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Test elements, retracts and automorphic orbits

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A R T I C L E   I N F O

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A B S T R A C T

Let A_2 be a free associative or polynomial algebra of rank two over a field K of characteristic zero. Based on the degree estimate of Makar-Limanov and J.-T. Yu, we prove: (1) An element $p \in A_2$ is a test element if p does not belong to any proper retract of A_2; (2) Every endomorphism preserving the automorphic orbit of a nonconstant element of A_2 is an automorphism.

1. Introduction and main results

In the sequel, K always denotes a field of characteristic zero. Automorphisms (endomorphisms) always mean K-automorphisms (K-endomorphisms).

Let A_n be a free associative or polynomial algebra of rank n over K. An element $p \in A_n$ is called a *test element* if every endomorphism of A_n fixing p is an automorphism. A subalgebra R of A_n is called a *retract* if there is an idempotent endomorphism $\pi(\pi^2 = \pi)$ of A_n (called a *retraction* or a *projection*) such that $\pi(A_n) = R$. Test elements and retracts of groups and other algebras are defined in a similar way. Test elements and retracts of algebras and groups have recently been studied in [3,5–7,12,16,18–22,27,28,30,31].

A test element does not belong to any proper retract for any algebra or group as the corresponding noninjective idempotent endomorphism is not an automorphism. The converse is proved by Turner [32] for free groups, by Mikhalev and Zolotykh [22] and by Mikhalev and J.-T. Yu [19,20]

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for free Lie algebras and free Lie superalgebras respectively, and by Mikhalev, Umirbaev and J.-T. Yu [17] for free nonassociative algebras. See also Mikhalev, Shpilrain and J.-T. Yu [16].

In view of the above, we may raise the following

Conjecture 1. *If an element $p \in A_n$ does not belong to any proper retract of A_n, then p is a test element.*

Recently, V. Shpilrain and J.-T. Yu [31] proved Conjecture 1 for $\mathbb{C}[x, y]$. A key lemma in their proof is the degree estimate of Shestakov and Umirbaev [24], which plays a crucial role in the recent celebrated solution of the Nagata conjecture [25,26] and the Strong Nagata conjecture [33].

More recently, Makar-Limanov and J.-T. Yu [15] developed a new combinatorial method based on the lemma on radicals and obtained a sharp degree estimate for the 'free' case, namely, for a free associative algebra or a polynomial algebra over a field of characteristic zero. It has found applications for automorphisms and coordinates of polynomial and free associative algebras. See S.-J. Gong and J.-T. Yu [9].

Now we consider another related problem. In an algebra or a group, certainly an automorphism preserves the automorphic orbit of an element p. The converse is proved by Shpilrain [29] and Ivanov [10] for free groups of rank two, by D. Lee [14] for free groups of any rank, by Mikhalev and J.-T. Yu [20] for free Lie algebras and by Mikhalev, Umirbaev and J.-T. Yu [17] for free nonassociative algebras, by van den Essen and Shpilrain [7] for A_2 when p is a coordinate, by Jelonek [11] for polynomial algebras over $\mathbb{C}$ when p is a coordinate. For the related linear coordinate preserving problem, see, for instance, S.-J. Gong and J.-T. Yu [8]. See also the book [16].

In view of the above, we may raise the following

Conjecture 2. *Let $p \in A_n - K$. Then any endomorphism of A_n preserving the automorphic orbit of p must be an automorphism.*

Conjecture 2 has recently been settled affirmatively by J.-T. Yu [34] for $A_2 = \mathbb{C}[x, y]$ based on Shpilrain and J.-T. Yu's characterization of test elements of $\mathbb{C}[x, y]$ in [31] and the main result in Drensky and J.-T. Yu [6].

In this paper, based on the recent degree estimate of Makar-Limanov and J.-T. Yu [15], the main ideals and techniques in Drensky and J.-T. Yu [6], Shpilrain and J.-T. Yu [30,31], and J.-T. Yu [34], we prove both Conjectures 1 and 2 for $n = 2$. Our main results are

Theorem 1.1. *If an element $p \in A_2$ does not belong to any proper retract of A_2, then p is a test element of A_2.*

Theorem 1.1 was proved by Shpilrain and J.-T. Yu [31] for $A_2 = \mathbb{C}[x, y]$.

Theorem 1.2. *If an endomorphism ϕ of A_2 preserves the automorphic orbit of a nonconstant element $p \in A_2$, then ϕ is an automorphism of A_2.*

Theorem 1.2 was proved by J.-T. Yu [34] for $A_2 = \mathbb{C}[x, y]$.

Crucial to the proofs of the above two theorems are the following two results, which have their own interests.

Theorem 1.3. *Let $p \in A_2$ have outer rank two. Then any injective endomorphism ϕ of A_2 is an automorphism if $\phi(p) = p$.*

Theorem 1.3 may be viewed as an analogue of a result in Turner [32] for free groups. It was proved for $A_2 = \mathbb{C}[x, y]$ in J.-T. Yu [34] based on a result in Shpilrain and J.-T. Yu [31].

Theorem 1.4. *An element $p(x, y) \in A_2$ belongs to a proper retract of A_2 if $p(x, y)$ is fixed by a noninjective endomorphism ϕ of A_2. Moreover, in this case there exists a positive integer m such that ϕ^m is a retraction of A_2.*

Theorem 1.4 was proved for $A_2 = \mathbb{C}[x, y]$ in Drensky and J.-T. Yu [6].

2. Proofs

The following two lemmas are Theorem 1.1 and Proposition 1.2 in Makar-Limanov and J.-T. Yu [15].

Lemma 2.1. *Let $A_n = K\langle x_1, \ldots, x_n\rangle$ be a free associative algebra over a field K of characteristic zero, $f, g \in A$ be algebraically independent, f^+ and g^+ are algebraically independent, or f^+ and g^+ are algebraically dependent and neither* $\deg(f) \mid \deg(g)$ *nor* $\deg(g) \mid \deg(f)$, *$p \in K\langle x, y\rangle$. Then*

$$\deg\bigl(p(f, g)\bigr) \geqslant \frac{\deg[f, g]}{\deg(fg)} w_{\deg(f),\deg(g)}(p).$$

Here deg is the total degree, $w_{\deg(f),\deg(g)}(p)$ is the weighted degree of p when the weight of the first variable is $\deg(f)$ and the weight of the second variable is $\deg(g)$, f^+ and g^+ are the highest homogeneous components of f and g respectively, and $[f, g] = fg - gf$ is the commutator of f and g.

Lemma 2.2. *Let $A_n = K[x_1, \ldots, x_n]$ be a polynomial algebra over a field K of characteristic zero, $f, g \in A$ be algebraically independent, $p \in K[x, y]$. Then*

$$\deg\bigl(p(f, g)\bigr) \geqslant w_{\deg(f),\deg(g)}(p)\left[1 - \frac{(\deg(f), \deg(g))(\deg(fg) - \deg(J(f, g)) - 2)}{\deg(f)\deg(g)}\right].$$

Here deg is the total degree, $w_{\deg(f),\deg(g)}(p)$ is the weighted degree of p when the weight of the first variable is $\deg(f)$ and the weight of the second variable is $\deg(g)$, $(\deg(f), \deg(g))$ is the greatest common divisor of $\deg(f)$ and $\deg(g)$, $\deg(J(f, g))$ is the largest degree of nonzero Jacobian determinants of f and g with respect to two of $x_1, \ldots, x_n$.

The following characterization of a proper retract of A_2 was obtained by Shpilrain and J.-T. Yu [30] based on a result of Costa [3].

Lemma 2.3. *Let R be a proper retract of A_2. Then $R = K[r]$ for some $r \in A_2$. Moreover, there exists an automorphism α of A_2 such that $\alpha(r) = x + w(x, y)$, where $w(x, y)$ belongs to the ideal of A_2 generated by y.*

Lemma 2.4. *Let $p \in A_2$ with outer rank* 2 *and $f, g \in A_n$. Then* $w_{\deg(f),\deg(g)}(p) \geqslant \deg(f) + \deg(g)$. *If every monomial of p contains both x and y and* $\deg(p) > 2$, *then* $w_{\deg(f),\deg(g)}(p) > \deg(f) + \deg(g)$.

Proof. (1) If p contains a monomial containing both x and y, where $i \neq 0, j \neq 0$, $w_{\deg(f),\deg(g)}(p) \geqslant i(\deg(f)) + j(\deg(g)) \geqslant \deg(f) + \deg(g)$. If every monomial of p contains both x and y and $\deg(p) > 2$, then the second inequality becomes strict.

(2) Otherwise p must contain monomials x^i and y^j where $i \geqslant 2$, $j \geqslant 2$. Then $w_{\deg(f),\deg(g)}(p) \geqslant 2\max\{\deg(f), \deg(g)\} \geqslant \deg(f) + \deg(g)$. □

Lemma 2.5. *Let $A_n = K\langle x_1, \ldots, x_n\rangle$ be a free associative algebra over an arbitrary field K of zero characteristic, $f, g \in A_2$ be algebraically independent, $p \in K\langle x, y\rangle$ have outer rank two. Then*

$$\deg\bigl(p(f, g)\bigr) \geqslant \deg[f, g].$$

If every monomial of p contains both x and y and $\deg(p) > 2$, *then*

$$\deg\bigl(p(f, g)\bigr) > \deg[f, g].$$

Proof. Let (1) If f^+ and g^+ are algebraically independent; or f^+, g^+ are algebraically dependent, but $\deg(f) \nmid \deg(g)$ and $\deg(g) \nmid \deg(f)$. Then by Lemmas 2.1 and 2.4, $\deg(p(f, g)) \geqslant \deg[f, g]$. If, in addition, every monomial of p contains both x and y and $\deg(p) > 2$, then by Lemmas 2.1 and 2.4, $\deg(p(f,g)) > \deg[f,g]$.

(2) Otherwise there exists an automorphism α, which is the composition of a sequence of elementary automorphisms, such that $\alpha(f) = \bar{f}$, $\alpha(g) = \bar{g}$, $\bar{p} = \alpha^{-1}(p)$ satisfying the condition in (1). Then $\deg(p(f,g)) = \deg(\bar{p}(\bar{f},\bar{g})) \geqslant \deg[\bar{f},\bar{g}] = \deg[f,g]$. □

Lemma 2.6. *Let $A_n = K[x_1,\dots,x_n]$ be a polynomial algebra over an arbitrary field K of zero characteristic, $f, g \in A_n$ be algebraically independent, $p \in K[x,y]$ has outer rank two. Then*

$$\deg\bigl(p(f,g)\bigr) \geqslant \deg\bigl(J(f,g)\bigr) + 2.$$

Proof. We may assume $\deg(f) = m$, $\deg(g) = n$. As p has outer rank 2, by Lemma 2.4 then p contains a monomial with both x and y, or contains monomials x^i and y^j where $i \geqslant 2$, $j \geqslant 2$.

(1) Let f^+ and g^+ be algebraically independent.

(a) If there exists a monomial in p containing both x and y, then $\deg(p(f,g)) \geqslant \deg(f) + \deg(g) \geqslant \deg(J(f,g)) + 2$.

(b) Otherwise p must have a monomial of x^i where $i \geqslant 2$, and another monomial y^j where $j \geqslant 2$, then $\deg(p(f,g)) \geqslant 2\max\{m,n\} \geqslant \deg(f) + \deg(g) \geqslant \deg(J(f,g)) + 2$.

(2) Let f^+, g^+ be algebraically dependent, and $m \nmid n$ and $n \nmid m$.

(c) If $w_{\deg(f),\deg(g)}(p) < \operatorname{lcm}(m,n)$, then in $p(f,g)$, f^+ and g^+ cannot cancel out, hence similar to the case 1(a), $\deg(p(f,g)) \geqslant \deg(f) + \deg(g) \geqslant \deg(J(f,g)) + 2$.

(d) Otherwise $w_{\deg(f),\deg(g)}(p) \geqslant \operatorname{lcm}(m,n) = mn/(m,n)$. We also have $mn = (m,n)\operatorname{lcm}(m,n) \geqslant (m,n)(m+n)$. Hence $\deg(p(f,g)) \geqslant \deg(J(f,g)) + 2$ by Lemma 2.2.

(3) Let f^+, g^+ be algebraically dependent, but $m \mid n$ or $n \mid m$. Then by same process in the Proof (2) of Lemma 2.4, we may reduce to the above cases (1) or (2). □

Lemma 2.7. *Let $\phi = (f,g)$ be an injective endomorphism of $K\langle x,y\rangle$ but not an automorphism. Then $\deg([\phi^k(x),\phi^k(y)]) \geqslant k+2$ for $k \geqslant 0$.*

Proof. We use induction. $\deg[\phi^0(x),\phi^0(y)] = \deg[x,y] = 0 + 2$. Assuming $\deg[\phi^{k-1}(x),\phi^{k-1}(y)] \geqslant (k-1)+2$. Define $p(x,y) := [f(x,y), g(x,y)]$. As $\phi = (f,g)$ is injective, every monomial of $p(x,y)$ contains both x and y. Since $\phi = (f,g)$ is not an automorphism, by the well-known result of Dicks (see, Dicks [4], or Cohn [2]), $\deg(p(x,y)) > \deg(x) + \deg(y) = 2$. Applying Lemma 2.5, $\deg(p(u,v)) > \deg[u,v]$ for $u = \phi^{k-1}(x)$, $v = \phi^{k-1}(y)$, hence $\deg[\phi^k(x),\phi^k(y)] = \deg(p(\phi^{k-1}(x),\phi^{k-1}(y))) > \deg[\phi^{k-1}(x),\phi^{k-1}(y)] \geqslant (k-1)+2 = k+1$. Therefore, $\deg[\phi^k(x),\phi^k(y)] \geqslant (k+1)+1 = k+2$. □

Lemma 2.8. *Let $\phi = (f,g)$ be an injective endomorphism of $K[x,y]$ but not an automorphism and there exists an element $p \in K[x,y]$ fixed by ϕ. Then $\deg(J(\phi^k(x),\phi^k(y))) \geqslant k$ for $k \geqslant 0$.*

Proof. As ϕ fixes p, ϕ is not an automorphism, by a result of Kraft [13] (see also Shpilrain and J.-T. Yu [30]), $\deg(J(\phi(x),\phi(y))) = \deg(J(f,g)) \geqslant 1$. By the chain rule for the Jacobian,

$$\begin{aligned}\deg\bigl(J\bigl(\phi^k(x),\phi^k(y)\bigr)\bigr) &= \deg\bigl(J(f,g)\bigl(\phi^{k-1}(x),\phi^{k-1}(y)\bigr)\bigl(J\bigl(\phi^{k-1}(x),\phi^{k-1}(y)\bigr)\bigr)\bigr)\\ &\geqslant \deg\bigl(J\bigl(\phi^{k-1}(x),\phi^{k-1}(y)\bigr)\bigr) + 1.\end{aligned}$$

The proof is concluded by induction. □

Lemma 2.9. *Let $\phi = (f,g)$ be an injective endomorphism of A_2 but not an automorphism. Then any element $p \in A_2$ with outer rank 2 cannot be fixed by ϕ.*

Proof. If $p \in A_2$ with outer rank two fixed by ϕ, then $\deg(p(f,g)) = \deg(p(\phi^k(x), \phi^k(y))) \geqslant k+2$ for all $k \geqslant 0$, by Lemmas 2.5 and 2.7 for noncommutative case; and by Lemmas 2.6 and 2.8 for polynomial case. The contradiction completes the proof. □

Proof of Theorem 1.3. By Lemma 2.9. □

Proof of Theorem 1.4. The proof presented here is similar to the proof of the main theorem in Drensky and J.-T. Yu [6].

Let $p \in A_2 - \{0\}$ fixed by a noninjective endomorphism of A_2. Then $\phi(x)$ and $\phi(y)$ are algebraically dependent over K. Let us denote the image of $\phi(A_2)$ by $S = K[\phi(x), \phi(y)]$ (since $\phi(x)$ and $\phi(y)$ are algebraically dependent, $\phi(x)$ and $\phi(y)$ are in a polynomial algebra of rank one over K as a consequence of a result of Bergman [1] for noncommutative case and as a consequence of a result of Shestakov and Umirbaev [24] for polynomial case) and by $Q(S)$ the field of fractions of S. Therefore the transcendence degree of $Q(S)$ over K is 1. Let $0 \neq q(x,y) \in (\mathrm{Ker}(\phi)) \cap S$. Since $p(x,y)$ also belongs to S, the polynomials p and q are algebraically dependent and

$$h(p,q) = a_0(q)p^n + a_1(q)p^{n-1} + \cdots + a_{n-1}(q)p + a_n(q) = 0$$

for an irreducible polynomial $h(u,v) \in K[u,v]$ and $a_i(t) \in K[t]$, $i = 0, 1, \ldots, n$. Hence $\phi(h(p,q)) = h(\phi(p), \phi(q)) = h(p, 0)$,

$$a_0(0)p^n + a_1(0)p^{n-1} + \cdots + a_{n-1}(0)p + a_n(0) = 0.$$

Therefore $a_0(0) = a_1(0) = \cdots = a_n(0) = 0$. Now the polynomials $a_i(t)$ have no constant terms and $h(u,v)$ is divisible by v which contradicts to the irreducibility of $h(u,v)$. Therefore $(\mathrm{Ker}(\phi)) \cap S = 0$ and ϕ acts injectively on its image S. Hence we may extend the action of ϕ on $Q(S)$ (because $a_1/b_1 = a_2/b_2$ in $Q(S)$ is equivalent to $a_1b_2 = a_2b_1$ and hence $\phi(a_1/b_1) = \phi(a_1)/\phi(b_1) = \phi(a_2)/\phi(b_2) = \phi(a_2/b_2)$). By Lüroth's theorem (see, for instance, Schinzel [23]), $Q(S) = K(w)$ for some $w \in Q(S)$. The automorphism ϕ fixes $p(x,y)$ and its extension $\bar{\phi}$ on $Q(S)$ fixes $K(p)$. Since w is algebraic over $K(p)$, $Q(S)$ is a finite dimensional vector space over $K(p)$ and $\bar{\phi}$ is a $K(p)$-linear operator of $Q(S)$ with trivial kernel. Hence $\bar{\phi}$ is invertible on $Q(S)$ and we may consider $\bar{\phi}$ as an automorphism of the finite field extension $Q(S)$ over $K(p)$ which fixes $K(p)$. By Galois theory ($\bar{\phi}$ interchanges the roots of the minimal polynomial of w over $K(p)$ and there are finite number of possibilities for $\bar{\phi}(w)$), $\bar{\phi}$ has finite order. Let $\bar{\phi}^m = 1$. Then $\phi^{m+1}(r) = \phi^m(\phi(r)) = \bar{\phi}^m(\phi(r)) = \phi(r)$ for every $r \in A_2$ and $(\phi^m)^2 = \phi^{m+1}\phi^{m-1} = \phi\phi^{m-1} = \phi^m$. Therefore $\pi = \phi^m$ is a retraction (idempotent endomorphism) of A_2 with a nontrivial kernel and $\pi(p) = p$. Hence $p(x,y)$ is in the image of π which is a proper retract $\pi(A_2)$ of A_2. □

Proof of Theorem 1.1. As $p \in A_2$ does not belong to any proper retract of A_2, by Theorem 1.4, any endomorphism ϕ of A_2 fixing p must be injective. By Lemma 2.3, obviously p must have outer rank two, otherwise p would belong to a proper retract of A_2. By Theorem 1.3, ϕ is an automorphism. Hence p is a test element of A_2. □

Proof of Theorem 1.2. The proof presented here is similar to the proof of the main result Theorem 1.4 in J.-T. Yu [34].

We may assume that $\phi(p) = p$. By the definition of the test element, we may assume p is not a test element. By Theorem 1.1, we may assume p belongs to a proper retract $K[r]$ of A_2. By a result in J.-T. Yu [34], we may assume p has outer rank 2. By Theorem 1.3, we may assume ϕ is noninjective. Suppose that $p = f(r)$, where $f \in K[t] - K$, $\deg(f) = m$. By Theorem 1.4, $\pi = \phi^m$ is a retraction of A_2 to $K[r]$. As ϕ preserves the automorphic orbit of p, so does $\pi = \phi^m$. Applying Lemma 2.3 (suppose $\alpha(r) = x + w(x,y)$, where $w(x,y) \notin K[y]$ belongs to the ideal of A_2 generated by y, α is some automorphism of A_2, replace r by $\alpha(r)$, and π by $\alpha\pi\alpha^{-1}$), we have reduced our proof to the following

Lemma 2.10. *Let $r = x + w(x, y)$, where $w(x, y)$ belongs to the ideal of A_2 generated by y and $w(x, y) \notin K[y]$, π the retraction of A_2 onto $K[r]$ defined by $\pi(x) = x + w(x, y)$, $\pi(y) = 0$, $f \in K[t] - K$. Then π does not preserve the automorphic orbit of $f(r)$.*

Proof. Suppose on the contrary, π preserves the automorphic orbit of $f(r)$. Then for any automorphism α of A_2, $\pi\alpha(f(r)) = \beta(f(r)) \in K[r]$ for some automorphism β of A_2. Note that $\pi\beta(f(r)) = \pi^2\beta(f(r)) = \pi\alpha(f(r)) = \beta(f(r))$. By Theorem 1.4, $\pi^{\deg(f)} = \pi$ is the retraction of A_2 onto the retract $K[\beta(r)]$ taking $\beta(r)$ to $\beta(r)$. By hypothesis, π is also a retraction of A_2 onto the retract $K[r]$ taking r to r. This forces that $\beta(r) = cr + d$ for some $c \in K^*$, $d \in K$. We have concluded that for any automorphism α of A_2, there exists some $c \in K^*$, $d \in K$, such that $\pi\alpha(f(r)) = f(cr + d)$. □

Now we proceed the proof in two cases.

1. Noncommutative case: $A_2 = K\langle x, y\rangle$.

Denote by $\mathcal{C}$ the commutator ideal of $K\langle x, y\rangle$.

(a) If $w(x, y) \in \mathcal{C}$, then take α to be the automorphism of $K\langle x, y\rangle$ defined by $\alpha(x) = y + x^2$, $\alpha(y) = x$. Direct calculation shows that $\pi\alpha(f(r)) = f(r^2 + w(r^2, r)) = f(r^2) \neq f(cr + d)$, a contradiction.

(b) If $w(x, y) \notin \mathcal{C}$, then $w^a(x, y) = yv(x, y)$ for some $v(x, y) \in K[x, y] - \{0\}$. Here $w^a(x, y) \in K[x, y]$ is the image of $w(x, y)$ under the abelianization from $K\langle x, y\rangle$ onto $K[x, y]$. Let M be a positive integer greater than $\deg(v(x, y))$, it is easy to see that $x^M - y$ does not divide $v(x, y)$ in $K[x, y]$. Let α be the automorphism of $K\langle x, y\rangle$ defined by $\alpha(x) = x$, $\alpha(y) = y + x^M$. Then $\pi\alpha(f(r)) = f(r + w(r, r^M)) = f(r + r^M v(r, r^M))$. As $x^M - y$ does not divide $v(x, y)$, $v(r, r^M) \neq 0$. Therefore $\pi\alpha(f(r)) = f(r + r^M v(r, r^M)) \neq f(cr + d)$, a contradiction.

2. Polynomial case: $A_2 = K[x, y]$.

In this case we write $w(x, y) = yq(x, y)$ where $q(x, y) \notin K[y]$. Let M be a positive integer greater than $\deg(q(x, y))$, it is easy to see that $x^M - y$ does not divide $q(x, y)$ in $K[x, y]$. Let α be the automorphism of $K[x, y]$ defined by $\alpha(x) = x$, $\alpha(y) = y + x^M$. Then easy calculation shows that $\pi\alpha(f(r)) = f(r + r^M q(r, r^M))$. As $x^M - y$ does not divide $q(x, y)$, $q(r, r^M) \neq 0$. Therefore $\pi\alpha(f(r)) = f(r + r^M q(r, r^M)) \neq f(cr + d)$. The contradiction completes the proof. □

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