On Miki’s identity for Bernoulli numbers

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Abstract

We give a short proof of Miki’s identity for Bernoulli numbers,

\[ \sum_{i=2}^{n-2} \beta_i \beta_{n-i} - \sum_{i=2}^{n-2} \binom{n}{i} \beta_i \beta_{n-i} = 2H_n \beta_n \]

for \( n \geq 4 \) where, \( \beta_i = B_i / i \), \( B_i \) is the \( i \)th Bernoulli number, and \( H_n = 1 + 1/2 + \cdots + 1/n \).

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1. Introduction

The Bernoulli numbers \( B_n \) are defined by

\[ \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}. \]
There are many identities involving binomial convolutions of Bernoulli numbers, the best known being Euler’s identity

\[ \sum_{i=2}^{n-2} \binom{n}{i} B_i B_{n-i} = -(n+1)B_n \]

for \( n \geq 4 \). Euler’s identity is an easy consequence of the formula

\[ b(x)^2 = (1 - x)b(x) - xb'(x), \]

where \( b(x) = x/(e^x - 1) \). Many similar identities for Bernoulli numbers can be proved in the same way. (See, e.g., \([1,4]\).)

A more mysterious identity was proved by Hiroo Miki \([5]\). Let \( \beta_n = B_n/n \) and let \( H_n \) be the harmonic number \( 1 + 1/2 + \cdots + 1/n \). Then

\[ \sum_{i=2}^{n-2} \beta_i \beta_{n-i} - \sum_{i=2}^{n-2} \binom{n}{i} \beta_i \beta_{n-i} = 2H_n \beta_n. \]  

(1)

This identity is unusual because it involves both a binomial convolution and an ordinary convolution.

Miki gave a complicated proof of his identity that was based on a formula for the Fermat quotient \((a^p - a)/p\) modulo \( p^2 \). He showed that both sides of (1) are congruent modulo \( p \) for every sufficiently large prime \( p \), which implies that they are equal. Another proof of Miki’s identity, using \( p \)-adic analysis, was given by Shiratani and Yokoyama \([6]\).

We give here a simple proof of Miki’s identity, based on two different expressions for Stirling numbers of the second kind \( S(n, k) \), which may be defined by the ordinary generating function

\[ \sum_{n=0}^{\infty} S(n, k)x^n = \frac{x^k}{(1 - x)(1 - 2x) \cdots (1 - kx)} \]  

(2)

or by the exponential generating function

\[ \sum_{n=0}^{\infty} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}. \]  

(3)
The equivalence of (2) and (3) follows by comparing the partial fraction expansion
\[ \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{1-jx} \]
with the binomial theorem expansion
\[ \frac{(e^x - 1)^k}{k!} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{jx}. \]

From each of (2) and (3) one can derive a formula that for fixed \(n\) expresses \(S(m+n, m)\) as a polynomial in \(m\). (For some combinatorial applications of these polynomials, see [3].) Equating coefficients of \(m^2\) in these formulas gives Miki’s identity. Similarly, equating coefficients of higher powers of \(m\) gives an infinite sequence of related, though more complicated, identities, of which the next one is (for \(n \geq 4\))

\[ \sum_{\substack{i+j+k=n \\ i,j,k \geq 2}} \binom{n}{i,j,k} \beta_i \beta_j \beta_k + 3H_n \sum_{i=2}^{n-2} \binom{n}{i} \beta_i \beta_{n-i} + 6H_{n,2} \beta_n \]
\[ = \sum_{\substack{i+j+k=n \\ i,j,k \geq 2}} \beta_i \beta_j \beta_k + \frac{n^2 - 3n + 5}{4} \beta_{n-2}, \tag{4} \]

where \(H_{n,2} = \sum_{1 \leq i < j \leq n} (ij)^{-1}\).

2. Lemmas

The \textit{Nörlund polynomials} \(B_n^{(z)}\) are defined by
\[ \sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!} = \left( \frac{x}{e^x - 1} \right)^z. \]

(If \(z\) is a nonnegative integer, then \(B_n^{(z)}\) is called a \textit{Bernoulli number of order} \(z\).) Note that \(B_n^{(1)} = B_n\) and that for fixed \(n\), \(B_n^{(z)}\) is a polynomial in \(z\) of degree \(n\). As is well known, the Stirling numbers of the second kind can be expressed in terms of these polynomials:

\textbf{Lemma 1.}

\[ S(m+n, m) = \binom{m+n}{n} B_n^{(-m)}. \]
Proof. From (3) we have

\[ \sum_{n=0}^{\infty} \frac{B_n(-m)}{n!} x^n = \left( \frac{e^x - 1}{x} \right)^m = \sum_{n=0}^{\infty} \frac{S(m+n, m)}{(m+n)!} \frac{m! n!}{n!} x^n, \]

and the result follows. □

To find a formula for coefficients of powers of \( m \) in \( S(m+n, m) \) we need to expand both \( \binom{m+n}{n} \) and \( B_n(-m) \) in powers of \( m \).

Let us define generalized harmonic numbers \( H_{n,j} \) by

\[ H_{n,j} = \sum_{1 \leq k_1 < k_2 < \cdots < k_j \leq n} \frac{1}{k_1 k_2 \cdots k_j}, \]

with \( H_{n,0} = 1 \). Then \( H_n = H_{n,1} \). The numbers \( H_{n,j} \) are closely related to the unsigned Stirling numbers of the first kind \( c(n, j) \) which may be defined by

\[ x(x+1) \cdots (x+n-1) = \sum_{j=0}^{n} c(n, j)x^j; \]

we have \( H_{n,j} = c(n+1, j+1)/n! \).

Lemma 2.

\[ \binom{m+n}{n} = \sum_{j=0}^{n} H_{n,j} m^j. \]

Proof. This follows easily from

\[ \binom{m+n}{n} = \frac{m+n}{n} \cdot \frac{m+n-1}{n-1} \cdots \frac{m+1}{1} = \left( 1 + \frac{m}{1} \right) \left( 1 + \frac{m}{2} \right) \cdots \left( 1 + \frac{m}{n} \right). \]

Now let us define \( \beta_n \) to be \( (-1)^n B_n/n \) for \( n \geq 1 \). (Note that \( \beta_n = B_n/n \) except for \( \beta_1 = -B_1 = 1/2 \), since \( B_n = 0 \) when \( n \) is odd and greater than 1.) We define \( \beta_n^{(j)} \) by

\[ \beta_n^{(j)} = \frac{1}{j!} \sum_{i_1+i_2+\cdots+i_j=n} \binom{n}{i_1, \ldots, i_j} \beta_{i_1} \cdots \beta_{i_j}, \quad (5) \]

where the sum is over positive integers \( i_1, \ldots, i_j \).
Lemma 3. For \( n > 0 \),

\[
B_n^{(-m)} = \sum_{j=1}^{n} \beta_n^{(j)} m^j.
\]

Proof. We have

\[
\sum_{n=0}^{\infty} B_n^{(-m)} \frac{x^n}{n!} = \left( \frac{e^x - 1}{x} \right)^m = \exp \left( m \log \left( \frac{e^x - 1}{x} \right) \right)
\]

\[
= \sum_{j=0}^{\infty} \frac{m^j}{j!} \left[ \log \left( \frac{e^x - 1}{x} \right) \right]^j.
\]

Since

\[
d \log \left( \frac{e^x - 1}{x} \right) = \frac{1}{x} \left( \frac{e^{-x}}{e^x - 1} - 1 \right) = \sum_{n=1}^{\infty} (-1)^n B_n \frac{x^{n-1}}{n!},
\]

we have

\[
\log \left( \frac{e^x - 1}{x} \right) = \sum_{n=1}^{\infty} (-1)^n \frac{B_n x^n}{n n!} = \sum_{n=1}^{\infty} \beta_n \frac{x^n}{n!}
\]

and thus

\[
\frac{1}{j!} \left[ \log \left( \frac{e^x - 1}{x} \right) \right]^j = \sum_{n=0}^{\infty} \beta_n^{(j)} \frac{x^n}{n!}.
\]

□

Now we apply formula (2), which we rewrite as

\[
\frac{1}{(1-x)(1-2x) \cdots (1-mx)} = \sum_{n=0}^{\infty} S(m+n, n)x^n. \tag{6}
\]

Lemma 4.

\[
\sum_{n=0}^{\infty} S(m+n, n)x^n = \exp \left[ \sum_{k=1}^{\infty} \frac{x^k}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{m^{j+1}}{j+1} B_{k-j} \right].
\]
Proof. The left-hand side of (6) may be written as
\[ \exp \left[ \sum_{k=1}^{\infty} \frac{(1^k + \cdots + m^k)x^k}{k} \right]. \]

There is a well-known formula for expressing the power sum \(1^k + \cdots + m^k\) in terms of Bernoulli numbers which may be derived as follows:
\[
\sum_{k=0}^{\infty} \frac{(1^k + \cdots + m^k)x^k}{k!} = e^x + \cdots + e^{mx} = \frac{e^{(m+1)x} - e^x}{e^x - 1} = \frac{e^{mx} - 1}{1 - e^{-x}} = \frac{e^{mx} - 1}{xe^{-x} - 1}. \tag{7}
\]

Equating coefficients of \(x^k\) in (7) gives
\[
1^k + \cdots + m^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{m^{j+1}}{j+1} B_{k-j}. \tag{8}
\]

3. The Proof

Combining Lemmas 1–3 gives our first formula for expressing \(S(m + n, m)\) as a polynomial in \(m\): For \(n > 0\),
\[
S(m + n, m) = \beta_n m + \left( \beta_n^{(2)} + H_n \beta_n \right) m^2 + \left( \beta_n^{(3)} + H_n \beta_n^{(2)} + H_{n,2} \beta_n \right) m^3 + \cdots. \tag{8}
\]

On the other hand, expanding the right-hand side of Lemma 4 gives the alternative formula (for \(n \geq 2\))
\[
S(m + n, n) = \beta_n m + \left( \frac{1}{2} (-1)^{n-1} B_{n-1} + \frac{1}{2} \sum_{i=1}^{n-1} \beta_i \beta_{n-i} \right) m^2 + C m^3 + \cdots, \tag{9}
\]
where
\[
C = \frac{1}{6} \sum_{i+j+k=n} \beta_i \beta_j \beta_k + \frac{1}{2} \sum_{i=1}^{n-1} (-1)^{n-i-1} \beta_i B_{n-i-1} + \frac{1}{6} (-1)^n (n-1) B_{n-2}. \]
Equating coefficients of $m^2$ in (8) and (9) yields the identity

$$\beta_n^{(2)} + H_n \beta_n = \frac{1}{2}(-1)^{n-1}B_{n-1} + \frac{1}{2} \sum_{i=1}^{n-1} \beta_i \beta_{n-i},$$

which for $n$ even and greater than 2 is Miki’s identity. (For $n$ odd, the coefficient of $m^2$ in both (8) and (9) is $\frac{n}{2} \beta_{n-1}$.)

Equating coefficient of $m^3$ in (8) and (9) yields the identity

$$\beta_n^{(3)} + H_n \beta_n^{(2)} + H_n \beta_n = \frac{1}{6} \sum_{i+j+k=n} \beta_i \beta_j \beta_k + \frac{1}{2} \sum_{i=1}^{n-1} (-1)^{n-i-1} \times \beta_i B_{n-i-1} + \frac{1}{6} (-1)^n (n-1) B_{n-2}. \quad (10)$$

For $n$ odd and greater than 3, (10) reduces to Miki’s identity. For $n$ even, we may simplify (10) by separating all occurrences of $\beta_1$ or $B_1$, and multiplying by 6, obtaining (4) for $n \geq 4$.

4. A generalization

An identity related to Miki’s was found by Faber and Pandharipande, and proved by Zagier [2]. Their identity may be written

$$\frac{n}{2} \sum_{i=2}^{n-2} \frac{B_i(\frac{1}{2})}{i} B_{n-i}(\frac{1}{2}) \frac{n+i-3}{n-i} - \sum_{i=0}^{n-2} \binom{n}{i} B_i(\frac{1}{2}) \beta_{n-i} = H_{n-1} B_n(\frac{1}{2}). \quad (11)$$

As before, $\beta_n = (-1)^n B_n / n$, and $B_n(\lambda)$ is the Bernoulli polynomial, defined by

$$\sum_{n=0}^{\infty} B_n(\lambda) \frac{x^n}{n!} = \frac{x e^{\lambda x}}{e^x - 1}.$$

Thus $B_n(0) = B_n$ and it is well known that $B_n(\frac{1}{2}) = (2^{1-n} - 1) B_n$.

By using the approach of this paper, one can prove a common generalization of the Faber–Pandharipande–Zagier identity and Miki’s identity:

$$\frac{n}{2} \left( B_{n-1}(\lambda) + \sum_{i=1}^{n-1} \frac{B_i(\lambda)}{i} B_{n-i}(\lambda) \frac{n+i-3}{n-i} \right) - \sum_{i=0}^{n-1} \binom{n}{i} B_i(\lambda) \beta_{n-i} = H_{n-1} B_n(\lambda) \quad (12)$$
for all $n \geq 1$. To derive (11) from (12), we observe that $B_i(\frac{1}{2})$ is 0 for $i$ odd, so for $\lambda = \frac{1}{2}$ and $n$ even, the terms in (12) that do not appear in (11) are all 0 (and for $n$ odd all terms in (11) are 0).

We can derive Miki’s identity (1) from the case $\lambda = 0$ of (12). We first observe that

$$ \frac{1}{i(n-i)} = \frac{1}{n} \left( \frac{1}{i} + \frac{1}{n-i} \right) $$

we get

$$ \sum_{i=2}^{n-2} \binom{n}{i} B_i B_{n-i} = \frac{2}{n} \sum_{i=2}^{n-2} \binom{n}{i} B_i B_{n-i}. $$

Thus (1) multiplied by $n/2$ is equivalent to

$$ \frac{n}{2} \sum_{i=2}^{n-2} B_i B_{n-i} - \sum_{i=2}^{n-2} \binom{n}{i} B_i B_{n-i} = H_n B_n. $$

For $n$ even and greater than 2, this can be obtained from the case $\lambda = 0$ of (12) by adding $\binom{n}{0} B_0 B_n = B_n/n$ to both sides and deleting some terms that are 0.

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References