Semilinear elliptic problems near resonance
with a nonprincipal eigenvalue

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Abstract
We consider the Dirichlet problem for the equation $-\Delta u = \lambda u \pm f(x,u) + h(x)$ in a bounded domain, where $f$ has a sublinear growth and $h \in L^2$. We find suitable conditions on $f$ and $h$ in order to have at least two solutions for $\lambda$ near to an eigenvalue of $-\Delta$.

A typical example to which our results apply is when $f(x,u)$ behaves at infinity like $a(x) |u|^{q-2}u$, with $M > a(x) > \delta > 0$, and $1 < q < 2$.

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1. Introduction

We will mainly consider the problem

\begin{equation}
\begin{cases}
-\Delta u = \lambda u \pm f(x,u) + h(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial \Omega$, $h \in L^2(\Omega)$ and the term $f$ is sublinear, namely

(f1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function,
there exist $C > 0$ and $q \in (1, 2)$ such that $|f(x,t)| \leq C(1 + |t|^{q-1})$.

We will refer to problem (1.1) as (1.1+) and (1.1−), based on the sign before the nonlinearity $f$.

We denote throughout the paper by $\sigma(-\Delta)$ the spectrum of the Laplacian in $H^1_0(\Omega)$, that is the set of the eigenvalues $\lambda_k$ where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$, and by $\phi_k$ ($k = 1, 2, \ldots$), the corresponding eigenfunctions, which
will be taken orthogonal and normalized with \( \|\phi_k\|_{H^1_0} = 1 \); finally, we denote by \( H_{\lambda_k} \) the eigenspace corresponding to an eigenvalue \( \lambda_k \).

Observe that for \( \lambda \notin \sigma(-\Delta) \) there always exists a solution of problem (1.1), which (for \( \lambda > \lambda_1 \)) may be obtained through the saddle point theorem; in fact, consider for example the case of a simple eigenvalue \( \lambda_k \): one uses a saddle point structure of order \( k - 1 \) when \( \lambda \in (\lambda_{k-1}, \lambda_k) \) and of order \( k \) when \( \lambda \in (\lambda_k, \lambda_{k+1}) \) (such solutions correspond to \( u_{k-1} \) in the proof of Theorem 1.1 part (a) and to \( u_k \) in the proof of Theorem 1.1, part (b)).

This means that the geometry of the functional quite changes when \( \lambda \) passes from below to above an eigenvalue \( \lambda_k \), so that it turns out to be interesting the study of this geometry when \( \lambda \) is very close (or coincident) to \( \lambda_k \). In particular, our aim is to find suitable additional hypotheses on the perturbation \( f \) in order to guarantee the existence of more solutions. In fact, observe that if we had \( f \equiv 0 \), then the situation would be the following: we would have a unique solution for any \( \lambda \notin \sigma(-\Delta) \), and infinite solutions for \( \lambda = \lambda_k \), provided that \( \int_\Omega h\phi_k = 0 \). We will show that a suitable perturbation will turn the almost resonant situation (\( \lambda \) near to \( \lambda_k \)) in a situation where the solutions are at least two: we will ask one of the following sets of hypotheses (here \( F(x,t) = \int_0^t f(x,s)ds \))

\[
\begin{align*}
(H1) \quad (f2): \quad & \lim_{t \to \pm\infty} f(x,t) = \pm \infty \quad \text{uniformly with respect to } x \in \Omega; \\
(H2) \quad (f3): \quad & \lim_{|x| \to \infty} F(x,t) = +\infty \quad \text{uniformly with respect to } x \in \Omega, \\
(f4): \quad & \text{there exists a constant } C_F > 0 \text{ such that } F(x,t) \geq -C_F, \\
(h1): \quad & \int_\Omega h\phi dx = 0 \quad \forall \phi \in H_{\lambda_k}.
\end{align*}
\]

Our main result is the following theorem:

**Theorem 1.1.** Let \( \lambda_k \) \((k \geq 2)\) be an eigenvalue of multiplicity \( m \) and \( h \in L^2(\Omega) \). Under hypothesis (f1) and one of the sets of hypotheses (H1) or (H2), one gets:

(a) there exists \( \epsilon_0 > 0 \) such that for \( \lambda \in (\lambda_k - \epsilon_0, \lambda_k) \) there exist two solutions of (1.1+);

(b) there exists \( \epsilon_1 > 0 \) such that for \( \lambda \in (\lambda_k, \lambda_k + \epsilon_1) \) there exist two solutions of (1.1−).

**Remark 1.2.**

(i) The paper is written assuming \( k \geq 2 \): for \( k = 1 \) one should be able to prove the same result with few changes in the proofs, but this is not really interesting since for this case better results are already known (see the references in Section 2).

(ii) Hypothesis (f2) is stronger than (f3), and in fact with hypothesis (f2) we do not need any additional “nonresonance” condition on the forcing term \( h \), like we do with (f3)–(f4).

Moreover, observe that, in order to obtain the multiplicity result, the sign of the perturbation \( f \) should be different when we consider the case slightly above or slightly below the eigenvalue.

(iii) All the given hypotheses only deal with the asymptotical behavior of \( f \): no condition in the origin is required for our multiplicity result.

(iv) A sufficient condition for a Carathéodory function \( f \) to satisfy the hypotheses (f1) and (f2) is \( \lim_{|s| \to \infty} \frac{f(x,s)}{|s|^{q-1}} = a(x) \) uniformly, with \( 0 < \delta < a(x) < M \) and \( q \in (1,2) \): so a model for such a function could be

\[
f(x,u) = a(x)|u|^q - u.
\]

However, the hypotheses (f3)–(f4) are much weaker, so that a model for such a function could be, for example, the bounded nonlinearity

\[
f(x,u) = a(x) \arctan(u);
\]

in fact, even if \( \lim_{|s| \to \infty} f(x,s) = 0 \), hypotheses (f3)–(f4) may still be satisfied provided \( f \) goes to zero in such a way that its primitive still diverges.

Since, as mentioned above, all hypotheses deal with the behavior at infinity, a perturbation of lower order may always be added at the above model nonlinearities.
In Section 6 we will also briefly consider the case \( h(x) = 0 \) and \( f(x, 0) = 0 \) (that is, when problem (1.1) admits the trivial solution \( u = 0 \)), and the problem at resonance, i.e. when \( \lambda \) is an eigenvalue of \((-\Delta, H^1_0(\Omega))\).

2. Literature and techniques

Multiplicity results like those in Theorem 1.1 are known for the first eigenvalue and were studied by many authors since the work of Mawhin and Schmitt [1], where the problem in dimension one is considered using bifurcation from infinity and degree theory; we cite [2,3], which also consider the one-dimensional case, and [4,5], which deal with the higher dimension problems; these works are all based on bifurcation theory.

In [6,7], the same kind of problems are analyzed from a variational point of view: at least three solutions are found when approaching the first eigenvalue from below and from above, under conditions which are basically our set of hypotheses (H2). The variational approach was later exploited in [8] to obtain a similar result for the \( p \)-Laplacian operator (see also [9]).

Results for higher eigenvalues were obtained in [3], again using bifurcation from infinity and degree theory, but only for the one-dimensional case and making use of the fact that in this case all the eigenvalues are simple.

For what concerns the multiplicity result that we give in Theorem 6.1, that is when \( h(x) = f(x, 0) = 0 \) and with some additional condition at the origin, we remark that the existence of a nontrivial solution was proved in [10] for an even more general class of nonlinearity. See also [11,12] for some related problems.

This paper may be seen as a continuation of the work in [6,7], in the sense that we will study problem (1.1) in any spatial dimension and using variational techniques too, but we will consider eigenvalues above the principal one (even when they are multiple eigenvalues).

The result in Theorem 1.1 will be obtained by finding two saddle point geometries, once with a linking of order \( k - 1 \), and another time with a linking of order \( k \) (or \( k + m - 1 \) if \( m \) is the multiplicity of the eigenvalue \( \lambda_k \)). Then one obtains two solutions, which will be shown to be distinct since they lie at different levels.

This picture is coherent with the situation we described in the introduction, since we are considering a nonlinearity whose asymptotic behavior would give one solution through a saddle point theorem of order \( k - 1 \) when \( \lambda < \lambda_k \) (respectively, of order \( k + m - 1 \) if \( \lambda > \lambda_k \)), but hypotheses (H1) or (H2) give rise to another saddle point geometry of order \( k + m - 1 \) when \( \lambda < \lambda_k \) (respectively, \( k - 1 \) when \( \lambda > \lambda_k \)).

In the case of point (a) in Theorem 1.1, we will indeed apply two times the classical saddle point theorem (this will also imply the nontriviality of the critical groups of such solutions, which will be exploited in the proof of Theorem 6.1).

In the case of point (b) the geometry will be more complicated, so that one of the critical points will be obtained by a local saddle point geometry: we will use the following theorem from [13]:

**Theorem 2.1.** (From Theorem 8.1 of [13].) Let \( H = X_1 \oplus X_2 \) be a Hilbert space where \( X_1 \) has finite dimension, \( J \in C^1(H, \mathbb{R}) \) satisfying the PS condition and such that, for given \( \rho_1, \rho_2 > 0 \),

\[
\sup_{u \in \rho_1 S_1} J(u) < a = \inf_{u \in \rho_2 B_2} J(u) \le b = \sup_{u \in \rho_1 B_1} J(u) < \inf_{\rho_2 S_2} J(u),
\]

where \( B_i \) and \( S_i \) represent the unit ball and the unit sphere in \( X_i \): \( i = 1, 2 \).

Then there exists a critical point \( u_0 \) such that \( J(u_0) \in [a, b] \).

More general versions of this theorem can be found in [14].

The paper is structured as follows: in Section 3 we will give the proof of Theorem 1.1 based on several estimates, whose proof will be presented in Sections 4 and 5. Finally, Section 6 deals with the case when a trivial solution exists, and with a resonant problem.

3. Proof of Theorem 1.1

In this section we will first set the variational setting for our problem, then we will produce the estimates needed to apply the saddle point theorems and, based on these estimates, we will show how Theorem 1.1 is proved.
3.1. Variational setting

We will consider the $C^1$ functional

$$J^\pm : H^1_0(\Omega) \to \mathbb{R}; J^\pm(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda u^2) \, dx \mp \int_\Omega F(x, u) \, dx - \int_\Omega hu \, dx \quad (3.1)$$

since the problem in Theorem 1.1 is not resonant, $J^\pm$ satisfies the Palais–Smale condition of compactness (see for example in [15]).

We will denote by $\|\cdot\|$ the usual norm in $H^1_0$, we set

$$V = \text{span}\{\phi_1, \ldots, \phi_{k-1}\}, \quad Z = \text{span}\{\phi_k, \ldots, \phi_{k+m-1}\} = H_{\lambda k}, \quad W = (V \oplus Z)^\perp, \quad (3.2)$$

and we define

$$B_V = \{u \in V: \|u\| \leq 1\}, \quad B_{VZ} = \{u \in V \oplus Z: \|u\| \leq 1\}, \quad (3.3)$$

and $S_V, S_{VZ}, S_W$, respectively, their relative boundaries.

3.2. Proof of Theorem 1.1, part (a)

Theorem 1.1 will be a consequence of the geometry in Propositions 3.1 and 3.2 stated below, whose proofs will be postponed to Section 4.

Proposition 3.1. If $\lambda \in (\lambda_{k-1}, \lambda_k)$ and hypothesis (f1) is satisfied, then there exist constants $D_{\lambda}$ and $\rho^+_{\lambda} > 0$ such that

$$J^+(u) \geq D_{\lambda} \quad \text{for } u \in Z \oplus W, \quad (3.5)$$

$$J^+(u) < D_{\lambda} \quad \text{for } u \in \rho^+_{\lambda} S_V. \quad (3.6)$$

Moreover, if one of the sets of hypotheses (H1) or (H2) is satisfied, then there exists $\varepsilon_0 > 0$ such that for $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$ there exist $D_W, D_{\lambda} \in \mathbb{R}, \rho^+_{\lambda} > R^+ > 0$ such that, in addition to (3.5)–(3.6),

$$J^+(u) \geq D_W \quad \text{for } u \in W, \quad (3.7)$$

$$J^+(u) < D_W \quad \text{for } u \in R^+ S_{VZ}, \quad (3.8)$$

$$J^+(u) < D_W \quad \text{for } u \in V, \quad \|u\| \geq R^+. \quad (3.9)$$

(The values with index $\lambda$ depend on $\lambda$, the others may be fixed uniformly.)

Based on this geometry we give

Proof of Theorem 1.1, part (a). Since the functional $J^+$ satisfies the PS condition, we can apply two times the saddle point theorem (see for example in [16]), let

$$\Gamma_{k-1} = \{\gamma \in C^0(\rho^+_{\lambda} B_V; H^1_0) \text{ s.t. } \gamma|_{\rho^+_{\lambda} S_V} = \text{Id}\}, \quad (3.10)$$

$$\Gamma_k = \{\gamma \in C^0(R^+ B_{VZ}; H^1_0) \text{ s.t. } \gamma|_{R^+ S_{VZ}} = \text{Id}\}. \quad (3.11)$$

The first solution, which we denote by $u_{k-1}$ and may be obtained for any $\lambda \in (\lambda_{k-1}, \lambda_k)$ with just hypothesis (f1), corresponds to a critical point at the level

$$c_{k-1} = \inf_{\gamma \in \Gamma_{k-1}} \sup_{v \in \rho^+_{\lambda} B_V} J^+(\gamma(v));$$

the criticality of this level is guaranteed by the estimates (3.5) and (3.6), since $\rho^+_{\lambda} S_V$ and $Z \oplus W$ link (that is, the image of any map in $\Gamma_{k-1}$ intersects $Z \oplus W$).
The second solution, which we denote by \( u_k \), corresponds to a critical point at the critical level
\[
c_k = \inf_{\gamma \in I_k} \sup_{v \in R^+ B_{VZ}} J^+(\gamma(v));
\]
actually, this is a critical level because of the estimates (3.7) and (3.8), since \( R^+ S_{VZ} \) and \( W \) link.

To conclude the proof, we need to show that these two solutions are distinct.

We observe first that by estimate (3.7) we have that \( c_k \geq D_W \), then we observe that we may build a map \( \tilde{\gamma} \in \Gamma_{k-1} \) in such a way that its image is the union between the annulus \( \{ u \in V, ||u|| \in [R^+ , \rho_\lambda^+] \} \) and the image of a \((k-1)\)-dimensional ball in \( R^+ S_{VZ} \) whose boundary is \( R^+ S_V \). By the estimates (3.8) and (3.9), we deduce that \( \sup_{v \in \rho_\lambda^+ B_V} J^+(\gamma(v)) < D_W \), and as a consequence \( c_{k-1} < D_W \), proving that the two solutions are distinct, for being at different critical levels. \( \Box \)

3.3. Proof of Theorem 1.1, part (b)

**Proposition 3.2.** If \( \lambda \in (\lambda_k , \lambda_k + m) \) and hypothesis \((f1)\) is satisfied, then there exist constants \( K_\lambda \) and \( \rho_\lambda^- > 0 \) such that
\[
J^-(u) \geq K_\lambda \quad \text{for} \quad u \in W, \tag{3.12}
\]
\[
J^-(u) < K_\lambda \quad \text{for} \quad u \in \rho_\lambda^- S_{VZ}. \tag{3.13}
\]
Moreover, if one of the sets of hypotheses \((H1)\) or \((H2)\) is satisfied, then there exists \( \varepsilon_1 > 0 \) such that for \( \lambda \in (\lambda_k , \lambda_k + \varepsilon_1) \) there exist \( K_\lambda , K_V , E \in \mathbb{R} , \rho_\lambda^- > R^- > 0 , \xi > 0 \) such that, in addition to (3.12)–(3.13),
\[
J^-(u) < K_V \quad \text{for} \quad u \in V, \tag{3.14}
\]
\[
J^-(u) > K_V \quad \text{for} \quad u \in R^- S_{ZW}, \tag{3.15}
\]
\[
J^-(u) > E \quad \text{for} \quad u \in R^- B_{ZW}, \tag{3.16}
\]
\[
J^-(u) < E \quad \text{for} \quad u \in \xi S_V. \tag{3.17}
\]
(The values with index \( \lambda \) depend on \( \lambda \), the others may be fixed uniformly.)

This geometry, along with Lemma 4.6, allows us to give

**Proof of Theorem 1.1, part (b).** Since the functional \( J^- \) satisfies the PS condition, we can apply the saddle point theorem and Theorem 2.1.

The first solution, which we denote by \( w_k \) and may be obtained for any \( \lambda \in (\lambda_k , \lambda_k + m) \) with just hypothesis \((f1)\), is again obtained through the saddle point theorem and corresponds to a critical point at the critical level
\[
d_k = \inf_{\gamma \in I_k} \sup_{v \in \rho_\lambda^- B_{VZ}} J^-(\gamma(v)),
\]
where now
\[
I_k = \{ \gamma \in C^0(\rho_\lambda^- B_{VZ}; H^1_V) \text{ s.t. } \gamma|_{\rho_\lambda^- S_{VZ}} = 1d \}; \tag{3.19}
\]
the criticality is guaranteed by estimates (3.12) and (3.13), since \( \rho_\lambda^- S_{VZ} \) and \( W \) link.

The second solution, which we denote by \( w_{k-1} \), comes from Theorem 2.1, where we set \( X_1 = V \) and \( X_2 = Z \oplus W \), actually we have the structure
\[
\sup_{\xi \in S_V} J^- (u) < E \leq \inf_{R^- B_{ZW}} J^- (u) \leq \sup_{\xi B_{V}} J^- (u) < K_V \leq \inf_{R^- S_{ZW}} J^- (u)
\]
and then we have a critical point \( w_{k-1} \) at the level \( d_{k-1} \leq K_V \).

Finally, in order to prove that these two solutions are distinct, we need a sharper estimate for \( d_k \) than that given by (3.13). For this we use Lemma 4.6 to guarantee that for any map \( \gamma \in I_k \), since \( \rho_\lambda^- > R^- \), one has that the image of \( \gamma \) either intersects \( R^- S_{ZW} \) or has a point \( u \in W \) with \( ||u|| \geq R^- \). This implies that \( \sup_{v \in \rho_\lambda^- B_{VZ}} J^- (\gamma(v)) > K_V \), by estimates (3.15) and (3.16), and then \( d_k > K_V \) proving that the two solutions are distinct, for being at different critical levels. \( \Box \)
4. Proof of the estimates

In this section we will prove all the estimates in Propositions 3.1 and 3.2 and Lemma 4.6. We will use several times the estimates below (\(C\) will denote various constants throughout the proofs). By hypothesis (f1) and the compact immersion of \(L^q\) in \(H^1_0\), one may estimate

\[
\left| \int_{\Omega} F(x,u) \, dx \right| \leq C \left( 1 + \|u\|^q \right),
\]

moreover, standard estimates give

\[
\int_{\Omega} u^2 \, dx \geq \frac{1}{\lambda_j} \|u\|^2 \quad \text{for} \quad u \in \text{span}\{\phi_1, \ldots, \phi_j\},
\]

\[
\int_{\Omega} u^2 \, dx \leq \frac{1}{\lambda_{j+1}} \|u\|^2 \quad \text{for} \quad u \in \text{span}\{\phi_1, \ldots, \phi_j\}^\perp,
\]

\[
\left| \int_{\Omega} hu \, dx \right| \leq \|h\|_{L^2} \|u\|_{L^2} \leq \|h\|_{L^2} \|u\|.
\]

4.1. Estimates of the saddle geometry

Lemma 4.1. Under hypothesis (f1), one gets:

- for \(\lambda \in (\lambda_{k-1}, \lambda_k)\), there exists \(D_{\lambda} \in \mathbb{R}\) satisfying (3.5) and \(D_W \in \mathbb{R}\) satisfying (3.7);
- for \(\lambda \in (\lambda_k, \lambda_{k+m})\):
  - there exists \(K_{\lambda} \in \mathbb{R}\) satisfying (3.12),
  - for a given \(R^- > 0\), there exists \(E \in \mathbb{R}\) satisfying (3.17).

Proof. Let \(u \in W\): using estimates (4.1), (4.3) and (4.4) we get

\[
J^\pm(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_{k+m}} \right) \|u\|^2 - C \left( 1 + \|u\|^q \right) - \|h\|_{L^2} \|u\|.
\]

If \(\lambda \in (\lambda_{k-1}, \lambda_k)\), then \(1 - \frac{\lambda}{\lambda_{k+m}} > 1 - \frac{\lambda_k}{\lambda_{k+m}} > 0\) and since 2 is the higher power, there exists a \(D_W\) as in (3.7).

If \(\lambda \in (\lambda_k, \lambda_{k+m})\), then the same estimate holds but the constant cannot be made independent of \(\lambda\), giving (3.12).

In the same way, let \(u \in Z \oplus W\) and set \(\epsilon = \lambda_k - \lambda > 0\), we get

\[
J^\pm(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_k} \right) \|u\|^2 - C \left( 1 + \|u\|^q \right) - \|h\|_{L^2} \|u\|
\]

\[
\geq \frac{\epsilon}{2\lambda_k} \|u\|^2 - C \|u\|^q - \|h\|_{L^2} \|u\| - C \geq D_\lambda,
\]

that is Eq. (3.5), where again the constant \(D_\lambda\) depends on \(\epsilon\), that is on \(\lambda\).

Finally, Eq. (4.6) with \(\lambda \in (\lambda_k, \lambda_{k+m})\) implies

\[
J^\pm(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_{k+m}}{\lambda_k} \right) \|u\|^2 - C \left( 1 + \|u\|^q \right) - \|h\|_{L^2} \|u\|;
\]

then, no matter the value of \(\lambda\), \(J^\pm\) is bounded from below in any bounded subset of \(Z \oplus W\), giving (3.17) for a suitable value of \(E\). \(\Box\)

Lemma 4.2. Under hypothesis (f1), one gets:

- for \(\lambda \in (\lambda_{k-1}, \lambda_k)\), given the constant \(D_{\lambda} \in \mathbb{R}\), there exists \(\rho^+_{\lambda} > 0\) satisfying (3.6);
• for $\lambda \in (\lambda_k, \lambda_{k+m})$:
  - there exists $K_V \in \mathbb{R}$ satisfying (3.14),
  - for a given $K_\lambda \in \mathbb{R}$, there exists $\rho_\lambda^+ > 0$ satisfying (3.13),
  - for a given $E \in \mathbb{R}$, there exists $\xi > 0$ satisfying (3.18).

Moreover, given the values $R^\pm$, one may always choose $\rho^\pm > R^\pm$, as claimed in Propositions 3.1 and 3.2.

**Proof.** For $u \in V$, by estimates (4.1), (4.2) and (4.4),
\[
J^\pm(u) \leq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_k} \right) \|u\|^2 + C \left( 1 + \|u\| \right)^q + \|h\|_{L^2} \|u\|.
\]

For $\lambda \in (\lambda_{k-1}, \lambda_k)$ one has $1 - \frac{\lambda}{\lambda_{k-1}} < 0$ and then obtains (3.6) for suitably large $\rho^+_\lambda > R^+$. For $\lambda \in (\lambda_k, \lambda_{k+m})$, since $1 - \frac{\lambda}{\lambda_k} < 1 - \frac{\lambda}{\lambda_{k-1}} < 0$, one obtains, for suitable $K_V$ and $\xi > 0$, Eqs. (3.14) and (3.18).

Finally, let $u \in V \oplus Z$ and set $\epsilon = \lambda - \lambda_k > 0$, we get
\[
J^\pm(u) \leq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_k} \right) \|u\|^2 + C \left( 1 + \|u\|^q \right) + \|h\|_{L^2} \|u\| + C;
\]

it is clear that (once that $\epsilon$ is fixed) this goes to $-\infty$ and then we may find the claimed $\rho^-_\lambda > R^-$ such that (3.13) holds.

Observe that $K_V$ and $E$ can be chosen uniformly for $\lambda \in (\lambda_k, \lambda_{k+m})$, while $\rho^\pm_\lambda$ will in fact depend on $\lambda$. \qed

### 4.2. Estimating the effect of the nontrivial perturbation

In this section we will prove the remaining inequalities in Propositions 3.1 and 3.2, those which rely on the hypotheses (H1) or (H2), which, roughly speaking, say that the perturbation $f$ is nontrivial in such a way that a new solution arises when $\lambda$ is sufficiently near to the eigenvalue $\lambda_k$.

The proof is simpler for problem (1.1$^+$), since we need to estimate the functional in the compact set $SVZ$, while for problem (1.1$^-$) the same kind of estimate is required in the noncompact set $SZW$.

#### 4.2.1. Estimating $J^+$ in $SVZ$

For the next estimates, we will need the following lemma:

**Lemma 4.3.** Hypotheses (f3) and (f4) imply that there exists a nondecreasing function $D : (0, +\infty) \to \mathbb{R}$ such that
\[
\lim_{R \to +\infty} D(R) = +\infty \quad \text{and} \quad \inf_{u \in RSVZ} \int_{\Omega} F(x, u) \, dx > D(R).
\]

**Proof.** First we claim that there exists a constant $\delta > 0$ such that the sets $\Omega_u = \{ x \in \Omega : |u(x)| > \delta \}$ have measure $|\Omega_u| > \delta$, for all $u \in S_{VZ}$.

Actually, the functions $u \in SVZ$ are smooth, they are uniformly bounded and then (since their $L^2$ norm is at least $\lambda_k^{-1/2}$ by (4.2)), the claim follows.

Now, fixed a value $H > 0$, we will show that we can find a $\widetilde{R}$ large enough so that $\int_{\Omega} F(x, Ru) \, dx \geq H$ for any $u \in SVZ$ and $R \geq \widetilde{R}$, this means that
\[
\lim_{R \to +\infty} \inf_{u \in RSVZ} \int_{\Omega} F(x, u) \, dx = +\infty.
\]

In order to do this, we set $M = (H + |\Omega| C_F) \delta^{-1}$: by (f3) we have that there exists $s_0$ such that $F(x, s) > M$ for $|s| > s_0$. 
For \( R > s_0 / \delta \), one has \( \Omega_u \subseteq \{ x \in \Omega : | Ru(x) | > s_0 \} \), and then one gets

\[
\int_{| Ru | \geq s_0} F(x, Ru) \, dx \geq M \delta.
\]

Since by (f4) \( \int_{| Ru | < s_0} F(x, Ru) \, dx \geq -| \Omega | C_F \) one finally obtains

\[
\int_{\Omega} F(x, Ru) \, dx \geq M \delta - | \Omega | C_F = H.
\]

To conclude, since \( \int_{\Omega} F(x, u) \, dx \geq -| \Omega | C_F \) for any \( u \in H^1_0 \), it is elementary that

\[
D(R) := \inf_{\rho \geq R} \inf_{u \in RSVZ} \int_{\Omega} F(x, u) \, dx
\]

is well defined and satisfies the claim. \( \square \)

Now we may prove:

**Lemma 4.4.** Consider problem (1.1+) with one of the sets of hypotheses (H1) or (H2). Given the constant \( D_W \in \mathbb{R} \), there exist \( R^+, \varepsilon > 0 \) such that, for any \( \lambda \in (\lambda_k - \varepsilon_0, \lambda_k) \), Eqs. (3.8) and (3.9) hold.

**Proof.** In the case (H1), (f1) implies that

\[
\text{for any } M > 0 \text{ there exists } C_M \text{ such that } F(x, t) \geq M |t| - C_M; \quad (4.11)
\]

in particular we set \( M = 1 + \| h \|_{L^2} \).

Let \( u \in RSVZ \): for being in a finite-dimensional subspace, all the norms are equivalent, so that (set \( \varepsilon = \lambda_k - \lambda > 0 \) and use estimates (4.1), (4.2) and (4.4))

\[
J^+(u) \leq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_k} \right) \| u \|^2 - \frac{1}{2} \int_{\Omega} F(x, u) \, dx + \| h \|_{L^2} \| u \|
\]

\[
\leq \frac{\varepsilon}{2 \lambda_k} \| u \|^2 - M \| u \| + D_M + \| h \|_{L^2} \| u \|
\]

\[
\leq \frac{\varepsilon}{2 \lambda_k} R^2 - R + D_M. \quad (4.12)
\]

In case (H2), let \( D(R) \) be as in Lemma 4.3; for \( \| u \| = R \), let \( u = v + \phi \) with \( v \in V \) and \( \phi \in Z = H_{\lambda_k} \),

\[
J^+(u) \leq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_{k-1}} \right) \| v \|^2 + \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_k} \right) \| \phi \|^2 - \frac{1}{2} \int_{\Omega} F(x, u) \, dx - \int_{\Omega} hu \, dx.
\]

Since \( \int_{\Omega} hu = \int_{\Omega} hv \) by (h1), and we may suppose that \( 1 - \frac{\lambda}{\lambda_{k-1}} \leq -\delta < 0 \), we estimate

\[
J^+(u) \leq \frac{\varepsilon}{2 \lambda_k} \| \phi \|^2 - \delta \| v \|^2 - D(R) + \| h \|_{L^2} \| v \|;
\]

using \( \| h \|_{L^2} \| v \| \leq C + \delta \| v \|^2 \), we conclude

\[
J^+(u) \leq \frac{\varepsilon}{2 \lambda_k} \| \phi \|^2 + C - D(R)
\]

\[
\leq \frac{\varepsilon}{2 \lambda_k} R^2 - D(R) + C. \quad (4.13)
\]

Considering Eqs. (4.12) and (4.13), we see that since \( \lim_{R \to \infty} D(R) = +\infty \) by Lemma 4.3, we may fix \( R^+ \) so that \( C - D(R^+) < D_W - 1 \) (or \( D_M - R^+ < D_W - 1 \) for the case (H1)) and then for \( 0 < \varepsilon < 2 \lambda_k / (R^+)^2 \) one gets Eq. (3.8).

To obtain Eq. (3.9), we observe that (since \( \lambda > \lambda_{k-1} \)) if \( \phi = 0 \) (that is, if \( u \in V \)), then in estimates (4.12) and (4.13) we may avoid the term \( \frac{\delta}{2 \lambda_k} R^2 \) so that (remember that \( D(R) \) is nondecreasing) \( J^+(u) < D_W - 1 \) for \( \| u \| > R^+ \). \( \square \)
4.2.2. Estimating $J^-$ in $S_{ZW}$

We consider the corresponding of the previous lemma, for problem (1.1–).

**Lemma 4.5.** Consider problem (1.1–) with one of the sets of hypotheses (H1) or (H2). Given the constant $K_V \in \mathbb{R}$, there exist $R^-$, $\varepsilon_1 > 0$ such that for any $\lambda \in (\lambda_k, \lambda_k + \varepsilon_1)$, Eqs. (3.15) and (3.16) hold.

**Proof.** First, we observe that $\varepsilon_1 < \varepsilon_2$ implies that $J_{\varepsilon_1}^- \leq J_{\varepsilon_2}^-$, where $J_{\varepsilon}^-$ is the functional (3.1) with $\lambda = \lambda_k + \varepsilon$; hence it will be sufficient to show that there exists an $\varepsilon > 0$ satisfying the claim.

Then we see from Eq. (4.5), that property (3.16) will be satisfied provided that $R^-$ is large enough (say $R^- > \tilde{R}$); observe that this value can be made independent from $\lambda$ once $\varepsilon$ is small enough.

Now we go into the proof of Eq. (3.15). Let us suppose, for sake of contradiction, that for any two sequences $R_n > 0$ and $\varepsilon_n \to 0^+$ there exist $u_n \in Z \oplus W$ with $\|u_n\| = R_n$ such that $J_{\varepsilon_n}^-(u_n) \leq K_V$. In particular, it is no loss of generality to suppose that these sequences are such that $R_n > \tilde{R}$, $R_n \to +\infty$ and $\varepsilon_n R_n^2 \to 0$.

We write $u_n = w_n + \phi_n$, with $w_n \in W$ and $\phi_n \in Z$, so

$$K_V \geq J_{\varepsilon_n}^-(u_n) \geq \frac{1}{2} \left( 1 - \frac{\lambda_k + \varepsilon_n}{\lambda_k + m} \right) \|w_n\|^2 - \frac{\varepsilon_n}{2\lambda_k} \|\phi_n\|^2 + \int \Omega F(x, u_n) \, dx - \int \Omega h u_n \, dx, \quad (4.14)$$

we divide (4.14) by $R_n^2$, obtaining $\frac{J_{\varepsilon_n}^-(u_n)}{R_n^2} \leq K_V \to 0$; since

$$\varepsilon_n \frac{\|\phi_n\|^2}{R_n^2} \to 0, \quad \int \Omega F(x, u_n) \frac{R_n^2}{R_n^2} \to 0, \quad \int \Omega h u_n \frac{R_n^2}{R_n^2} \to 0,$$

we obtain (estimating $1 - \frac{\lambda_k + \varepsilon_n}{\lambda_k + m} > \delta_1 > 0$)

$$\delta_1 \frac{\|w_n\|^2}{R_n^2} \leq o(1), \quad (4.15)$$

that is $\frac{\|w_n\|^2}{R_n^2} \to 0$, and since $\frac{\|w_n\|^2}{R_n^2} = 1 - \frac{\|\phi_n\|^2}{R_n^2}$ we deduce that $\|\phi_n\| \to R_n$.

Our aim is now to show that this last result implies that the $L^1$-norm and the $H^1$-norm of $u_n$ may be interchanged for $n$ large enough. For this purpose consider $U_n = \frac{w_n}{R_n} = z_n + \tau_n$ where $z_n = w_n/R_n \to 0$ and, since $Z$ is finite-dimensional, up to a subsequence $\tau_n = \frac{\phi_n}{R_n} \to \tau$ uniformly, with $\tau \in Z$ and $\|\tau\| = 1$.

We claim that

there exist $\delta > 0$ and an integer $n$ such that for $n > \tilde{n}$,

the sets $\Omega_n = \{ x \in \Omega : |U_n(x)| > \delta \}$ have measure $|\Omega_n| > \delta$. \quad (4.16)

Actually, since $\tau_n \to \tau$ uniformly and $z_n \to 0$ in $L^2$, there exists a $\delta > 0$ such that for $n$ large enough, there exist $\Omega_n$ with $|\Omega_n| > \delta$ such that $|\tau_n(x)| > 2\delta$ and $|z_n(x)| < \delta$ for almost every $x \in \Omega_n$, so that $|U_n| > \delta$ a.e. in $\Omega_n$.

Now we consider the two sets of hypotheses separately.

- In case (H1), by Eq. (4.11) where we set $M = (1 + \|h\|_{L^2})/\delta^2$, we have

$$\int \Omega F(x, R_n U_n) \, dx \geq M R_n \int \Omega |U_n| \, dx - D_M \geq M R_n \delta^2 - D_M,$$

then we get $\int \Omega F(x, R_n U_n) \, dx - \int \Omega h R_n U_n \, dx \geq M R_n \delta^2 - D_M - \|h\|_{L^2} R_n \geq R_n - D_M$. Since, we suppose that $\varepsilon_n R_n^2 \to 0$, Eq. (4.14) becomes

$$K_V + D_M + o(1) \geq R_n \to +\infty. \quad (4.17)$$

- In case (H2), going back to the proof of Lemma 4.3, we see that condition (4.16) implies

$$\lim_{R \to +\infty} \inf_{n > \tilde{n}} \int \Omega F(x, R_n U_n) \, dx = +\infty; \quad (4.18)$$
also, by hypothesis (h1), $\int_\Omega h u_n = \int_\Omega h w_n$ and we may estimate
\[
\frac{1}{2} \left( 1 - \frac{\lambda_k + \epsilon_n}{\lambda_{k+m}} \right) \| u_n \|^2 - \int_\Omega h w_n \, dx \geq -\delta_2
\]
for a suitable $\delta_2 > 0$; with this estimate and since we suppose $\epsilon_n R_n^2 \to 0$, Eq. (4.14) becomes (using also (4.18))
\[
K_V + \delta_2 + o(1) \geq \int_\Omega F(x, R_n U_n) \, dx \to +\infty.
\]
Eqs. (4.17) and (4.19) provide the contradiction which proves our claim. \qed

4.3. Linking condition

We conclude this section with the proof of the linking condition that we used at the end of the proof in Section 3.3, in order to distinguish the level of the two solutions.

Lemma 4.6. For $\rho > R > 0$, the set $\rho S_{VZ}$ links with the set
\[
\hat{W} = \{ u \in W : \| u \| \geq R \} \cup RS_{ZW},
\]
that is, the image of any continuous map $\psi : \rho B_{VZ} \to H_0^1$ with $\psi|_{\rho S_{VZ}} = \text{Id}$ intersects $\hat{W}$.

Proof. We will prove a slightly different statement: we set an arbitrary $\phi_k \in H_{l_k} \setminus \{0\}$ and we will prove that the above intersection property holds with the subset of $\hat{W}$ given by
\[
\tilde{W} = \{ u \in W : \| u \| \geq R \} \cup \{ u = t\phi_k + w \in Z \oplus W : \| u \| = R \text{ and } t > 0 \}.
\]
Consider the decomposition $H_0^1 = H = V \oplus Z \oplus W$ and denote by $P_W : H \to W$ and $P_{VZ} : H \to V \oplus Z$ the orthogonal projections.

The map $M : \tilde{W} \to W; \tilde{u} \mapsto P_W(\tilde{u})$ is a continuous bijection, in fact, it is a homeomorphism.

Now observe that the action of the map $M$ is a translation parallel to the subspace $V \oplus Z$ (in which lies $\rho S_{VZ}$) and that $W$ is orthogonal to this subspace. Then we may extend the map $M$ to the map
\[
\tilde{M} : H \to H : v + z + w \mapsto v + z + w + (w - M^{-1}(w))
\]
which is still a homeomorphism and which translates each plane parallel to $V \oplus Z$ by the same quantity. Since the plane containing $\rho S_{VZ}$ intersects $W$ in the origin and $0 - M^{-1}(0) = -R\phi_k$, this plane is translated exactly by this vector.

Finally, consider any map $\psi : \rho B_{VZ} \to H$ with $\psi|_{\rho S_{VZ}} = \text{Id}$ and consider the composition $\Theta = P_{VZ} \circ \tilde{M} \circ \psi$; $\psi$ is the identity on $\rho S_{VZ}$ and so $\Theta|_{\rho S_{VZ}} = \text{Id} - R\phi_k$ and the topological degree $\text{deg}(\Theta, \rho B_{VZ}, 0) = \text{deg}(\text{Id} - R\phi_k, \rho B_{VZ}, 0) = \text{deg}(\text{Id}, \rho B_{VZ}, R\phi_k) = 1$, since $\rho > R$ implies that $R\phi_k \in \rho B_{VZ}$. This implies that there exists $p \in \rho B_{VZ}$ such that $\Theta(p) = 0$, that is $\psi(p) \in M^{-1}(W) = \tilde{W}$, giving the claimed linking property. \qed

5. Proof of the geometry in Propositions 3.1 and 3.2

We finally give the proof of Propositions 3.1 and 3.2, which is nothing but a resume of the lemmata above, verifying that all the constants can be chosen sequentially without contradictions.

Proof of Proposition 3.1. Under hypothesis (f1), if we fix a value $\lambda \in (\lambda_{k-1}, \lambda_k)$, then we obtain the constant $D_\lambda$ from Lemma 4.1 and with this we get $\rho_\lambda^+$ from Lemma 4.2.

If we consider also one of the two sets of hypotheses (H1) or (H2), then we proceed as follows: first of all, we determine (once for ever) the constant $D_W$ from Lemma 4.1; with this we obtain from Lemma 4.4 the values $R^+$ and $\epsilon_0$. Then, for any (now fixed) $\lambda \in (\lambda_k - \epsilon_0, \lambda_k)$, we obtain from Lemma 4.1 the value $D_\lambda$. Finally, we can get from Lemma 4.2 the corresponding value of $\rho_\lambda^+ > R^+$. \qed
Proof of Proposition 3.2. Under hypothesis (f1), if we fix a value \( \lambda \in (\lambda_k, \lambda_k + m) \), then we obtain the constant \( K_\lambda \) from Lemma 4.1 and with this we get \( \rho_\lambda^- \) from Lemma 4.2.

If we consider also one of the two sets of hypotheses (H1) or (H2), then we proceed as follows: first of all, we determine (once for ever) the constant \( KV \) from Lemma 4.2; with this we obtain from Lemma 4.5 the values \( R^- \) and \( \varepsilon_1 \). Since we have \( R^- \), we can get from Lemma 4.1 the constant \( E \) and with this obtain \( \xi \) from Lemma 4.2.

Finally, for any (now fixed) \( \lambda \in (\lambda_k, \lambda_k + \varepsilon_1) \), we obtain from Lemma 4.1 the constant \( K_\lambda \) and with this we get from Lemma 4.2 the corresponding value of \( \rho_\lambda^- > R^- \).

6. Further results

6.1. The case with trivial solution

Now we will assume that \( h(x) \equiv 0 \) and \( f(x, 0) = 0 \), so that the problem (1.1+) becomes the following one and has the trivial solution \( u \equiv 0 \):

\[
\begin{align*}
-\Delta u &= \lambda u + f(x, u) \quad \text{in} \ \Omega, \\
u &= 0 \quad \text{on} \ \partial\Omega.
\end{align*}
\] (6.1)

In order to prove that the solutions found in Theorem 1.1 are nontrivial, we may consider some hypotheses on the behavior of the nonlinearity at the origin, which will allow us to estimate and compare the critical groups of these solutions with those of the trivial one.

We obtain the following result:

**Theorem 6.1.** In the hypotheses of Theorem 1.1, part (a), if \( f(x, 0) = 0 \) and

\[
\lim_{t \to 0} \frac{f(x, t) - m(x)t}{t} = 0
\]

with \( |m(x)| = 0 \) and \( 1 < p < 2 \), then the two solutions of (6.1) given by Theorem 1.1 are nontrivial.

**Proof.** For \( \lambda \in (\lambda_k - \varepsilon_0, \lambda_k) \) the solutions \( u_k \) and \( u_{k-1} \) found in Theorem 1.1 come from the classical saddle point theorem, so it is known (see [17]) that they have at least a nontrivial critical group. Then they are nontrivial since, with the given additional hypothesis, the critical groups of the trivial solution are all zero, by Theorem 2.1 in [10].

6.2. The resonant problem

The same techniques that we used for Theorem 1.1, can be exploited in the resonant case \( \lambda = \lambda_k \), provided that a suitable sublinear term takes the place of the small perturbation \( (\lambda - \lambda_k)u \) which avoids resonance in problem (1.1); in particular we consider the following problems (6.2±):

\[
\begin{align*}
-\Delta u &= \lambda_k u + \eta g(x, u) \pm f(x, u) + h(x) \quad \text{in} \ \Omega, \\
u &= 0 \quad \text{on} \ \partial\Omega,
\end{align*}
\] (6.2)

and we obtain:

**Theorem 6.2.** Let \( \lambda_k (k \geq 2) \) be an eigenvalue of multiplicity \( m \), \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function satisfying

\[
g(x, t)t \geq 0 \quad \text{and} \quad C_1(|r|^{r-1} - 1) \leq |g(x, t)| \leq C_2(1 + |r|^{r-1})
\] (6.3)

for suitable \( C_{1,2} > 0, r \in (q, 2) \), and let \( h \in L^r(\Omega) \).

Under hypothesis (f1) and one of the sets of hypotheses (H1) or (H2), one gets:

(a) there exists \( \varepsilon_0 > 0 \) such that for \( \eta \in (-\varepsilon_0, 0) \) there exist two solutions of (6.2+);
(b) there exists \( \varepsilon_1 > 0 \) such that for \( \eta \in (0, \varepsilon_1) \) there exist two solutions of (6.2−).

A model problem in the case of Theorem 6.2 could be with any of the \( f \) given in Remark 1.2 and \( g(x, u) = |u|^{r-2}u \), with \( q < r < 2 \).
Proof. We will consider the $C^1$ functional defined for $u \in H^1_0(\Omega)$
\[ \mathcal{F}^\pm (u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda_k u^2) \, dx - \eta \int_\Omega G(x, u) \, dx \mp \int_\Omega F(x, u) \, dx - \int_\Omega hu \, dx, \] (6.4)
where $G(x, t) = \int_0^t g(x, s) \, ds$.

The PS condition for $\mathcal{F}^\pm$ was proved in [18] under the given hypotheses, provided $\eta \neq 0$.

From (6.3) we get
\[ G(u) \geq 0, \] (6.5)
\[ |G(x, u)| \leq C(1 + |u|^r), \] (6.6)
\[ G(x, u) \geq C(|u|^r - 1), \] (6.7)
so that $\int_\Omega G(x, u) \, dx$ is of order lower than $\|u\|^2$, but higher than $\|u\|_{L^q}^2$: this is what allows us to obtain our result: we sketch below the main differences from the proof of Theorem 1.1.

In Eqs. (4.5), (4.7) and (4.8) the term in $\|u\|^2$ is still dominant, so that we may obtain the same conclusions from these equations, once that we fix a bound for $|\eta|$.

In Eqs. (4.6), (4.9) and (4.12)–(4.13) the term $\frac{\eta}{2} \|u\|^2$ disappears but it is substituted by a term of the form $|\eta| \int_\Omega G(x, u) \, dx$.

For Eq. (4.6) we estimate in the $L^r$-norms instead of the $H^1_0$-norms and we get, with $\eta < 0$ and $u \in Z \oplus W$,
\[ \mathcal{F}^\pm (u) \geq (-\eta) \int_\Omega G(x, u) \, dx - C(1 + \|u\|_{L^q}^q) - \|h\|_{L^r} \|u\|_{L^r} \] (6.8)
\[ \geq (-\eta)C(\|u\|_{L^r} - 1) - C(1 + \|u\|_{L^q}^q) - \|h\|_{L^r} \|u\|_{L^r} \geq D_\eta, \]
providing the analogue of (3.5).

For Eq. (4.9) we use (6.7), while for Eqs. (4.12)–(4.13) we use (6.6), and in both cases we can still use the $H^1_0$-norms since in $V \oplus Z$ all the norms are equivalent.

Finally, in Lemma 4.5 we still have the property $\tilde{\mathcal{F}}_{\eta_2} \leq \tilde{\mathcal{F}}_{\eta_1}$ for $\eta_1 < \eta_2$ by (6.5); then, one works with $\eta_n \to 0^+$ in place of $\epsilon_n$ and chooses $\epsilon_n \eta_n R_n^R \to 0$, so that Eq. (4.14) becomes
\[ K \geq \tilde{\mathcal{F}}_{\eta_n}(u_n) \] (6.9)
\[ \geq \frac{1}{2} \left(1 - \frac{\lambda_k}{\lambda_{k+m}}\right) \|w_n\|^2 - C\eta_n \left(\|w_n\|^2 + \|\phi_n\|^2\right)^{1/2} + \int_\Omega F(x, u_n) \, dx - \int_\Omega hu_n \, dx; \]
from which again one gets (4.15) by dividing by $R_n^2$ and the rest of the proof goes on the same lines. \qed

References


