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Annihilators of quadratic and bilinear forms over fields of characteristic two

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Abstract

Let F be a field with $2 = 0$, $W(F)$ the Witt ring of symmetric bilinear forms over F and $W_q(F)$ the $W(F)$-module of quadratic forms over F. Let $I_F \subset W(F)$ be the maximal ideal. We compute explicitly in I_F^m and $I^m W_q(F)$ the annihilators of n-fold bilinear and quadratic Pfister forms, thereby answering positively, in the case $2 = 0$, certain conjectures stated by Krüskemper in [M. Krüskemper, On annihilators in graded Witt rings and in Milnor's K-theory, in: B. Jacob et al. (Eds.), Recent Advances in Real Algebraic Geometry and Quadratic Forms, in: Contemp. Math., vol. 155, 1994, pp. 307–320].

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1. Introduction

Let F be a field with $2 = 0$. We denote by $W(F)$ the Witt ring of symmetric nonsingular bilinear forms over F and by $W_q(F)$ the $W(F)$-module of nonsingular quadratic forms over F (see [3,4,11]).

For $a_i \in F^* = F - \{0\}$, $1 \leqslant i \leqslant n$, we denote by $\langle a_1, \ldots, a_n \rangle$ the bilinear form with diagonal Gramm matrix and entries a_i on the diagonal. The quadratic form $x^2 + xy + ay^2$,

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$a \in F$, is denoted by $[1, a]$. The maximal ideal I_F of $W(F)$ is additively generated by the forms $\langle 1, a\rangle = \langle\langle a\rangle\rangle$, $a \in F^*$, so that the powers I_F^n, $n \geqslant 1$, are additively generated by the n-fold bilinear forms $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, a_1\rangle \cdots \langle 1, a_n\rangle$, $a_i \in F^*$. The submodules $I^n W_q(F)$, $n \geqslant 1$, are generated by the n-fold quadratic Pfister forms $\langle\langle a_1, \dots, a_n; a|] = \langle\langle a_1, \dots, a_n\rangle\rangle \cdot [1, a]$, $a_i \in F^*$, $a \in F$.

We have the filtrations $W(F) \supset I_F \supset I_F^2 \supset \cdots$ and $W_q(F) \supset IW_q(F) \supset \cdots$. The graded objects I_F^n/I_F^{n+1} and $I^n W_q(F)/I^{n+1} W_q(F)$ are denoted by $\bar{I}_F^n$ respectively $\bar{I}^n W_q(F)$. I_F^0 means $W(F)$ and $I^0 W_q(F)$ means $W_q(F)$.

In this paper we will study annihilators of n-fold Pfister forms. Let $x = \langle\langle a_1, \dots, a_n\rangle\rangle$ be an n-fold bilinear Pfister form. For any $m \geqslant 0$ we set

$$\begin{aligned} \mathrm{annb}_m(x) &= \{y \in I_F^m \mid xy = 0\}, \\ \mathrm{annq}_m(x) &= \{y \in I^m W_q(F) \mid xy = 0\}, \\ \overline{\mathrm{ann}}\mathrm{b}_m(x) &= \{\bar{y} \in \bar{I}_F^m \mid x\bar{y} = 0\}, \\ \overline{\mathrm{ann}}\mathrm{q}_m(x) &= \{\bar{y} \in \bar{I}^m W_q(F) \mid x\bar{y} = 0\}. \end{aligned}$$

If $x = \langle\langle a_1, \dots, a_n; a|]$ is a quadratic n-fold Pfister form, we set

$$\begin{aligned} \mathrm{annb}_m(x) &= \{y \in I_F^m \mid yx = 0\}, \\ \overline{\mathrm{ann}}\mathrm{b}_m(x) &= \{\bar{y} \in \bar{I}_F^m \mid \bar{y}x = 0\}. \end{aligned}$$

The main results of this paper are contained in the following two theorems.

1.1. Theorem.

(i) *Let $x = \langle\langle a_1, \dots, a_n\rangle\rangle$ be a bilinear n-fold Pfister form over F with $x \neq 0$ in $W(F)$. Then for any $m \geqslant 1$*

$$\begin{aligned} \overline{\mathrm{ann}}\mathrm{b}_m(x) &= \overline{\mathrm{ann}}\mathrm{b}_1(x)\bar{I}_F^{m-1}, \\ \overline{\mathrm{ann}}\mathrm{q}_m(x) &= \bar{I}_F^m \cdot \overline{\mathrm{ann}}\mathrm{q}_0(x) + \overline{\mathrm{ann}}\mathrm{b}_1(x)\bar{I}^{m-1} W_q(F). \end{aligned}$$

(ii) *Let $x = \langle\langle a_1, \dots, a_n; a|]$ be a quadratic n-fold Pfister form over F with $x \neq 0$ in $W_q(F)$. Then for $m \geqslant 1$*

$$\overline{\mathrm{ann}}\mathrm{b}_m(x) = \overline{\mathrm{ann}}\mathrm{b}_1(x)\bar{I}_F^{m-1}.$$

And the much stronger:

1.2. Theorem.

(i) *Let $x = \langle\langle a_1, \dots, a_n\rangle\rangle$ be a bilinear n-fold Pfister form over F with $x \neq 0$ in $W(F)$. Then for any $m \geqslant 1$*

$$\mathrm{annb}_m(x) = \mathrm{annb}_1(x) I_F^{m-1},$$
$$\mathrm{annq}_m(x) = I_F^m \cdot \mathrm{annq}_0(x) + \mathrm{annb}_1(x) I^{m-1} W_q(F).$$

(ii) *Let* $x = \langle\langle a_1, \ldots, a_n; a]]$ *be a quadratic* n*-fold Pfister form over* F *with* $x \neq 0$ *in* $W_q(F)$. *Then for* $m \geqslant 1$

$$\mathrm{annb}_m(x) = \mathrm{annb}_1(x) I_F^{m-1}.$$

These results were conjectured by M. Krüskemper in [9] for fields of characteristic different from 2. Recently in [11] Orlov, Vishik and Voevodsky announced the positive answer of Krüskemper's conjecture for the graded Witt ring of a field of characteristic $\neq 2$. Based on these results, Arason and Elman proved in [1] the ungraded version of this conjecture in the case $2 \neq 0$.

The proof of Theorem 1.1 will be given in Section 4 and it is based on Kato's correspondence between quadratic or symmetric bilinear forms and differential forms over F. We will shortly explain this correspondence in Section 3 (see [4,7]) and prove there some technical results needed in the proof of Theorem 1.1. In Section 2 we show that Theorem 1.2 follows from Theorem 1.1.

The terminology used in this paper is standard and we refer to [4,10,12] for details on basic facts needed in the paper.

2. Proof of Theorem 1.2

We will assume Theorem 1.1 and derive from it Theorem 1.2. Recall that a 2-basis of a field F of characteristic 2 is a set $\mathcal{B} = \{b_i \mid i \in I\} \subset F$ such that the elements $\prod_{i \in I} b_i^{\varepsilon_i}$, $\varepsilon_i \in \{0, 1\}$ and only finitely many $\varepsilon_i \neq 0$, form a basis of F over F^2. An n-fold bilinear Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ over F is $\neq 0$ in $W(F)$, i.e. it is anisotropic over F, if and only if $\{a_1, \ldots, a_n\}$ are part of a 2-basis of F (i.e. 2-independent). In this case the subfield $F^2(a_1, \ldots, a_n)$ of F consists of all elements of F represented by the form $\langle\langle a_1, \ldots, a_n\rangle\rangle$. The elements of F represented by the pure part $\langle\langle a_1, \ldots, a_n\rangle\rangle'$ of $\langle\langle a_1, \ldots, a_n\rangle\rangle$ form a subgroup denoted by $F^2(a_1, \ldots, a_n)'$. Recall that $\langle\langle a_1, \ldots, a_n\rangle\rangle'$ is defined by $\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle 1\rangle \perp \langle\langle a_1, \ldots, a_n\rangle\rangle'$. Moreover if F has a finite 2-basis, say $\{b_1, \ldots, b_N\}$, then $I_F^m = 0$ for all $m \geqslant N + 1$ (see [10]).

We will need the following

2.1. Lemma.

(i) *Let* x *be an* n*-fold bilinear Pfister form,* $x \neq 0$, *and* $z \in I_F$ *such that* $zx \in I_F^{n+2}$, *i.e.* $\bar{z} \in \overline{\mathrm{ann}}\mathrm{b}_1(x)$. *Then*

$$z = z_0 + w$$

with $z_0 \in I_F$, $z_0 x = 0$ *and* $w \in I_F^2$.

(ii) *Let x be an n-fold bilinear Pfister form, $x \neq 0$, and $z \in W_q(F)$ with $xz \in I^{n+1}W_q(F)$. Then*

$$z = z_0 + w$$

with $z_0 \in W_q(F)$, $xz_0 = 0$ and $w \in IW_q(F)$.

Proof. (i) For any $z \in I_F$ we can write $z = \langle 1, d\rangle + w$ with $d = \det(z)$ and $w \in I_F^2$. Then $xz \in I_F^{n+2}$ implies $\langle 1, d\rangle x \in I_F^{n+2}$, and since $\langle 1, d\rangle x$ is $(n+1)$-fold Pfister form, it follows $\langle 1, d\rangle x = 0$ in $W(F)$.

(ii) Any $z \in W_q(F)$ can be written as

$$z = [1, d] + w$$

with $d = \mathrm{Arf}(z) \in F$ and $w \in IW_q(F)$ (see [12]). Here Arf(z) means the Arf invariant of the form z (see [3] or [12]). From $xz, xw \in I^{n+1}W_q(F)$, it follows $x[1, d] \in I^{n+1}W_q(F)$ and hence $x[1, d] = 0$. □

Let us now prove Theorem 1.2. We assume first that F has a finite 2-basis, i.e. $I_F^{N+1} = 0$ for some integer N. Let $x \neq 0$ (in $W(F)$) be an n-fold bilinear Pfister form. The contentions $\supseteq$ in (i) (and (ii)) are obvious. Let $y \in \mathrm{annb}_m(x)$, i.e. $y \in I_F^m$, $yx = 0$. Hence $\bar{y} \in \overline{\mathrm{ann}}\mathrm{b}_m(x)$ and Theorem 1.1 implies $\bar{y} = \sum \bar{z}_i \bar{y}_{i,0}$ with $\bar{z}_i \in \overline{\mathrm{ann}}\mathrm{b}_1(x)$, $y_{i,0} \in I_F^{m-1}$. Then $y - \sum z_i y_{i,0} \in I_F^{m+1}$. Using Lemma 2.1(i) we can write $z_i = z_{i,0} + w_i$ with $z_{i,0} \in \mathrm{annb}_1(x)$ and $w_i \in I_F^2$. Then $y_1 = y - \sum z_{i,0} y_{i,0} \in I_F^{m+1}$ and moreover $y_1 x = 0$. The same argument implies $y_1 - \sum z_{i,1} y_{i,1} \in I_F^{m+2}$ with elements $z_{i,1} \in \mathrm{annb}_1(x)$, $y_{i,1} \in I_F^m$. Iterating this process we obtain, for any $k \geqslant 0$, elements $z_{i,l} \in \mathrm{annb}_1(x)$ and $y_{i,l} \in I_F^{m+l-1}$, $0 \leqslant l \leqslant k$ such that $y - \sum_{i,l} z_{i,l} y_{i,l} \in I_F^{m+k}$. Choosing $k \geqslant N + 1 - m$ we obtain $y = \sum_{i,l} z_{i,l} y_{i,l} \in \mathrm{annb}_1(x) I_F^{m-1}$, since $I_F^{N+1} = 0$.

Let now $y \in \mathrm{annq}_m(x)$, i.e. $y \in I^m W_q(F)$ with $xy = 0$. Theorem 1.1 implies $\bar{y} = \sum \bar{y}_i \bar{z}_i + \sum \bar{u}_j \bar{v}_j$ with $\bar{y}_i \in \bar{I}_F^m$, $\bar{z}_i \in \overline{\mathrm{ann}}\mathrm{q}_0(x)$, $\bar{u}_j \in \overline{\mathrm{ann}}\mathrm{b}_1(x)$, $\bar{v}_j \in \bar{I}^{m-1}W_q(F)$. Hence $y - \sum y_i z_i - \sum u_j v_j \in I^{m+1}W_q(F)$. Using Lemma 2.1 we can find $z_{i,0} \in \mathrm{annq}_0(x)$, $u_{j,0} \in \mathrm{annb}_1(x)$ such that $z_i = z_{i,0} + w_i$, $w_i \in IW_q(F)$ and $u_j = u_{j,0} + t_j$, $t_j \in I_F^2$. We obtain

$$y_1 = y - \sum y_i z_{i,0} - \sum u_{j,0} v_j \in I^{m+1} W_q(F)$$

with $y_1 x = 0$. Iterating this procedure we obtain after $k \geqslant N + 1 - m$ steps that

$$y \in I_F^m \mathrm{annq}_0(x) + \mathrm{annb}_1(x) I^{m-1} W_q(F).$$

The proof of part (ii) of Theorem 1.2 is similar and we omit the details. Thus we have proved Theorem 1.2 in the case $I_F^{N+1} = 0$ for some N. Let us now consider the general case.

Let x a bilinear n-fold Pfister form over F, $x \neq 0$ in $W(F)$ and take $y \in \mathrm{annb}_m(x)$, i.e. $y \in I_F^m$ with $yx = 0$. This relation involves only finitely many elements of F, say the finite set $S \subset F$. Let $F_0 := \mathbb{F}(S) \subset F$, where $\mathbb{F} =$ prime field contained in F. Then there exist an n-fold bilinear Pfister form x_0 over F_0 and $y_0 \in I_{F_0}^m$ such that $x = x_0 \otimes F$, $y = y_0 \otimes F$ and $x_0 y_0 = 0$. Since F_0 has a finite 2-basis, we obtain from the first part of the proof that $y_0 \in \mathrm{annb}_1(x_0) I_{F_0}^{m-1}$ and hence $y \in \mathrm{annb}_1(x) I_F^{m-1}$. The same argument applies for the other assertions in Theorem 1.2 and this concludes the proof of Theorem 1.2. □

2.2. Remark. If x is a bilinear n-fold Pfister form over F, then one can describe explicitly the annihilators $\mathrm{annb}_1(x) \subset W(F)$ and $\mathrm{annq}_0(x) \subset W_q(F)$ as follows

$$\mathrm{annb}_1(x) = \sum_{d \in D_F(x)^*} W(F)\langle 1, d\rangle, \tag{2.3}$$

$$\mathrm{annq}_0(x) = \sum_{d \in D_F(x)} W(F)[1, d]. \tag{2.4}$$

Here $D_F(z)$ denotes the set in F of elements represented by the form z. The result (2.3) is shown in [6] and (2.4) in [5]. If x denotes now a quadratic n-fold Pfister form over F, $x \neq 0$ in $W_q(F)$, then (see [8])

$$\mathrm{annb}_1(x) = \sum_{d \in D_F(x)^*} W(F)\langle 1, d\rangle. \tag{2.5}$$

In Section 4 we will give an independent proof of these facts based on Kato's correspondence (see Section 3) and on the arguments used in this section.

3. Quadratic, symmetric bilinear and differential forms

In this section we will briefly describe Kato's correspondence between quadratic, bilinear and differential forms over a field F with $2 = 0$ and prove a technical result needed in the proof of Theorem 1.1 (see [2,4,7]).

Let $\Omega_F^1 = F\,\mathrm{d}F$ be the F-space of 1-differential forms generated over F by the symbols $\mathrm{d}a$, $a \in F$, with $\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b$, $\mathrm{d}(ab) = a\,\mathrm{d}b + b\,\mathrm{d}a$. For any $n \geqslant 1$ set $\Omega_F^n = \bigwedge^n \Omega_F^1$ and let $\mathrm{d}: \Omega_F^n \to \Omega_F^{n+1}$ be the differential operator $\mathrm{d}(x\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = \mathrm{d}x \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, where $\wedge$ denotes exterior multiplication. For example if $c \in F$ is represented in a 2-basis $\mathcal{B} = \{b_i \mid i \in I\}$ as $c = \sum_\varepsilon c_\varepsilon^2 b^\varepsilon$, where $\varepsilon = (\varepsilon_i)$, $\varepsilon_i \in \{0, 1\}$ and almost all $\varepsilon_i = 0$, we have $\mathrm{d}c = \sum_i D_i(c)\,\mathrm{d}b_i$ where $D_i(c)$ is the partial derivative of c with respect to b_i, i.e. $D_i(c) = b_i^{-1} \sum_{\varepsilon, \varepsilon_i = 1} c_\varepsilon^2 b^\varepsilon$ (see [1]).

Let $\wp: \Omega_F^n \to \Omega_F^n / \mathrm{d}\Omega_F^{n-1}$ be the Artin–Schreier operator defined on generators by

$$\wp\left(x \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}\right) = \left(x^2 - x\right) \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \quad \mathrm{mod}\ \mathrm{d}\Omega_F^{n-1}$$

and denote by $\nu_F(n)$ its kernel and by $H^{n+1}(F)$ its cokernel (see [1]). In [7] it is shown that there are natural isomorphisms $\alpha : \nu_F(n) \simeq \bar{I}_F^n$ and $\beta : H^{n+1}(F) \simeq \bar{I}^n W_q(F)$ given on generators by $\alpha(\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}) = \langle\langle x_1, \ldots, x_n \rangle\rangle \bmod I_F^{n+1}$ and

$$\beta\left(\overline{x\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}}\right) = \langle\langle x_1, \ldots, x_n; x|] \mod I^{n+1}W_q(F).$$

The fact that $\nu_F(n)$ is additively generated by the pure logarithmic forms $\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$ follows from a result of Kato which we explain now. Let us fix a 2-basis $\mathcal{B}$ of F, $\mathcal{B} = \{b_i \mid i \in I\}$, and endow I with a total ordering. For any $j \in I$, let F_j, respectively $F_{<j}$, be the subfields of F generated over F^2 by b_i, $i \leqslant j$, respectively b_i, $i < j$. For any $n \geqslant 1$ let Σ_n be the set of maps $\alpha : \{1, \ldots, n\} \to I$ such that $\alpha(i) < \alpha(j)$ whenever $1 \leqslant i < j \leqslant n$, and endow Σ_n with the lexicographic ordering.

We obtain a filtration of Ω_F^n given by the subspaces $\Omega_{F,\alpha}^n$, respectively $\Omega_{F,<\alpha}^n$, which are generated by the elements $\frac{\mathrm{d}b_\beta}{b_\beta} = \frac{\mathrm{d}b_{\beta(1)}}{b_{\beta(1)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\beta(n)}}{b_{\beta(n)}}$ with $\beta \leqslant \alpha$, respectively $\beta < \alpha$. An important result of Kato, named here as Kato's lemma, asserts that for any $\alpha \in \Sigma_n$, $y \in F$, if $\wp(y\frac{\mathrm{d}b_\alpha}{b_\alpha}) \in \Omega_{F,<\alpha}^n + \mathrm{d}\,\Omega_F^{n-1}$, then there exist $v \in \Omega_{F,<\alpha}^n$ and $a_i \in F_{\alpha(i)}^*$, $1 \leqslant i \leqslant n$, such that $y\frac{\mathrm{d}b_\alpha}{b_\alpha} = v + \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}$ (see [6]). This implies that any $u \in \Omega_{F,\alpha}^n$ satisfying $\wp(u) \in \mathrm{d}\,\Omega_F^{n-1}$, can be written as

$$u = \sum_{\gamma \leqslant \alpha} \frac{\mathrm{d}a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}a_{\gamma(n)}}{a_{\gamma(n)}} \tag{3.1}$$

with $a_{\gamma(i)} \in F_{\gamma(i)} \setminus F_{<\gamma(i)}$. Then the following result will be used in Section 4 during the proof of Theorem 1.1.

3.2. Lemma. *Let $\mathcal{B} = \{b_i \mid i \in I\}$ be a 2-basis of F with a given ordering on I. Let $\alpha \in \Sigma_n$ and $\sum_{\gamma \leqslant \alpha} c_\gamma \frac{\mathrm{d}b_\gamma}{b_\gamma}$ be a differential form with $c_\alpha \neq 0$ such that $\sum_{\gamma \leqslant \alpha} c_\gamma \frac{\mathrm{d}b_\gamma}{b_\gamma} \in \mathrm{d}\,\Omega_F^{n-1}$. Then there exist elements $M_i \in F_{<\alpha(i)}$, $1 \leqslant i \leqslant n$, such that*

$$c_\alpha = b_{\alpha(1)}M_1 + \cdots + b_{\alpha(n)}M_n.$$

Proof. Let $k \in I$ be the index with $c_\alpha \in F_k \setminus F_{<k}$. We claim that $k = \alpha(i)$ for some $1 \leqslant i \leqslant n$. Otherwise we have $k > \alpha(n)$ or $k < \alpha(1)$ or $\alpha(j) < k < \alpha(j+1)$ for some $1 \leqslant j \leqslant n$. From the choice of k we have $c_\alpha = b_k A + B$ with $A, B \in F_{<k}$, $A \neq 0$. Then

$$\mathrm{d}t = (b_k A + B)\frac{\mathrm{d}b_\alpha}{b_\alpha} + \sum_{\gamma < \alpha} c_\gamma \frac{\mathrm{d}b_\gamma}{b_\gamma}$$

and applying the operator d to this form, since $\mathrm{d}^2 = 0$, we get

$$b_k A\frac{\mathrm{d}b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}b_k}{b_k} + b_k A\frac{\mathrm{d}b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}A}{A} + B\frac{\mathrm{d}b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}B}{B} + \sum_{\gamma < \alpha}\sum_{i \in I} b_i D_i(c_\gamma)\frac{\mathrm{d}b_\gamma}{b_\gamma} \wedge \frac{\mathrm{d}b_i}{b_i} = 0,$$

where $D_i(c_\gamma)$, as mentioned above, denotes the partial derivative of c_γ with respect to b_i. Looking at the coefficient of $\frac{\mathrm{d}b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}b_k}{b_k}$ we obtain

$$b_k A = \sum_{(\alpha,k)=(\gamma_i,i)} b_i D_i(c_\gamma),$$

where (α, k) respectively (γ_i, i) denotes the unique $\lambda \in \Sigma_{n+1}$ with $\mathrm{Im}(\lambda) = \mathrm{Im}(\alpha) \cup \{k\}$ respectively $\mathrm{Im}(\lambda) = \mathrm{Im}(\gamma_i) \cup \{i\}$. Since for those i we have $i > k$, $A \in F_{<k}$ and $D_i(D_i(c_{\gamma_i})) = 0$, we conclude $A = 0$, which is a contradiction. Thus $k = \alpha(i)$ for some $1 \leqslant i \leqslant n$.

Let $c_\alpha = b_{\alpha(i)} M_i + B$ with $M_i, B \in F_{<\alpha(i)}$. Then

$$\mathrm{d}t = (b_{\alpha(i)} M_i + B)\frac{\mathrm{d}b_\alpha}{b_\alpha} + \sum_{\gamma<\alpha} c_\gamma \frac{\mathrm{d}b_\gamma}{b_\gamma}.$$

But

$$\begin{aligned} b_{\alpha(i)} M_i \frac{\mathrm{d}b_\alpha}{b_\alpha} &= b_{\alpha(i)} M_i \frac{\mathrm{d}b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(i)}}{b_{\alpha(i)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(n)}}{b_{\alpha(n)}} \\ &= \mathrm{d}(b_{\alpha(i)} M_i) \wedge \frac{\mathrm{d}b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(i-1)}}{b_{\alpha(i-1)}} \wedge \frac{\mathrm{d}b_{\alpha(i+1)}}{b_{\alpha(i+1)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(n)}}{b_{\alpha(n)}} \\ &\quad + b_{\alpha(i)} M_i \frac{\mathrm{d}b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}M_i}{M_i} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(n)}}{b_{\alpha(n)}} \end{aligned}$$

so that replacing t by

$$t' = t + b_{\alpha(i)} M_i \frac{\mathrm{d}b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(i-1)}}{b_{\alpha(i-1)}} \wedge \frac{\mathrm{d}b_{\alpha(i+1)}}{b_{\alpha(i+1)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(n)}}{b_{\alpha(n)}},$$

and since

$$b_{\alpha(i)} M_i \frac{\mathrm{d}b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}M_i}{M_i} \wedge \cdots \wedge \frac{\mathrm{d}b_\alpha(n)}{b_\alpha(n)} \in \Omega_<\alpha^n,$$

we get

$$\mathrm{d}t' = B\frac{\mathrm{d}b_\alpha}{b_\alpha} + \sum_{\gamma<\alpha} c'_\gamma \frac{\mathrm{d}b_\gamma}{b_\gamma}$$

with certain $c'_\gamma \in F$ and $B \in F_<\alpha(i)$. We proceed again as before with B instead of c_α and the lemma follows by induction. □

An immediate generalization of Lemma 3.2 is

3.3. Proposition. *Let*

$$\sum_{\gamma \leqslant \alpha} c_\gamma \frac{\mathrm{d} b_\gamma}{b_\gamma} = \mathrm{d}(t) + \wp(w)$$

with $c_\alpha \neq 0$, where $\mathcal{B} = \{b_i \mid i \in I\}$ is a given 2-basis of F (and a fixed ordering in I) and $t \in \Omega_F^{n-1}$, $w \in \Omega_F^n$. Then there exist elements $u \in F$, $M_i \in F_{<\alpha(i)}$, $1 \leqslant i \leqslant n$, such that

$$c_\alpha = \wp u + b_{\alpha(1)} M_1 + \cdots + b_{\alpha(n)} M_n.$$

Proof. Let us write w as $\sum_{\gamma \leqslant \delta} f_\gamma \frac{\mathrm{d} b_\gamma}{b_\gamma}$ with $f_\delta \neq 0$. Then we have

$$\sum_{\gamma \leqslant \max\{\alpha,\delta\}} \bigl(c_\gamma - \wp(f_\gamma)\bigr) \frac{\mathrm{d} b_\gamma}{b_\gamma} \in \mathrm{d}\,\Omega_F^{n-1}.$$

If $\delta > \alpha$, we have

$$\wp(f_\delta) \frac{\mathrm{d} b_\gamma}{b_\gamma} \in \mathrm{d}\,\Omega_F^{n-1} + \Omega_{F,<\delta}^n$$

and, by Kato's lemma (see [7]), we conclude that

$$f_\delta \frac{\mathrm{d} b_\gamma}{b_\gamma} = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} + u',$$

where $u' \in \Omega_{F,<\delta}^n$. Since $\wp(\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}) \in \mathrm{d}\,\Omega_F^{n-1}$, we replace w by $w' = w - \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$ which has lower maximal multi-index δ'. By iterative application of the above procedure we may assume that $\delta \leqslant \alpha$. In this case we have

$$\sum_{\gamma \leqslant \alpha} \bigl(c_\gamma - \wp(f_\gamma)\bigr) \frac{\mathrm{d} b_\gamma}{b_\gamma} \in \mathrm{d}\,\Omega_F^{n-1}.$$

Now using Lemma 3.2 we obtain the desired conclusion. □

4. Annihilators of differential forms in $\nu_F(m)$ and $H^{m+1}(F)$

The groups $\nu_F(m)$ act on the groups $H^{n+1}(F)$ through exterior multiplication

$$\wedge : \nu_F(m) \times H^{n+1}(F) \to H^{m+n+1}(F),$$
$$\wedge : \nu_F(m) \times \nu_F(n) \to \nu_F(m+n),$$

and we can define for any $x \in \nu_F(n)$ the annihilators

$$\mathrm{annb}_m(x) = \{y \in \nu_F(m) \mid xy = 0 \text{ in } \nu_F(m+n)\},$$
$$\mathrm{annq}_m(x) = \{y \in H^{m+1}(F) \mid xy = 0 \text{ in } H^{n+m+1}(F)\}.$$

Also if $x \in H^{n+1}(F)$, we define

$$\mathrm{annb}_m(x) = \{y \in \nu_F(m) \mid yx = 0 \text{ in } H^{n+m+1}(F)\}.$$

Through Kato's isomorphisms (see Section 3) these annihilators are isomorphic to the corresponding graded annihilators of bilinear and quadratic forms, namely, if $x \in \nu_F(n)$

$$\alpha : \mathrm{annb}_m(x) \simeq \overline{\mathrm{annb}}_m(\alpha(x)),$$
$$\beta : \mathrm{annq}_m(x) \simeq \overline{\mathrm{annq}}_m(\alpha(x)),$$

and if $x \in H^{n+1}(F)$,

$$\alpha : \mathrm{annb}_m(x) \simeq \overline{\mathrm{annb}}_m(\beta(x)).$$

Thus, Theorem 1.1 is equivalent to the following

4.1. Theorem.

(i) *Let* $x = \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n} \in \nu_F(n)$ *be a pure logarithmic differential form,* $x \neq 0$*. Then for any* $m \geqslant 1$

$$\mathrm{annb}_m(x) = \mathrm{annb}_1(x) \wedge \nu_F(m-1),$$
$$\mathrm{annq}_m(x) = \nu_F(m) \wedge \mathrm{annq}_0(x) + \mathrm{annb}_1(x) \wedge H^m(F).$$

(ii) *If* $x = \overline{a\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}} \neq 0$ *in* $H^{n+1}(F)$*, then in* $\nu_F(m)$

$$\mathrm{annb}_m(x) = \mathrm{annb}_1(x) \wedge \nu_F(m-1).$$

Proof. Let $\mathcal{B} = \{b_i \mid i \in I\}$ be a 2-basis of F such that $a_1, \ldots, a_n \in \mathcal{B}$ are the first elements in some ordering of I. Let $y \in \mathrm{annb}_m(x)$. Using Kato's lemma we can write

$$y = \sum_{\gamma \in \Sigma_m} \varepsilon_\gamma \frac{\mathrm{d}a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}a_{\gamma(m)}}{a_{\gamma(m)}}$$

with $a_{\gamma(i)} \in F_{\gamma(i)} \setminus F_{<\gamma(i)}$, $\varepsilon_\gamma \in \{0, 1\}$. Let $\alpha \in \Sigma_m$ be maximal with $\varepsilon_\alpha \neq 0$. Then

$$y \equiv \frac{\mathrm{d}a_\alpha}{a_\alpha} \mod \Omega^m_{F,<\alpha}.$$

The assumption $xy = 0$ means

$$\left(\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}\right) \wedge \frac{\mathrm{d}a_\alpha}{a_\alpha} + \left(\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}\right) \wedge \sum_{\gamma<\alpha} \varepsilon_\gamma \frac{\mathrm{d}a_\gamma}{a_\gamma} = 0.$$

Assume first $\alpha(1) > n$ and define $\delta = (1, \dots, n, \alpha(1), \dots, \alpha(m)) \in \Sigma_{n+m}$. It follows $\delta > (1, \dots, n, \gamma)$ for all $\gamma \in \Sigma_m$ with $\gamma < \alpha$. From the last relation we conclude

$$\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n \wedge \mathrm{d}a_{\alpha(1)} \wedge \cdots \wedge \mathrm{d}a_{\alpha(m)} = 0$$

which is a contradiction to the fact that $a_1, \dots, a_n, a_{\alpha(1)}, \dots, a_{\alpha(m)}$ are 2-independent. Thus we have $\alpha(1) \leqslant n$, and this implies $x \wedge \frac{\mathrm{d}a_{\alpha(1)}}{a_{\alpha(1)}} = 0$, i.e. $\frac{\mathrm{d}a_{\alpha(1)}}{a_{\alpha(1)}} \in \mathrm{annb}_1(x)$. Hence $y - \frac{\mathrm{d}a_\alpha}{a_\alpha} \in \mathrm{annb}_m(x)$ and moreover $y - \frac{\mathrm{d}a_\alpha}{a_\alpha} \in \Omega^m_{F,<\alpha}$. Proceeding by induction on α we get the first assertion in (i).

Take now $\bar{y} \in \mathrm{annq}_m(x) \subset H^{m+1}(F)$. Then

$$y \equiv \sum_{\gamma \in \Sigma_m} c_\gamma \frac{\mathrm{d}b_\gamma}{b_\gamma} \quad \mathrm{mod}\ \wp\Omega^m_F + \mathrm{d}\,\Omega^{m-1}_F$$

with $x \wedge y \in \wp\Omega^{m+n}_F + \mathrm{d}\,\Omega^{m+n-1}_F$, i.e.

$$\sum_{\gamma \in \Sigma_m} c_\gamma \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n} \wedge \frac{\mathrm{d}b_\gamma}{b_\gamma} \in \wp\Omega^{m+n}_F + \mathrm{d}\,\Omega^{m+n-1}_F. \tag{4.2}$$

(Here the elements $b_{\gamma(i)}$ belong to $\mathcal{B}$.) Let $\alpha \in \Sigma_m$ be maximal with $c_\alpha \neq 0$. If $\alpha(1) \leqslant n$, then $\frac{\mathrm{d}b_{\alpha(1)}}{b_{\alpha(1)}} \in \mathrm{annb}_1(x)$ and $c_\alpha \frac{\mathrm{d}b_\alpha}{b_\alpha} = \frac{\mathrm{d}b_{\alpha(1)}}{b_{\alpha(1)}} \wedge c_\alpha \frac{\mathrm{d}b_{\alpha(2)}}{b_{\alpha(2)}} \wedge \cdots \wedge \frac{\mathrm{d}b_{\alpha(m)}}{b_{\alpha(m)}} \in \mathrm{annb}_1(x) \wedge H^m(F)$, and $y - c_\alpha \frac{\mathrm{d}b_\alpha}{b_\alpha} \in \Omega^m_{F,<\alpha}$. Hence we may proceed by induction on α. Thus we can assume $\alpha(1) > n$ and we define $\delta = (1, \dots, n, \alpha(1), \dots, \alpha(m)) \in \Sigma_{n+m}$. We see in (4.2) that δ is the maximal multi-index with coefficient $c_\alpha \neq 0$. Using now Proposition 3.3, we conclude from (4.2) that

$$c_\alpha = \wp(u) + E_\alpha$$

with $E_\alpha = \sum_{i=1}^n a_i M_i + \sum_{j=1}^m b_{\alpha(j)} M_{\alpha(j)}$ and $M_k \in F_{<k}$. Here we have chosen the ordering of $\mathcal{B}$ such that $a_1, \dots, a_n$ are the first elements.

Inserting c_α in y we get

$$y \equiv c_\alpha \frac{\mathrm{d}b_\alpha}{b_\alpha} \quad \mathrm{mod}\ \wp\Omega^m_F + \mathrm{d}\,\Omega^{m-1}_F + \Omega^m_{F,<\alpha},$$

$$y \equiv \left[\wp(u) + \sum_{i=1}^n a_i M_i + \sum_{j=1}^m b_{\alpha(j)} M_{\alpha(j)}\right] \frac{\mathrm{d}b_\alpha}{b_\alpha},$$

$$y \equiv \left[\sum_{i=1}^{n} a_i M_i\right] \frac{\mathrm{d}\, b_\alpha}{b_\alpha} + \left[\sum_{j=1}^{m} b_{\alpha(j)} M_{\alpha(j)}\right] \frac{\mathrm{d}\, b_\alpha}{b_\alpha}.$$

Since $M_k \in F_{<k}$, we have $a_i M_i \frac{\mathrm{d} b_\alpha}{b_\alpha} \in \nu_F(m) \wedge \mathrm{annq}_0(x)$ because $a_i M_i \frac{\mathrm{d} a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d} a_n}{a_n} = \mathrm{d}(a_i M_i \frac{\mathrm{d} a_1}{a_1} \wedge \cdots \overset{i}{\wedge} \cdots \wedge \frac{\mathrm{d} a_n}{a_n}) \in \mathrm{d}\, \Omega_F^{n-1}$ implies $a_i M_i \in \mathrm{annq}_0(x)$ (we have used $\mathrm{d}\, M_i \wedge x = 0$). The same argument shows, since $M_{\alpha(j)} \in F_{<\alpha(j)}$, that

$$b_{\alpha(j)} M_{\alpha(j)} \frac{\mathrm{d}\, b_{\alpha(j)}}{b_{\alpha(j)}} = \mathrm{d}(b_{\alpha(j)} M_{\alpha(j)}) + b_{\alpha(j)} M_{\alpha(j)} \frac{\mathrm{d}\, M_{\alpha(j)}}{M_{\alpha(j)}} \in \mathrm{d}\, F + \Omega^1_{F,<\alpha(j)}$$

and hence

$$\left(\sum_{j=1}^{m} b_{\alpha(j)} M_{\alpha(j)}\right) \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \in \mathrm{d}\, \Omega_F^{m-1} + \Omega^m_{F,<\alpha}.$$

Thus we have

$$y = y' + z \mod \wp\Omega_F^m + \mathrm{d}\, \Omega_F^{m-1}$$

with $y' \in \Omega^m_{F,<\alpha}$, $y' \in \mathrm{annq}_m(x)$ and $z \in \nu_F(m) \wedge \mathrm{annq}_0(x)$. Applying now the above procedure to y' we get our second assertion by induction on α. This proves (i).

(ii) Let $x = \overline{a \frac{\mathrm{d} a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d} a_n}{a_n}} \in H^{n+1}(F)$ be a pure element, $x \neq 0$. We fix as before a 2-basis $\mathcal{B} = \{b_i \mid i \in I\}$ of F such that $a_1, \ldots, a_n$ are the first elements in $\mathcal{B}$ in some ordering of I. Let $y \in \mathrm{annb}_m(x) \subset \nu_F(m)$. From Kato's lemma we have $y = \sum_{\gamma \in \Sigma_m} \varepsilon_\gamma \frac{\mathrm{d} a_\gamma}{a_\gamma}$ with $\varepsilon_\gamma \in \{0, 1\}$ and $a_{\gamma(i)} \in F_{\gamma(i)} \backslash F_{<\gamma(i)}$, $1 \leqslant i \leqslant m$. We write

$$y = \sum_{\substack{\gamma \in \Sigma_m \\ \gamma(1) \leqslant n}} \varepsilon_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma} + \sum_{\substack{\gamma \in \Sigma_m \\ \gamma(1) > n}} \varepsilon_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma}.$$

For $\gamma \in \Sigma_m$ with $\gamma(1) \leqslant n$ we have $\frac{\mathrm{d} a_{\gamma(1)}}{a_{\gamma(1)}} \in \mathrm{annb}_1(x)$ since $a_{\gamma(1)} \in F_n = F^2(a_1, \ldots, a_n)$ and hence the first summand in this decomposition is in $\mathrm{annb}_1(x) \wedge \nu_F(m-1)$. Thus the second summand is in $\mathrm{annb}_m(x)$ and we can assume $y = \sum_{\gamma \in \Sigma_m} \varepsilon_\gamma \frac{\mathrm{d} a_\gamma}{a_\gamma}$ with all γ such that $\gamma(1) > n$. Let α be maximal in this sum with $\varepsilon_\alpha \neq 0$. We can replace $\mathcal{B}$ by a new 2-basis $\mathcal{B}' = \{c_i \mid i \in I\}$ such that $c_{\alpha(j)} = a_{\alpha(j)}$, $1 \leqslant j \leqslant m$ and $c_i = b_i$ for all $i \notin \{\alpha(1), \ldots, \alpha(m)\}$. Let $\delta = (1, \ldots, n, \alpha(1), \ldots, \alpha(m)) \in \Sigma_{n+m}$. Hence

$$0 \equiv y \wedge x \equiv a \frac{\mathrm{d}\, c_\delta}{c_\delta} \mod \wp\Omega_F^{n+m} + \mathrm{d}\, \Omega_F^{n+m-1} + \Omega^{n+m}_{F,<\delta}.$$

Then Proposition 3.3 implies

$$a = \wp(u) + \sum_{i=1}^{n} c_i M_i + \sum_{j=1}^{m} c_{\alpha(j)} M_{\alpha(j)}$$

with $M_k \in F_{<k}$. Let $s \in \{1, \ldots, m\}$ be maximal with $M_{\alpha(s)} \neq 0$ and set $Q = a + \wp u + \sum_{i=1}^n c_i M_i$, i.e. $Q = \sum_{j=1}^m c_{\alpha(j)} M_{\alpha(j)}$. Then $c_{\alpha(s)} = M_{\alpha(s)}^{-1}(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)})$. Inserting in y we get modulo $\nu_{F,<\alpha}(m)$

$$\begin{aligned} y &\equiv \frac{\mathrm{d}\, c_{\alpha(1)}}{c_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(s)}}{c_{\alpha(s)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \quad \operatorname{mod} \nu_{F,<\alpha}(m) \\ &\equiv \frac{\mathrm{d}\, c_{\alpha(1)}}{c_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, M_{\alpha(s)}^{-1}(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)})}{M_{\alpha(s)}^{-1}(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)})} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \\ &\equiv \frac{\mathrm{d}(c_{\alpha(1)} M_{\alpha(1)})}{(c_{\alpha(1)} M_{\alpha(1)})} \wedge \cdots \wedge \frac{\mathrm{d}(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)})}{Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}}. \end{aligned}$$

Here we have inserted $M_{\alpha(j)}$ whenever it is $\neq 0$, without altering the congruence modulo $\nu_{F,<\alpha}(m)$. Use now the relation $\frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}b}{b} = \frac{\mathrm{d}(ab)}{ab} \wedge \frac{\mathrm{d}(a+b)}{a+b}$ to conclude

$$\begin{aligned} y &\equiv \frac{\mathrm{d}(c_{\alpha(1)} M_{\alpha(1)})}{(c_{\alpha(1)} M_{\alpha(1)})} \wedge \cdots \wedge \frac{\mathrm{d}(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)})}{Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \quad \operatorname{mod} \nu_{F,<\alpha}(m) \\ &\equiv \frac{\mathrm{d}\, f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}\, Q}{Q} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \end{aligned}$$

with certain $f_1, \ldots, f_{s-1} \in F$. Since $\frac{\mathrm{d}\, Q}{Q} \in \operatorname{annb}_1(x)$ (we can assume $a \in F^2$ without restriction), we get $\frac{\mathrm{d}\, f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}\, Q}{Q} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \in \operatorname{annb}_1(x) \wedge \nu_F(m-1)$. Thus we have shown $y \in \operatorname{annb}_1(x) \wedge \nu_F(m-1) + \nu_{F,<\alpha}(m)$. We apply now induction on α to conclude the proof of (ii). □

Let us briefly compute the annihilators $\operatorname{annb}_1(x)$ and $\operatorname{annq}_0(x)$ for $x = \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n} \in \nu_F(n)$ and $\operatorname{annb}_1(x)$ for $x = \overline{a \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}} \in H^{n+1}(F)$.

4.3. Proposition.

(i) *Let* $x = \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n} \in \nu_F(n)$, $x \neq 0$. *Then*

$$\begin{aligned} \operatorname{annb}_1(x) &= \left\{ \frac{\mathrm{d}z}{z} \;\middle|\; z \in F^2(a_1, \ldots, a_n)^* \right\}, \\ \operatorname{annq}_0(x) &= \left\{ \bar{z} \in F/\wp F \mid z \in F^2(a_1, \ldots, a_n)' \right\}, \end{aligned}$$

where $F^2(a_1, \ldots, a_n)'$ *are the pure elements in* $F^2(a_1, \ldots, a_n)$ *(notice that* $H^1(F) = F/\wp F$*).*

(ii) *Let* $x = \overline{a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}} \in H^{n+1}(F)$, $x \neq 0$. *Then*

$$\operatorname{annb}_1(x) = \left\{ \frac{dz}{z} \;\Big|\; z \in D_F\big(\langle\langle a_1, \dots, a_n; a|]\big)^* \right\},$$

where $D_F(q)$ *denotes the elements represented in* F *by the quadratic form* q.

Proof. (i) Let $x = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \neq 0$ in $\nu_F(n)$. If $\frac{dz}{z} \in \operatorname{annb}_1(x) \subset \nu_F(1)$, then

$$\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \wedge \frac{dz}{z} = 0$$

in $\nu_F(n+1)$, which means that $a_1, \dots, a_n, z$ are 2-dependent, and since $a_1, \dots, a_n$ are 2-independent, this means $z \in F^2(a_1, \dots, a_n)^*$ (which is the set in F^* of elements represented by the n-fold Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$).

Let now $\bar{y} \in H^1(F) = F/\wp F$ be in $\operatorname{annq}_0(x)$. Then $\overline{y\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}} = 0$ in $H^{n+1}(F)$, and this means

$$y\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \in \wp\Omega_F^n + d\,\Omega_F^{n-1}.$$

Taking a 2-basis of F so that $a_1, \dots, a_n$ are the first elements of it (in some ordering), we conclude from Proposition 3.3

$$y = \wp u + b$$

with $u \in F$ and $b \in F^2(a_1, \dots, a_n)'$. This proves (i).

(ii) Let $x = \overline{a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}} \in H^{n+1}(F)$, $x \neq 0$ and take $\frac{dz}{z} \in \operatorname{annb}_1(x) \subset \nu_F(1)$. This means

$$a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \wedge \frac{dz}{z} \in \wp\Omega_F^{n+1} + d\,\Omega_F^n.$$

If $\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \wedge \frac{dz}{z} = 0$, then we get as before $z \in F^2(a_1, \dots, a_n)^* \subset D_F(\langle\langle a_1, \dots, a_n; a|])^*$. Assume $\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \wedge \frac{dz}{z} \neq 0$. Then we can assume that $a_1, \dots, a_n, z$ are the first elements of some 2-basis of F (in some ordering), and applying now Proposition 3.3 we obtain $a = \wp u + b$ with $b \in F^2(a_1, \dots, a_n, z)'$, i.e. $b = z \cdot h + g$ with $h \in F^2(a_1, \dots, a_n)^*$ and $g \in F^2(a_1, \dots, a_n)'$.

Thus $z = h^{-1}(\wp u + a + g) \in D_F(\langle\langle a_1, \dots, a_n; a|])^*$. This proves (ii). □

The isomorphisms $\nu_F(m) \simeq \bar{I}_F^m$ and $H^{m+1}(F) \simeq \bar{I}^m W_q(F)$ enable us to translate this result into the language of bilinear and quadratic forms.

Let $x = \langle\langle a_1, \dots, a_n \rangle\rangle$ be a bilinear anisotropic n-fold Pfister form. Then we have

$$\overline{\text{ann}}\mathrm{b}_1(x) = \left\{ \overline{\langle\langle z \rangle\rangle} \mid z \in D_F(x)^* \right\},$$
$$\overline{\text{ann}}\mathrm{q}_0(x) = \left\{ \bar{z} \in F/\wp F \mid z \in D_F(x')^* \right\},$$

where we identify $\bar{I}^0 W_q(F)$ with $F/\wp F$ through the Arf-invariant.

If $x = \langle\langle a_1, \dots, a_n; a]]$ is a quadratic anisotropic n-fold Pfister form, then

$$\overline{\text{ann}}\mathrm{b}_1(x) = \left\{ \overline{\langle\langle z \rangle\rangle} \mid z \in D_F(x)^* \right\}.$$

Now the technique used in Section 2 enables us to compute the full annihilators $\text{annb}_1(x)$, $\text{annq}_0(x)$ if $x = \langle\langle a_1, \dots, a_n \rangle\rangle$ and $\text{annb}_1(x)$ if $x = \langle\langle a_1, \dots, a_n; a]]$, thereby obtaining the results (2.3), (2.4) and (2.5). Let us prove for example (2.3) (the others cases are left as exercises). Let $x = \langle\langle a_1, \dots, a_n \rangle\rangle$ and take $y \in \text{annb}_1(x) \subset I_F$. Then $\bar{y} \in \overline{\text{ann}}\mathrm{b}_1(x)$ and hence $\bar{y} = \overline{\langle\langle z \rangle\rangle}$ for some $z \in D_F(x)^*$. Thus $y - \langle\langle z \rangle\rangle \in I^2$ and $(y - \langle\langle z \rangle\rangle)x = 0$, i.e. $y - \langle\langle z \rangle\rangle \in \text{annb}_2(x) = \text{annb}_1(x) \cdot I_F$. Write $y - \langle\langle z \rangle\rangle = \sum y_i v_i$ with $y_i \in \text{annb}_1(x)$, $v_i \in I_F$. Then $y_i - \langle\langle z_i \rangle\rangle \in I_F^2$ for some $z_i \in D_F(x)^*$ and hence

$$y - \langle\langle z \rangle\rangle - \sum \langle\langle z_i \rangle\rangle v_i \in I_F^3.$$

Iterating this procedure and assuming $I_F^{N+1} = 0$ for some N, we get (2.3). The general case can be reduced to the assumption $I_F^{N+1} = 0$ using the trick of Section 2. This proves (2.3). The same argument applies for (2.4) and (2.5). Thus we have a complete description of the annihilators of Pfister forms over a field F with $2 = 0$.

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