_C L \to H$. Then the map $\bar{\theta} = (\theta, 1).1_G : (1_G, t)$:

\[
G \xrightarrow{(1_G, t)} G \times_C L \xrightarrow{\overline{(\theta, 1)}} H
\]
from \( g \) to \( h \) in \( AbxC \) has its direction equal to

\[
d_C G \xrightarrow{(1,0)} d_C G \times_C L \xrightarrow{(\theta,1)} H
\]

which is \( \theta \). Let us show that this map has the universal property of a cocartesian map. So let \( \chi \) a map between \( g \) and \( h' \) in \( AbxC \) such that there is a factorization \( v \) in \( SaxC \) satisfying \( d_C(\chi) = v.\theta \). First let us consider the following factorization \( \bar{\chi} \) which makes commute the following diagram of split epimorphisms:

\[
\begin{array}{ccc}
G \times_C d_C(H') & \longrightarrow & d_C(H') \\
pG & \downarrow & d_1 \\
(1, s_{h'}g) & \bar{\chi} & h' \\
g & \chi & C
\end{array}
\]

Thus we have \( d_1.\bar{\chi}.(1, s_{h'}g) = d_1.s_0.\chi = \chi \). On the other hand, a straightforward calculation in the additive category \( SaxC \) shows that \( d_C(d_1.\bar{\chi}) = (d_C(\chi), 1) \). Then consider the following plain diagram in \( AbxC \):

\[
\begin{array}{ccc}
G \times_C L & \longrightarrow & H \\
1 \times_C v & \downarrow & u \\
G \times_C d_C(H') & \longrightarrow & R[h'] \\
\bar{\chi} & \longrightarrow & H'
\end{array}
\]

whose image by the functor \( d_C \) is completed into a commutative diagram:

\[
\begin{array}{ccc}
d_C(G) \times_C L & \longrightarrow & L \\
1 \times_C v & \downarrow & v \\
d_C(G) \times_C d_C(H') & \longrightarrow & R[h'] \\
\tilde{\chi} & \longrightarrow & H'
\end{array}
\]

The map \( (\theta, 1) \) being cocartesian, there is a unique factorization \( u \) above \( v \). Moreover, \( u.\tilde{\theta} = u.(\theta, 1). (1, t,g) = d_1.\tilde{\chi}.1 \times_C v.(1, t,g) = d_1.\tilde{\chi}.(1, s_{h'}g) = \chi \). The unicity of this factorization is a consequence of Lemma 2.4. Moreover, any map in \( AbxC \) is cocartesian since any morphism in \( AbxC \) mapped by \( d_C \) to an identity is an isomorphism. \( \square \)

2.3. Extensions with central kernel relation

Unlike the case above where the map \( \theta : d_C(G) \to L \) is a regular epimorphism, there is no reason why, in general, its associated cocartesian map \( \tilde{\theta} : G \to H \) would produce a pushout with
the left-hand side square of the following diagram:

\[
\begin{array}{cccccc}
1 & \rightarrow & A & \xrightarrow{\alpha} & G & \xrightarrow{g} & C & \rightarrow & 1 \\
\tau & \downarrow & \bar{\theta} & & \downarrow & 1_C \\
1 & \rightarrow & B & \xrightarrow{\beta} & H & \xrightarrow{h} & C & \rightarrow & 1 \\
\end{array}
\]

However, in the case of extensions \(g\) with \textit{central} kernel relation, it is possible to explicit a specific universal property for this square. For that we must recall the intrinsic notion of central map in a pointed protomodular category [10]. Consider the following square:

\[
\begin{array}{ccc}
X & \xrightarrow{l_X} & X \times Y \\
\downarrow & & \downarrow p_Y \\
1 & \xrightarrow{r_Y} & Y \\
\end{array}
\]

It is a pullback which makes the pair \((l_X, r_Y)\) be jointly strongly epic, and thus jointly epic. Therefore a map \(\varphi : X \times Y \rightarrow Z\) is uniquely determined by the pair of maps \((f, g)\), \(f : X \rightarrow Z\) and \(g : Y \rightarrow Z\), with \(f = \varphi.l_X\) and \(g = \varphi.r_Y\). Consequently the existence of such a map \(\varphi\) becomes a property in respect to the pair \((f, g)\).

\textbf{Definition 2.2.} Given a pair \((f, g)\) of morphisms in any pointed protomodular category \(\mathcal{C}\), when such a map \(\varphi\) exists, we say that the maps \(f\) and \(g\) cooperate and that the map \(\varphi\) is the cooperator of the pair \((f, g)\). A map \(f : X \rightarrow Y\) is central when \(f\) and \(1_Y\) cooperate. An object \(X\) is said abelian when the map \(1_X : X \rightarrow X\) is central.

Let us begin by the following observation, see also [20, Proposition 2.2].

\textbf{Proposition 2.7.} Let \(\mathcal{C}\) be an efficiently homological category. Then any extension \(g\) has a central kernel equivalence relation \(R[g]\) if and only if it has a central kernel map \(\alpha : A \rightarrow G\).

\textbf{Proof.} Suppose \(g\) has a central kernel map \(\alpha\), and let \(\varphi : G \times A \rightarrow G\) be its cooperator. Then consider the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{r_A} & G \times A & \xrightarrow{\varphi} & G \\
\downarrow & & & \downarrow & \downarrow \\
1 & \xrightarrow{p_G} & 1_G & \xrightarrow{g} & C \\
\end{array}
\]
We have $\varphi.r_A = \alpha$. The whole rectangle is a pullback, in the same way as the left-hand side square; the middle vertical square is split. Accordingly, in the protomodular category $\mathbb{C}$, the right-hand square is a pullback [2]. The equation $\varphi.l_G = 1_G$ completes the kernel relation of $g$:

$$
\begin{array}{ccc}
G \times A & \xleftarrow{\varphi} & G \\
& \searrow{p_G} \downarrow{l_G} & \searrow{g} \\
& & C
\end{array}
$$

The diagonal $s_0$ is given by $l_G$ here, and this is a kernel map. Accordingly $g$ has a central kernel relation. Conversely suppose $R[g]$ is a central equivalence relation. Then, following Proposition 2.1, the direction of this extension is given by the following diagram:

$$
\begin{array}{ccc}
R[g] & \xrightarrow{q_g} & C \times A \\
\downarrow{s_0.\alpha} & \nearrow{g_0} & \searrow{g_1} \\
A & \xrightarrow{\alpha} & G & \xrightarrow{g} & C
\end{array}
$$

This implies that there is an isomorphism $\gamma : R[g] \to G \times A$ such that $p_G.\gamma = g_0, g \times 1_A.\gamma = q_g$ and $\gamma.s_0 = l_G$. Thanks to the last equation, the map $d_1.\gamma^{-1}$ becomes the cooperator of the (central) map $d_1.\gamma^{-1}.r_A$ which is necessarily a normalization of the equivalence relation $R[g]$, and thus a kernel of $g$. This implies that any other kernel of $g$, and in particular $\alpha$, is central. □

In this context, we can now explicit the universal property we were looking for:

**Proposition 2.8.** Let $\mathbb{C}$ be an efficiently homological category. Consider any extension $g$ with central kernel map in $\mathbb{C}$:

$$
1 \to A \xrightarrow{\alpha} G \xrightarrow{g} C \to 1
$$

and $\tau : A \to B$ any map between two abelian objects. Then there is a unique extension $(\beta, h)$ with central kernel map

$$
1 \to A \xrightarrow{\alpha} G \xrightarrow{g} C \to 1
$$

such that the left-hand side square has the following universal property:

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & G \\
\downarrow{\tau} & & \downarrow{g} \\
B & \xrightarrow{\beta} & H & \xrightarrow{h} & C
\end{array}
$$

\[\xymatrix{ & A \ar[dl]^{\tau} \ar[dr]_{\tau} & \\
B & H & C}
\]
given any pair \((\gamma, \chi)\) making the square commute and such that \(\gamma\) is a central map, there is a unique factorization \(\xi\).

**Proof.** In the pointed protomodular category \(\mathbb{C}\), any abelian object \(A\) is an internal group object, and any map \(\tau: A \to B\) between abelian objects is a group homomorphism. Accordingly the map \(\theta = 1_C \times \tau: C \times A \to C \times B\) determines a group homomorphism in \(\text{Sax } C\). Let us consider \(\bar{\theta}: G \to H\) its associated cocartesian map in \(\text{Abx } C\). The extension \(h\) has \((p_C, l_C): C \times B \rightrightarrows C\) as direction and has consequently a central kernel relation. Its kernel \(\beta: B \to H\) is central, and more importantly the map \(\bar{\theta}\) produces the diagram of extensions claimed by the proposition. Suppose we have any pair \((\gamma, \chi)\) such that \(\gamma\) is a central map and satisfies \(\gamma \cdot \tau = \chi \cdot \alpha\). According to the construction of \(\bar{\theta}\), let us consider the following diagram:

Let us denote by \(\varphi\) the cooperator of the central map \(\gamma\), and set \(\zeta = \varphi \cdot \chi \times 1_B : G \times B \to W\). We have then \(\zeta \cdot (1_G, 0) = \varphi \cdot \chi \times 1_B \cdot (1_G, 0) = \varphi \cdot (1_W, 0) \cdot \chi = \chi\). And we check that \(\zeta \cdot \alpha = 1_B = \beta \cdot (\tau, 1)\) by composition with the jointly epic pair \((l_A, r_B)\). The lower left-hand side square is a pushout, since the map \((\theta, 1)_C\) (which by construction produces the cocartesian map \((\theta, 1)\)) is a regular epimorphism. Thus there is a unique \(\xi: H \to W\) making the diagram commute. \(\square\)

When \(\mathbb{C}\) is semi-abelian, this result specifies the universal property of the construction described in [20, Corollary 3.3]. Of course, when the category \(\mathbb{C}\) is additive, any map is central, and the square in question in the proposition above is necessarily a pushout. Whence the following:

**Corollary 2.4.** Let \(\mathbb{A}\) be a finitely complete, efficiently regular, additive category. Then a kernel map admits a pushout along any map and this pushout is itself a kernel map.

3. The Baer sums

According to the general definition given in [7], the Baer sums of two extensions \(g\) and \(h\) with abelian kernel relation and same direction \((d, s): D \rightrightarrows C\) is the codomain of the cocartesian map above the map \((1, 1)_C: D \times_C D \to D\) in \(\text{Sax } C\) (which is nothing but the internal abelian group operation \(+\) on \((d, s): D \rightrightarrows C\) having their product \(g \times_C h\) as domain. Denote by \(a: A \rightrightarrows D\) a kernel of \(d\). Obviously a kernel of the map \((1, 1)_C\) is \((-a, a): A \rightrightarrows D \times_C D\), and certainly we have

\[ (-a, a) = a \times a.(-1, 1): A \rightrightarrows A \times A \rightrightarrows D \times_C D. \]
So according to Proposition 2.6, the Baer sum of the two extensions \( g \) and \( h \) is given by the right-hand side extension produced by the following 3 × 3 diagram, since \((-\alpha, \beta) = \alpha \times \beta.(-1, 1)\): 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 \xrightarrow{(1,1)} A \times A \xrightarrow{\alpha \times \beta} A \xrightarrow{\gamma} 1 \\
\downarrow & \downarrow & \downarrow \\
1 \xrightarrow{(-\alpha, \beta)} G \times_C H \xrightarrow{\tilde{q}} G \otimes_C H \xrightarrow{g \otimes_C h} 1 \\
\downarrow & \downarrow & \downarrow \\
1 \xrightarrow{1_C} C \xrightarrow{1_C} C \xrightarrow{1} 1
\end{array}
\]

The associativity of this tensor product and the fact that the direction \((d, s) : D \rightleftarrows C\) is the unit of this tensor product are consequences of [7, Theorem 9]. Let us show that the “inverse” of \( g \) is its dual \( g^*\):

**Proposition 3.1.** The Baer sum of an extension \( g : G \to C \) with its dual \( g^* \) is its direction \((d, s) : D \rightleftarrows C\), i.e. the unit element of the tensor product.

**Proof.** Assuming that \((-\alpha) \times \alpha.(-1, 1) = (\alpha, \alpha) = s_0.\alpha\), consider the following 3 × 3 diagram, where \( G \times_C G \) is nothing but \( R[g] \) by definition:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 \xrightarrow{(-1,1)} A \xrightarrow{s_0.\alpha} R[g] \xrightarrow{d} D \xrightarrow{1} 1 \\
\downarrow & \downarrow & \downarrow \\
1 \xrightarrow{1} C \xrightarrow{1_C} C \xrightarrow{1} 1
\end{array}
\]

Then \( d \) is the unique map which completes the diagram. According to the previous construction of the Baer sum, this achieves the proposition. \( \square \)
Consider now any split exact sequence with $d$ having an abelian kernel relation, i.e. any abelian group in $\mathbb{C}/\mathbb{C}$ or any object of $\text{Sax} \mathbb{C}$:

$$1 \longrightarrow A \stackrel{\alpha}{\longrightarrow} D \stackrel{d}{\longrightarrow} C \longrightarrow 1$$

Then clearly the Baer sum gives an abelian group structure to the set $\text{Ext}_d(C, A)$ of the isomorphism classes of extensions with abelian kernel relation and fixed direction $d$.

### 3.1. The categories $\text{GpTop}$ and $\text{GpHaus}$

Let us now have a special look at the categories $\text{GpTop}$ and $\text{GpHaus}$ of topological and Hausdorff groups which are pointed protomodular and efficiently regular. Suppose $\mathbb{C}$ and $\mathbb{A}$ are two topological (respectively Hausdorff) groups with $\mathbb{A}$ abelian, and $\phi: \mathbb{C} \to \text{Aut} \mathbb{A}$ is a group action such that the map $\mathbb{C} \times \mathbb{A} \to \mathbb{A}$ associating $\phi_c(a)$ with $(c, a)$ is continuous. We shall say that such a group action is continuous. It was classically noticed, see [5, Proposition 27] for instance, that the product topology makes the semi-direct product $\mathbb{C} \ltimes_{\phi} \mathbb{A}$ a topological group which is clearly Hausdorff when both $\mathbb{C}$ and $\mathbb{A}$ are Hausdorff. Actually this determines the continuous projection $\mathbb{C} \ltimes_{\phi} \mathbb{A} \to \mathbb{C}$ as an internal abelian group inside the slice category $\text{GpTop}/\mathbb{C}$ (respectively $\text{GpHaus}/\mathbb{C}$).

Now let $g: \mathbb{G} \to \mathbb{C}$ be a continuous extension with abelian kernel relation in $\text{GpTop}$ (respectively $\text{GpHaus}$). The category $\text{GpTop}$ (respectively $\text{GpHaus}$) being not only protomodular, but also strongly protomodular, as any category $\text{Gp} \mathbb{E}$ with $\mathbb{E}$ finitely complete, see [8], this is equivalent to say that the extension $g$ has an abelian kernel $\mathbb{A}$. Let $\phi: \mathbb{C} \to \text{Aut} \mathbb{A}$ be the group action associated with this continuous extension. It is a continuous action.

The direction of such a continuous extension $g$ coincides with the projection $\mathbb{C} \ltimes_{\phi} \mathbb{A} \to \mathbb{C}$, where the semi-direct product $\mathbb{C} \ltimes_{\phi} \mathbb{A}$ is endowed with the product topology which makes it a topological group and the projection $\mathbb{C} \ltimes_{\phi} \mathbb{A} \to \mathbb{C}$ continuous. The Baer sum construction described above gives a group structure to the set $\text{TOPext}(\mathbb{C}, \mathbb{A}, \phi)$ of continuous extensions of $\mathbb{A}$ by $\mathbb{C}$ with operator $\phi$, which says that the classical Baer sum construction $g \times C h : \mathbb{G} \otimes \mathbb{C} \mathbb{H} \to \mathbb{C}$ of two continuous extensions $(g, h)$ is still continuous, provided that its domain $\mathbb{G} \otimes \mathbb{C} \mathbb{H}$:

$$1 \longrightarrow A \overset{(-\alpha, \beta)}{\longrightarrow} G \times_{\mathbb{C}} H \overset{\tilde{q}}{\longrightarrow} G \otimes_{\mathbb{C}} \mathbb{H} \longrightarrow 1$$

is endowed with the quotient topology ($G \times_{\mathbb{C}} \mathbb{H}$ having the topology of the fibered product). The Hausdorff case is a particular case since, given any continuous extension:

$$1 \longrightarrow A \overset{\alpha}{\longrightarrow} G \overset{g}{\longrightarrow} \mathbb{C} \longrightarrow 1$$

the group $\mathbb{G}$ is clearly Hausdorff as soon as both $\mathbb{A}$ and $\mathbb{C}$ are so.

In particular this makes operate the classical additive Baer sum techniques in the additive and efficiently regular (but absolutely not abelian) categories $\text{AbTop}$ and $\text{AbHaus}$ of topological and Hausdorff abelian groups (thanks to Corollary 2.4).
3.2. The five terms exact sequence

Actually the Baer sum construction has some functorial properties.

**Proposition 3.2.** Pulling back along any map \( f : C' \to C \) determines a group homomorphism \( f^* : \text{Ext}^d(C, A) \to \text{Ext}^{f^*(d)} (C', A) \). Its kernel is the subgroup of \( \text{Ext}^d(C, A) \) whose elements are those extensions \( g \) such that there is a factorization \( t \):

\[
\begin{array}{c}
G \\
\downarrow \quad g \\
C' \\[2pt]
\rightarrow \\
\downarrow f \\
C
\end{array}
\]

**Proof.** Pulling back along any map \( f : C' \to C \) determines a left exact functor \( f^* : \mathbb{C}/C \to \mathbb{C}/C' \) which preserves the regular epimorphism. Accordingly it provides a left exact functor \( f^* : \text{Abx} \mathbb{C} \to \text{Abx} \mathbb{C}' \) which preserves the regular epimorphisms and the direction. This last functor preserves the cocartesian maps since any map is cocartesian. Accordingly this functor preserves (up to isomorphisms) the Baer sums and, in turn, provides a group homomorphism \( f^* : \text{Ext}^d(C, A) \to \text{Ext}^{f^*(d)} (C', A) \). The extension \( f^*(g) \) is trivial if and only if it is split, i.e. if and only if there is a factorization \( t \). \( \square \)

When we fix an extension \( g \), we get another interesting group homomorphism:

**Proposition 3.3.** Given any split extension with abelian kernel relation \( (d, s) : D \rightrightarrows C \) (with kernel \( B \)), the map \( \Gamma_g : \text{Hom}_{\text{Sax}} C(d_C(G), D) \to \text{Ext}^d(C, B) \) associating with any map \( \theta \in \text{Hom}_{\text{Sax}} C(d_C(G), D) \) the codomain of the cocartesian map \( \tilde{\theta} \), with fixed domain \( g \), above \( \theta \) is a group homomorphism. Its kernel is the image of the group \( \text{Hom}_{\text{Abx} \mathbb{C}} (g, d) \) (this is a group since \( d \) is a group object in \( \text{Abx} \mathbb{C} \)) by the direction functor.

**Proof.** The map \( \theta + \theta' \) is given by the following composition in \( \text{Sax} \mathbb{C} \):

\[
d_C(G) \xrightarrow{(\theta, \theta')} D \times_C D \xrightarrow{\cdot} D.
\]

Accordingly the cocartesian map above it is necessarily the following:

\[
G \xrightarrow{(\tilde{\theta}, \tilde{\theta}')} \Gamma_g \theta \times_C \Gamma_g \theta' \to \Gamma_g \theta \otimes \Gamma_g \theta',
\]

where \( \tilde{\theta} \) and \( \tilde{\theta}' \) are the cocartesian maps above \( \theta \) and \( \theta' \). Consequently \( \Gamma_g \) is a group homomorphism. Saying that \( \Gamma_g(\theta) = 0 \) is saying that its associated cocartesian map \( \tilde{\theta} : G \to D \) has \( D \) as codomain, which exactly means that \( \theta \) is the direction of a map:

\[
\begin{array}{c}
G \\
\downarrow \quad \tilde{\theta} \\
D \\[2pt]
\rightarrow \\
\downarrow g \\
C \\[2pt]
\leftarrow \\
\downarrow f \\
\end{array}
\]

\( \square \)
Actually the previous group homomorphisms fit in with an exact sequence. Indeed let the following extension with abelian kernel relation be given:

$$1 \xrightarrow{} A \xrightarrow{\alpha} G \xrightarrow{g} C \xrightarrow{} 1$$

as well the following split extension with abelian kernel relation:

$$1 \xrightarrow{} B \xrightarrow{\hat{\beta}} D \xrightarrow{d} C \xrightarrow{} 1.$$

**Theorem 3.1.** Suppose \( \mathbb{C} \) is efficiently homological. There is a five terms exact sequence of abelian groups:

$$0 \xrightarrow{} \text{Hom}_{\text{Abx}} \mathbb{C}(1_C, d) \xrightarrow{g^*} \text{Hom}_{\text{Abx}} \mathbb{G}(1_G, g^*(d)) \xrightarrow{\Gamma_g} \text{Hom}_{\text{Sax}} \mathbb{C}(d_C(g), d) \xrightarrow{} \text{Ext}_d(C, B)$$

where the second homomorphism is given by the direction functor (the group \( \text{Hom}_{\text{Abx}} \mathbb{G}(1_G, g^*(d)) \) being nothing but the group \( \text{Hom}_{\text{Abx}} \mathbb{C}(g, d) \)).

**Proof.** 1. The map \( \text{Hom}_{\text{Abx}} \mathbb{C}(1_C, d) \xrightarrow{g^*} \text{Hom}_{\text{Abx}} \mathbb{G}(1_G, g^*(d)) \) is a group homomorphism since pulling back is left exact and injective since \( g \) is a regular epimorphism.

2. The map \( \delta : \text{Hom}_{\text{Abx}} \mathbb{G}(1_G, g^*(d)) = \text{Hom}_{\text{Abx}} \mathbb{C}(g, d) \xrightarrow{\Gamma_g} \text{Hom}_{\text{Sax}} \mathbb{C}(d_C(G), d) \) associates with any map \( \chi : G \xrightarrow{} D \) its direction \( d_C(\chi) : d_C(G) \xrightarrow{} D \). It is a group homomorphism since \( t + t' = +.(t, t') : G \xrightarrow{} D \times_C D \xrightarrow{} D \) has the following map as direction: \( +.(d_C(t), d_C(t')) : d_C(G) \xrightarrow{} D \times_C D \xrightarrow{} D \). This is nothing but \( d_C(t) + d_C(t') \). According to Lemma 2.3, the kernel of \( \delta \) is given by the maps \( t = e.g : G \xrightarrow{} D \) where \( e \) is a global element of \( d : D \xrightarrow{} C \). This realizes precisely the image of the group \( \text{Hom}_{\text{Abx}} \mathbb{C}(1_C, d) \).

3. By Proposition 3.3, we know that \( \Gamma_g \) is a group homomorphism whose kernel is the image by \( \delta \) of the group \( \text{Hom}_{\text{Abx}} \mathbb{C}(g, d) \).

4. By Proposition 3.2, we know that \( g^* : \text{Ext}_d(C, B) \xrightarrow{} \text{Ext}_{g^*(d)}(G, B) \) is a group homomorphism whose kernel is the subgroup of \( \text{Ext}_d(C, B) \) whose elements are those extensions \( h \) such that there is a factorization \( \chi : H \xrightarrow{} G \xrightarrow{g} C \)

This precisely means that its direction \( d_C(\chi) : d_C(G) \xrightarrow{} D \) is in the group \( \text{Hom}_{\text{Sax}} \mathbb{C}(d_C(g), d) \), and consequently that \( h \) belongs to the image of the homomorphism \( \Gamma_g \). □
When \( \mathbb{C} \) is, moreover, finitely cocomplete, we can extend this result to any extension:

\[
1 \longrightarrow K \overset{k}{\longrightarrow} G \overset{g}{\longrightarrow} C \longrightarrow 1.
\]

The reason is that, \( \mathbb{C} \) being finitely cocomplete, we can associate with this extension, in a universal way, an extension \( g_\alpha \) with abelian kernel relation, see [2, Theorem 2.8.9]:

\[
\begin{array}{c}
G \overset{\eta}{\longrightarrow} G_\alpha \\
\downarrow g \\
C \overset{g_\alpha}{\longleftarrow}
\end{array}
\]

**Corollary 3.1.** Suppose \( \mathbb{C} \) is efficiently homological and finitely cocomplete. Then with any extension \( g \) as above and any split extension with abelian kernel relation:

\[
1 \longrightarrow B \overset{\hat{\beta}}{\longrightarrow} D \overset{d}{\longrightarrow} C \longrightarrow 1
\]

there is associated a five terms exact sequence of abelian groups:

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_{\text{Ab}\times C}(1_C, d) \overset{g^*}{\longrightarrow} \text{Hom}_{\text{Ab}\times G}(1_G, g^*(d)) \longrightarrow \text{Hom}_{\text{sax}\times C}(d_C(g_\alpha), d) \\
& & \text{Ext}_d(C, B) \longleftarrow \text{Ext}^{g_\alpha(d)}(G, B)
\end{array}
\]

**Proof.** We have \( \text{Hom}_{\text{Ab}\times G}(1_G, g^*(d)) = \text{Hom}_{\mathbb{C}/\mathbb{C}}(g, d) = \text{Hom}_{\text{Ab}\times C}(g_\alpha, d) \) by definition of \( g_\alpha \) since \( d \) has an abelian kernel relation. Accordingly, the four first terms of this sequence are exactly the ones of the sequence determined, thanks to the previous theorem, by the extension \( g_\alpha \) with abelian kernel relation. Moreover the kernel of \( g^* : \text{Ext}_d(C, B) \to \text{Ext}^{g_\alpha(d)}(G, B) \) is the same as the kernel of \( g_\alpha^* : \text{Ext}_d(C, B) \to \text{Ext}^{g_\alpha(d)}(G, B) \) since, once again, we have \( \text{Hom}_{\mathbb{C}/\mathbb{C}}(g, h) = \text{Hom}_{\text{Ab}\times C}(g_\alpha, h) \), for any extension \( h \) with abelian kernel relation.

**Remark.** 1. In the categories \( Gp \) of groups or \( R\text{-Lie} \) of Lie \( R \)-algebras, this five terms exact sequence is the classical one, see for instance [22, VI, Theorem 8.1] for groups and [22, VII, Theorem 3.2] for Lie \( R \)-algebras. This observation confirms that the notion of efficiently homological category is the root of the classically known (but not explained up to now) parallelism of treatment of homology theory for groups and Lie \( R \)-algebras.

2. The homological results of this section obviously apply to the topological material of the previous section.

**References**