**Elliptic systems involving multiple strongly coupled critical terms**

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**A b s t r a c t**

In this work, a singular elliptic system is investigated, which involves multiple strongly coupled critical terms. By means of variational methods and analytic techniques, the existence of positive solutions to the system is established.

The energy functional of (1.1) is defined in $H \times H$ by

$$
J(u, v) = \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 + |\nabla v|^2 - \mu \frac{u^2 + v^2}{|x|^2} - 2a_2 uv \right) dx - \frac{a_1}{q_1} \int_{\Omega} |u|^{q_1} - \frac{a_2}{q_2} \int_{\Omega} |v|^{q_2} - \frac{1}{2^{1+2}} \int_{\Omega} \left( \eta_1 |u|^{\alpha_1} |v|^{\beta_1} + \eta_2 |u|^{\alpha_2} |v|^{\beta_2} \right) dx.
$$

1. Introduction

In this work, we study the following elliptic system:

$$
\begin{align*}
Lu &= \frac{\eta_1 \alpha_1}{2^\ast} |u|^{\alpha_1 - 2} |v|^{\beta_1} u + \frac{\eta_2 \alpha_2}{2^\ast} |u|^{\alpha_2 - 2} |v|^{\beta_2} u + a_1 |u|^{q_1 - 2} u + a_2 v, \\
Lv &= \frac{\eta_1 \beta_1}{2^\ast} |u|^{\alpha_1 - 2} v + \frac{\eta_2 \beta_2}{2^\ast} |u|^{\alpha_2 - 2} v + a_2 u + a_3 |v|^{q_2 - 2} v,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain such that $0 \in \Omega$, $\mu < \bar{\mu}$, $L := \left( -\Delta - \mu \frac{\cdot}{|x|^2} \right)$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, $0 \leq \eta_j < +\infty$, $2 \leq q_j < 2^*$, $\alpha_j, \beta_j > 1$, $\alpha_j + \beta_j = 2^*$, $j = 1, 2$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, $\bar{\mu} := \left( \frac{N-2}{2} \right)^2$ is the best Hardy constant and $H^1_0(\Omega) := H$ denotes the completion of $C^\infty_0(\Omega)$ with respect to $\left( \int_{\Omega} |\nabla \cdot | dx \right)^{1/2}$.
Then $J \in C^1(H \times H, \mathbb{R})$ and the duality product between $H \times H$ and its dual space $(H \times H)^{-1}$ is defined as

$$
\langle f(u), (\varphi, \phi) \rangle := \int_{\Omega} \left( \nabla u \nabla \varphi + \nabla v \nabla \phi - \mu \frac{u \varphi + v \phi}{|x|^2} \right) \, dx
- \int_{\Omega} \left( a_1 |u|^{q_1-2} u \varphi + a_2 u \varphi + a_2 u \phi + a_3 |v|^{q_2-2} \phi \right) \, dx
- \int_{\Omega} \left( \frac{\eta_1 \alpha_1}{2^*} |u|^\alpha_1 |v|^\beta_1 u \varphi + \frac{\eta_2 \alpha_2}{2^*} |u|^\alpha_2 |v|^\beta_2 u \phi \right) \, dx
- \int_{\Omega} \left( \frac{\eta_1 \beta_1}{2^*} |u|^\alpha_1 |v|^\beta_1 u \phi + \frac{\eta_2 \beta_2}{2^*} |u|^\alpha_2 |v|^\beta_2 u \phi \right) \, dx,
$$

(1.3)

where $u, v, \varphi, \phi \in H$ and $f(u)$ denotes the Fréchet derivative of $J$ at $(u, v)$. A pair of functions $(u, v) \in H \times H$ is said to be a solution of (1.1) if

$$
(u, v) \neq (0, 0), \quad \langle f(u), (\varphi, \phi) \rangle = 0, \quad \forall (\varphi, \phi) \in H \times H.
$$

The solution of (1.1) is equivalent to a nonzero critical point of $J(u, v)$.

The following Hardy inequality is well known [1]:

$$
\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx \leq \frac{1}{\mu} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N).
$$

Therefore the operator $L$ is positive for all $\mu < \bar{\mu}$, and the first eigenvalue $\Lambda_1(\mu)$ of $L$ and the following best constant are well defined:

$$
S(\mu) := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left( |\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) \, dx}{\left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^2}, \quad \mu \in (-\infty, \bar{\mu}),
$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)^{1/2}$. Note that $S(0)$ is the well-known best Sobolev constant. For all $0 \leq \mu < \bar{\mu}$, the constant $S(\mu)$ is achieved by the following extremal functions [2]:

$$
V_\mu^\ast(x) := e^{\frac{2\mu}{N-2}} U_\mu(e^{-1} x), \quad \forall \varepsilon > 0,
$$

(1.4)

where

$$
U_\mu(x) = \left( \frac{2N(\bar{\mu} - \mu)}{\sqrt{\bar{\mu}}} \right)^{\frac{N}{4}} \left( |x|^{-\frac{N-\mu}{\sqrt{\bar{\mu}}} + \sqrt{\bar{\mu}}} + |x|^{-\frac{N+\mu}{\sqrt{\bar{\mu}}} - \sqrt{\bar{\mu}}} \right).
$$

For any $\mu < \bar{\mu}$, $\alpha_i, \beta_i > 1$ and $\alpha_i + \beta_i = 2^*$, $i = 1, 2$, by the Young and Sobolev inequalities the following best constants are well defined on the space $\mathcal{D} := (D^{1,2}(\mathbb{R}^N) \setminus \{0\})^2$:

$$
S_{\alpha_1, \alpha_2, \beta_1, \beta_2}(\mu) := \inf_{(u, v) \in \mathcal{D}} \frac{\int_{\mathbb{R}^N} \left( |\nabla u|^2 + |\nabla v|^2 - \mu \frac{|u|^{2^*}}{|x|^{2^*}} \right) \, dx}{\left( \int_{\mathbb{R}^N} (\eta_1 |u|^\alpha_1 |v|^\beta_1 + \eta_2 |u|^\alpha_2 |v|^\beta_2) \, dx \right)^2},
$$

(1.5)

We mention that in recent years, much attention has been paid to the singular equations involving the Hardy inequality. However, the elliptic systems involving the Hardy inequality have seldom been studied and we only find some results in [3–7]. Therefore it is necessary to investigate the related singular systems deeply. Since the case $q_1 = q_2 = 2$ had been studied in the references above, only the case $2 < q_1, q_2 < 2^*$ of (1.1) is considered in this work.

For any $\mu < \bar{\mu}$ we define

$$
C_N(\mu) := \max \left\{ \frac{N}{\sqrt{\mu} + \sqrt{\bar{\mu} - \mu}}, \frac{N - 2\sqrt{\mu} - \mu}{\sqrt{\bar{\mu}}} \right\},
$$

(1.6)

$$
\Lambda^*(\mu) := \inf_{u \in H_0^1(\Omega, |x|^{-2v}) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} |u|^2 \, dx},
$$

(1.7)
where \(v := \sqrt{\mu} + \sqrt{\mu - \mu} \) and \(H_0^1(\Omega, |x|^{-2v})\) is the usual weighted space. For all \(\eta_i > 0\), \(1 < \beta_i < 2^* - 1\), \(i = 1, 2\), we set
\[
\begin{align*}
f(\tau) &:= \frac{1 + \tau^2}{(\eta_1 \tau^{\beta_1} + \eta_2 \tau^{\beta_2})^{\frac{2}{\tau^2}}} \quad \tau \geq 0, \\
f(\tau_{\min}) &:= \min_{\tau \geq 0} f(\tau) > 0,
\end{align*}
\]
where \(\tau_{\min} \geq 0\) is a minimal point of \(f(\tau)\).

The following assumptions are needed in this work:
\(\mathcal{H}_1\) \(N \geq 3, 0 \leq \mu < \bar{\mu}, 0 \leq \eta_i < \infty, \alpha_i, \beta_i > 1, \alpha_i + \beta_i = 2^*\).
\(\mathcal{H}_2\) \(a_i > 0, i = 1, 2, 3, a_2 < A_1(\mu), 2 < q_1, q_2 < 2^*\).

The main results of this work are summarized in the following theorems. To the best of our knowledge, the conclusions are new.

**Theorem 1.1.** Suppose that \((\mathcal{H}_1)\) holds and \(V_{\mu}^{\epsilon}(\chi)\) are the minimizers of \(S(\mu)\) defined as in (1.4) Then \(S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu) = f(\tau_{\min})S(\mu)\), and has the minimizers
\[
(V_{\mu}^{\epsilon}(\chi), \tau_{\min}V_{\mu}^{\epsilon}(\chi)) \quad \forall \epsilon > 0.
\]

**Theorem 1.2.** Suppose that \((\mathcal{H}_1)\) and \((\mathcal{H}_2)\) hold, and one of the following conditions is satisfied:
(i) \(N \geq 4, 0 \leq \mu \leq \bar{\mu} - 1\).
(ii) \(N \geq 3, \bar{\mu} - 1 < \mu < \bar{\mu}, a_2 > A^*(\mu)\).
(iii) \(\max\{q_1, q_2\} > C_0(\mu)\).

Then the problem (1.1) has a positive solution.

In the following argument, \(\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{1/2}\) denotes the norm of the space \(H, \|u, v\|_{H \times H} = (\|u\|^2 + \|v\|^2)^{1/2}\) is the norm of the space \(H \times H\), \(O(\epsilon^\delta)\) denotes the quantity satisfying \(O(\epsilon^\delta)/\epsilon^\delta \leq C, o(\epsilon^\delta)\) means \(|o(\epsilon^\delta)|/\epsilon^\delta \to 0\) as \(\epsilon \to 0\) and \(o(1)\) is a generic infinitesimal value. In particular, the quantity \(O(\epsilon^\delta)\) means there exist constants \(C_1, C_2 > 0\) such that \(C_1 \epsilon^\delta \leq O(\epsilon^\delta) \leq C_2 \epsilon^\delta\) as \(\epsilon\) small. We always denote positive constants as \(C\) and omit \(dx\) in integrals for convenience.

**2. Proof of Theorem 1.1**

**Proof of Theorem 1.1.** We follow the argument in [8], where the constant \(S_{\eta_1, 0, \beta_1, \beta_2}(0)\) was investigated. Note that \(\min_{\tau \geq 0} f(\tau)\) must be achieved at finite \(\tau_{\min} \geq 0\). Suppose \(w \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}\). Choosing \((u, v) = (w, \tau_{\min}w)\) in (1.6) we have
\[
\frac{1 + (\tau_{\min})^2}{(\eta_1 (\tau_{\min})^{\beta_1} + \eta_2 (\tau_{\min})^{\beta_2})^{\frac{2}{\tau^2}}} \int_{\mathbb{R}^N} \left(|\nabla w|^2 - \mu \frac{w^2}{|x|^2}\right) \geq S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu).
\]
Taking the infimum as \(w \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}\) in (2.1), we have
\[
f(\tau_{\min})S(\mu) \geq S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu).
\]
Let \(\{(u_n, v_n)\} \subset D\) be a minimizing sequence of \(S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu)\) and define \(z_n = s_n v_n\), where
\[
s_n = \left(\left(\int_{\mathbb{R}^N} |v_n|^2\right)^{-1} \int_{\mathbb{R}^N} |u_n|^2\right)^{\frac{1}{2}}.
\]
Then
\[
\int_{\mathbb{R}^N} |z_n|^2 = \int_{\mathbb{R}^N} |u_n|^2.
\]
From the Young inequality and (2.3) it follows that
\[
\int_{\mathbb{R}^N} |u_n|^2 \leq \frac{\alpha_i}{2} \int_{\mathbb{R}^N} |u_n|^2 + \frac{\beta_i}{2} \int_{\mathbb{R}^N} |z_n|^2
\]
\[
= \int_{\mathbb{R}^N} |u_n|^2 = \int_{\mathbb{R}^N} |z_n|^2, \quad i = 1, 2.
\]
Consequently,
\[
\int_{\mathbb{R}^N} \left( |\nabla u_n|^2 + |\nabla v_n|^2 - \frac{\mu^2}{|x|^2} - \frac{\eta_1^2}{|x|^2} \right) \geq \int_{\mathbb{R}^N} \left( |\nabla u_n|^2 - \frac{\mu^2}{|x|^2} \right) \geq \int_{\mathbb{R}^N} \left( (\eta_1 s_n^{-\beta_1} + \eta_2 s_n^{-\beta_2}) \int_{\mathbb{R}^N} |u_n|^2 \right)^{\frac{\beta_1}{2}} \\
+ \int_{\mathbb{R}^N} (\eta_1 s_n^{-\beta_1} + \eta_2 s_n^{-\beta_2}) \int_{\mathbb{R}^N} |z_n|^2 \right)^{\frac{\beta_2}{2}} \\
\geq f(s_n^{-1}) S(\mu) \\
\geq f(\tau_{\min}) S(\mu).
\]

As \( n \to \infty \) we have
\[
S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu) \geq f(\tau_{\min}) S(\mu),
\tag{2.5}
\]
which together with (2.2) implies that
\[
S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu) = f(\tau_{\min}) S(\mu), \quad \forall \mu \in (-\infty, \bar{\mu}).
\]

By (1.4) and (1.5), \( S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu) \) has the minimizers \( (V_\mu(x), \tau_{\min} V_\mu(x)) \), \( 0 \leq \mu < \bar{\mu}, \varepsilon > 0 \). \( \square \)

### 3. Proof of Theorem 1.2

Let \( V_\mu(x) \) be defined as in (1.4) and set \( u_\varepsilon(x) = \psi(x)V_\mu^\varepsilon(x) \), where \( \psi(x) \) is a cutoff function:
\[
\psi(x) \in \mathcal{D}_0^\varepsilon(\Omega) \doteq \{ \psi \in C_0^\infty(\Omega) | \psi(x) \equiv 1 \text{ in a neighbourhood of } x = 0 \}.
\]

The following results are already known:

**Lemma 3.1** ([9]). As \( \varepsilon \to 0 \) we have the following estimates.
\[
\int_\Omega \left( |\nabla u_\varepsilon|^2 - \mu \frac{|u_\varepsilon|^2}{|x|^2} \right) = S(\mu)^{\frac{N}{2}} + O(\varepsilon^{2\sqrt{\bar{\mu}} - \mu}), \tag{3.1}
\]
\[
\int_\Omega |u_\varepsilon|^2 = S(\mu)^{\frac{N}{2}} + O(\varepsilon^{2\sqrt{\bar{\mu}} - \mu}), \tag{3.2}
\]
\[
\int_\Omega |u_\varepsilon|^2 = \begin{cases} 
O_1(\varepsilon^2), & 0 \leq \mu < \tilde{\mu} - 1, \\
O_1(\varepsilon^2), & \mu = \tilde{\mu} - 1, \\
O_1(\varepsilon^{2\sqrt{\mu} - \mu}), & \tilde{\mu} - 1 < \mu < \bar{\mu}.
\end{cases} \tag{3.3}
\]
\[
\int_\Omega |u_\varepsilon|^q = \begin{cases} 
O_1(\varepsilon^{N-q\sqrt{\bar{\mu}}}), & q < q^* = \frac{N}{\sqrt{\bar{\mu}} + \sqrt{\mu - \bar{\mu}}}, \\
O_1(\varepsilon^{N-q\sqrt{\bar{\mu}}}), & q = \frac{N}{\sqrt{\bar{\mu}} + \sqrt{\mu - \bar{\mu}}}, \\
O_1(\varepsilon^{2\sqrt{\mu} - \mu}), & 1 < q < \frac{N}{\sqrt{\bar{\mu}} + \sqrt{\mu - \bar{\mu}}}. \tag{3.4}
\end{cases}
\]

**Lemma 3.2.** Suppose that \((\mathcal{H}_1)\) and \((\mathcal{H}_2)\) hold. Then the functional \( J \) satisfies the \((PS)_c\) condition for all \( c < c^* := \frac{1}{N} S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu) \).

**Proof.** The proof is similar to that of [6] and is thus omitted. \( \square \)

**Lemma 3.3.** Under the assumptions of **Theorem 1.2**, there exists a pair of functions \((\bar{u}, \bar{v}) \in H \times H \setminus \{(0, 0)\}\) such that \( \sup_{t \geq 0} J(t\bar{u}, t\bar{v}) < c^* = \frac{1}{N} S_{\eta_1, \eta_2, \beta_1, \beta_2}(\mu) \).

**Proof.** (i) \( N \geq 4, 0 \leq \mu < \bar{\mu} - 1. \)

For all \( t \geq 0 \), define the function \( g_1(t) := J(tu_\varepsilon, t\tau_{\min}u_\varepsilon) \). Then
\[
g_1(t) \leq \frac{t^2}{2} (1 + \tau_{\min}^2) \int_\Omega \left( |\nabla u_\varepsilon|^2 - \mu \frac{|u_\varepsilon|^2}{|x|^2} \right) - \frac{t^2}{2^*} \left( \eta_1(\tau_{\min})^{\beta_1} + \eta_2(\tau_{\min})^{\beta_2} \right) \int_\Omega |u_\varepsilon|^{2^*}.
\]
Note that \( \sup_{t > 0} g_1(t) \) must be achieved at some finite \( t_\epsilon > 0 \) such that \( g_1'(t_\epsilon) = 0 \) and \( 0 < C' < t_\epsilon < C'' \), where \( C' \) and \( C'' \) are the constants independent of \( \epsilon \). Furthermore,

\[
0 \leq \mu < \tilde{\mu} - 1 \iff 2 < 2\sqrt{\tilde{\mu} - \mu}.
\]

From (3.5), Lemma 3.1 and Theorem 1.1 it follows that

\[
g_1(t_\epsilon) \leq \frac{1}{N} \left( (1 + \tau_{\min})^2 \int_\Omega \left( |\nabla u_\epsilon|^2 - \mu \frac{|u_\epsilon|^2}{|x|^2} - a_2 |u_\epsilon|^2 \right) \right)^{\frac{\eta}{2}}
\]

\[
\leq \frac{1}{N} \left( f(\tau_{\min}) \mathcal{S}(\mu) \right)^{\frac{\eta}{2}} + O(\epsilon^{2\sqrt{2}/\sqrt{\bar{\mu} - \mu})
\]

\[
\leq \frac{1}{N} \left( f(\tau_{\min}) \mathcal{S}(\mu) \right)^{\frac{\eta}{2}} + O(\epsilon^{2\sqrt{2}/\sqrt{\bar{\mu} - \mu})
\]

where

\[
\mathcal{S}(\mu) := \int_\Omega (\nabla \psi)^2 - a_2 \psi^2 ||V_\mu||^2 \quad \text{and} \quad \tilde{\mu} := \sqrt{\mu} - \mu.
\]

Note that \( \psi(x) \in D^*(\Omega) \), and \( v := \sqrt{\bar{\mu}} + \gamma < \frac{N}{2} \) for all \( \bar{\mu} - 1 < \mu < \tilde{\mu} \). Consequently,

\[
\left| \int_\Omega \left( \frac{|\nabla \psi|^2}{|x|^{2v}} - a_2 \frac{\psi^2}{|x|^{2v}} \right) \right| < \infty.
\]

Since \( a_2 > \Lambda^*(\mu) \), a standard density argument shows that there exists \( \psi \in D^*(\Omega) \) such that

\[
\left| \int_\Omega \left( \frac{|\nabla \psi|^2}{|x|^{2v}} - a_2 \frac{\psi^2}{|x|^{2v}} \right) \right| < 0.
\]
Furthermore, $V_{\mu}(x) = O_1(|x|^{-\psi})$ as $\varepsilon \to 0$. Then
\[
\int_\Omega \left( |\nabla \psi|^2 - a_2 \psi^2 \right) |V_{\mu}|^2 = \int_\Omega \left( \frac{|\nabla \psi|^2}{|x|^{2\psi}} - a_2 \frac{\psi^2}{|x|^{2\psi}} \right) O_1(\varepsilon^{2\gamma}) = -O_1(\varepsilon^{2\gamma}).
\]
(3.9)

Taking $\varepsilon$ small enough, from (3.7)–(3.9) it follows that
\[
g_1(t_0) \leq \frac{1}{N} \left( f(\min J) S(\mu) \beta \right)^{\frac{N}{\gamma}} + O(\varepsilon^{2\gamma}) - O_1(\varepsilon^{2\gamma}) < c^*.
\]

Without loss of generality, we may assume that $q_1 = \max\{q_1, q_2\}$. Note that $N - q_1 \sqrt{\mu} < 2\sqrt{\mu} - \mu$ when $q_1 > C_N(\mu)$.

As $\varepsilon \to 0$, from Lemma 3.1 it follows that
\[
\max_{t \geq 0} g_1(t) \leq \frac{1}{N} \left( f(\min J) \int_\Omega \left( |\nabla u_t|^2 - \mu \frac{|u_t|^2}{|x|^2} \right) \right)^{\frac{N}{\gamma}} - C \int_\Omega |u_t|^{q_1}
\]
\[
\leq \frac{1}{N} \left( f(\min J) S(\mu) \beta \right)^{\frac{N}{\gamma}} + O(\varepsilon^{2\gamma}) - C \int_\Omega |u_t|^{q_1}
\]
\[
\leq \frac{1}{N} f(\min J) S(\mu) \beta + O(\varepsilon^{2\gamma}) - C \varepsilon^{N-q_1} < c^*.
\]

The proof is thus complete. \bull

**Proof of Theorem 1.2.** For any $(u, v) \in H \times H \setminus \{(0, 0)\}$, by the Hardy and Sobolev inequalities we have
\[
J(u, v) \geq C\left(\|u\|_{L^q}^q - C\|u\|_{L^2}^2 - C\|u\|_{L^q}^{q_1} - C\|u\|_{L^q}^{q_2}\right) H^s H^s,
\]
and there exists a constant $\rho > 0$ small enough that
\[
b := \inf_{\|u\|_{L^q} = \rho} J(u, v) > 0 = J(0, 0).
\]

Set $c = \inf_{f \in \Gamma} \max_{t \in [0, 1]} f(\gamma(t))$, where
\[
\Gamma = \{ \gamma \in C([0, 1], H \times H) | \gamma(0) = (0, 0), J(\gamma(1)) < 0, \|\gamma(1)\| > \rho \}.
\]

Since $J(tu, tv) \to -\infty$ as $t \to \infty$, there exists $t_0 > 0$ such that $\|tu(t_0v)\|_{H^s} > \rho$ and $J(tu(t_0v)) < 0$. By the mountain pass theorem [11,12], there exists a sequence $(u_n, v_n) \subset H \times H$ such that $f(u_n, v_n) \to c$ and $f(u_n, v_n) \to 0$ as $n \to \infty$.

Let $(\bar{u}, \bar{v})$ be the testing functions obtained as in Lemma 3.2. Then
\[
0 < c \leq \sup_{t \in [0, 1]} \int_\Omega \left( t_0 t_0 \bar{u}, t_0 t_0 \bar{v} \right) \leq \sup_{t \in [0, 1]} \left( \bar{u}, \bar{v} \right) < c^*.
\]

From Lemma 3.2 it follows that there exists a subsequence of $(u_n, v_n)$, still denoted by $(u_n, v_n)$, such that $(u_n, v_n) \to (u, v)$ strongly in $H \times H$. We thus get a critical point $(u, v)$ of $J$ satisfying (1.1) and $c$ is the corresponding critical value.

Set $u_+ = \max(u, 0)$. Replacing the terms $\int_\Omega |u|^q|v|^q$, $\int_\Omega |u|^q|v|^q$, $\int_\Omega |u|^q|v|^q$ and $\int_\Omega |u|^q|v|^q$ in (1.2) by $\int_\Omega (u^+)^q(v^+)^q$, $\int_\Omega (u^+)^q(v^+)^q$, $\int_\Omega (u^+)^q(v^+)^q$ and $\int_\Omega (u^+)^q(v^+)^q$ respectively and repeating the above process, we can get a nonnegative solution $(u, v)$ of (1.1). From the maximum principle it follows that $u, v > 0$ in $\Omega$. \bull

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**References**