ABSTRACT

It has been known for a long time that the solutions of certain differential equation systems are orthogonal polynomials. Some of the properties of these orthogonal polynomials are traditionally derived from the differential equations. In this note a number of properties are derived just from the recurrence relation of the orthogonal polynomial. In particular we reestablish some recent results by Kwon and Littlejohn on the Laguerre polynomials \( \{L_n^{(-k)}(x)\}, \, k > 0 \), where the orthogonality is with respect to a weighted Sobolev inner product.

1. ROOTS OF ORTHOGONAL POLYNOMIALS

It is well known that the Legendre polynomials can be defined by Rodrigues's formula

\[
P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1 - x^2)^n.
\]

Since the function \((1 - x^2)^n\) is zero when \(x = \pm 1\) (repeated \(n\) times), it follows, by repeated application of Rolle's theorem, that \((d^n/dx^n)(1 - x^2)^n\) has \(n\) real roots which lie in the interval \((-1, 1)\).
Alternatively, the Legendre polynomials can be defined by the recurrence relation

\[ P_{n+1}(x) = \frac{2n + 1}{n + 1} x P_n(x) - \frac{n}{n + 1} P_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x. \] (2)

It is our intention to establish that the roots of the equation \( P_n(x) = 0 \) lie in \((-1, 1)\) and are distinct, without recourse to any results from the calculus.

The first step is to transform Equation (2) so that it corresponds to the recurrence relation generating the characteristic polynomial of the symmetric tridiagonal matrix, \( A \).

\[
A = \begin{bmatrix}
a_1 & b_2 \\
b_2 & a_2 & b_3 \\
& b_3 & a_3 & b_4 \\
& & \ddots & \ddots & \ddots \\
& & & b_{n-1} & a_{n-1} & b_n \\
& & & & b_n & a_n
\end{bmatrix}.
\]

Since the characteristic polynomial of \( A \) is generated by

\[
p_0(\lambda) = 1, \quad p_1(\lambda) = a_1 - \lambda, \\
p_{r+1}(\lambda) = (a_{r+1} - \lambda) p_r(\lambda) - b_{r+1}^2 p_{r-1}(\lambda), \quad r = 1, \ldots, n - 1,
\] (3)

we see that there is a correspondence between (2) and (3) if \( \lambda = -x \), \( a_r = 0 \) (all \( r \)), and we scale the polynomials so that

\[ P_r(x) = \frac{(2r - 1)(2r - 3) \cdots 3 \times 1}{r(r - 1) \cdots 2 \times 1} p_r(-x). \] (4)
for then

\[ p_{r+1}(\lambda) = -\lambda p_r(\lambda) - \frac{r^2}{4r^2-1} p_{r-1}(\lambda), \quad p_0(\lambda) = 1, \quad p_1(\lambda) = -\lambda. \]  

The roots of the Legendre polynomials are thus the roots of \( p_n(\lambda) = 0 \), i.e. the eigenvalues of the symmetric tridiagonal matrix \( A_1 \)

\[
A_1 = \begin{bmatrix}
0 & 1/\sqrt{3} & 0 & 0 & \cdots & 0 \\
1/\sqrt{3} & 0 & 2/\sqrt{15} & 0 & \cdots & 0 \\
2/\sqrt{15} & 0 & 3/\sqrt{35} & 0 & \cdots & 0 \\
3/\sqrt{35} & 0 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
g(n) & \cdots & 0 & g(n) & \cdots & 0 \\
g(n) & 0 & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]  

where \( g(n) = (n-1)/\sqrt{4(n-1)^2-1} \). Since the eigenvalues of a symmetric matrix are all real, it follows that the roots of the Legendre polynomial must all be real. Moreover, the absence of nonzero terms along the leading diagonal of the matrix \( A \) implies that the eigenvalues are symmetrically distributed about the origin — i.e., if \( \lambda \) is an eigenvalue, so is \(-\lambda\).

A convenient and numerically stable method of computing the eigenvalues of the matrix \( A \) is the QR transformation technique of Francis and Kublanovskaya (see Golub and van Loan [1]), although they can be computed by employing the separation properties of the sequence \( \{p_r(\lambda)\} \) and, in particular, the theorem:

**Theorem 1.** The number of agreements in sign between consecutive terms of the sequence \( \{p_r(\lambda)\} \), \( r = 0, 1, \ldots, n \), is the number of roots of the equation \( p_n(\lambda) = 0 \) which are greater than \( \lambda \).

(For proof see Wilkinson [2].)

As a second example, consider the Krall-Legendre polynomials, which are orthogonal on \([-1,1]\) with respect to the weight function

\[ w(x) = 1 + \frac{1}{A} \delta(x+1) + \frac{1}{A} \delta(x-1). \]  

(7)
where $\delta(x)$ is the Dirac delta function. These polynomials satisfy the three-term recurrence relation

$$P_{n+1}(x) = \frac{(2n + 1)[A + \frac{1}{2}n(n + 1)]}{(n + 1)[A + \frac{1}{2}n(n - 1)]} xP_n(x)$$

$$- n\frac{A + \frac{1}{2}(n + 1)(n + 2)}{(n + 1)[A + \frac{1}{2}n(n - 1)]} P_{n-1}(x),$$

$$P_0(x) = A, \quad P_1(x) = Ax; \quad (8)$$

see Everitt and Littlejohn [3]. We remark that as $A \to \infty$ the Krall-Legendre polynomials tend to the Legendre polynomials.

It can be shown in similar fashion that the roots of the Krall-Legendre polynomials are exactly the eigenvalues of the symmetric tridiagonal matrix

$$A_2 = \begin{bmatrix}
0 & \sqrt{\frac{A + 3}{3(A + 1)}} & & \\
\sqrt{\frac{A + 3}{3(A + 1)}} & 0 & \frac{2}{\sqrt{15}} \sqrt{\frac{A(A + 6)}{(A + 1)(A + 3)}} & \\
& \frac{2}{\sqrt{15}} \sqrt{\frac{A(A + 6)}{(A + 1)(A + 3)}} & 0 & \ddots \\
& & \ddots & \ddots \\
& & & 0 & g(n)h(n) \\
g(n)h(n) & & & & 0
\end{bmatrix}$$

(9)

where $g(n)$ is as before [in (6)] and

$$h(n) = \sqrt{\frac{[A + \frac{1}{2}n(n + 1)][A + \frac{1}{2}(n - 2)(n - 3)]}{[A + \frac{1}{2}n(n - 1)][A + \frac{1}{2}(n - 1)(n - 2)]}}.$$

As a final example, the Laguerre polynomials satisfy the recurrence relationship

$$L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2L_{n-1}(x),$$

$$L_0(x) = 1, \quad L_1(x) = 1 - x.$$
This is exactly in the form given by the equations (3), and consequently, without recourse to any transformations, the roots of the Laguerre polynomials are the eigenvalues of the symmetric tridiagonal matrix

\[
A_3 = \begin{bmatrix}
1 & 1 \\
1 & 3 & 2 \\
& 2 & 5 & 3 \\
& & \ddots & \ddots & \ddots \\
& & & n-2 & 2n-3 & n-1 \\
& & & & n-1 & 2n-1 \\
\end{bmatrix}
\] \qquad (11)

2. BOUNDS FOR THE ROOTS

In the previous section we established a connection between the roots of orthogonal polynomials and their associated tridiagonal matrices. Now we aim to develop methods for bounding the roots. One of the principal tools we shall employ is Gerschgorin's theorem (see Wilkinson [2]). This states:

**Theorem 2 (Gerschgorin).** The eigenvalues of the \( n \times n \) matrix \( A = (a_{ij}) \) lie in the union of the discs

\[
|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n. \tag{12}
\]

It follows that, for the matrix \( A_3 \) defined in (11), its eigenvalues lie in the union of the discs

\[
|\lambda - (2r - 1)| \leq 2r - 1, \quad r = 1, 2, \ldots, n - 1, \tag{13}
\]

\[
|\lambda - (2n - 1)| \leq n - 1. \tag{13a}
\]

The first \( n - 1 \) inequalities in (13) give

\[
0 \leq \lambda \leq 4r - 2, \quad r = 1, 2, \ldots, n - 1, \tag{13a}
\]

while the final one produces

\[
n \leq \lambda \leq 3n - 2. \tag{13b}
\]
It follows from Varga [4, Chapter 1] that $\lambda = 0$ is not attained, and consequently all the eigenvalues of $A_3$, and hence the roots of the Laguerre polynomials, are strictly positive. Equations (13a) and (13b) also show that the largest eigenvalue, and hence the largest root of the Laguerre polynomial, satisfies

$$|\lambda| \leq \max(4n - 6, 3n - 2),$$

i.e.

$$|\lambda| \leq 4n - 6, \quad n \geq 4.$$

When Gerschgorin's theorem is applied to the matrix $A_1$, we find that all the eigenvalues lie in the interval

$$|\lambda| < \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{15}} \approx 1.0937.$$

Clearly, this is not the best possible result: as already mentioned in Section 1, the roots of the Legendre polynomials are less than 1 in modulus. This property can however be established by means of Theorem 1, Equation (5), and the property

$$P_r(1) = 1, \quad P_r(-1) = (-1)^r, \quad \text{all } r,$$  \hspace{1cm} (14)

for this enables us to establish the table

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$p_0(\lambda)$</th>
<th>$p_1(\lambda)$</th>
<th>$p_2(\lambda)$</th>
<th>$\ldots$</th>
<th>$p_n(\lambda)$</th>
<th>No. of agreements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>$\ldots$</td>
<td>$\operatorname{sign}(-1)^n$</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>$\ldots$</td>
<td>+</td>
<td>$n$</td>
</tr>
</tbody>
</table>

It follows from the table that there are no eigenvalues greater than 1 and, as $P_0(1) = 1$, the value 1 is not a root of the polynomial. Also, all eigenvalues and hence all roots of the Legendre polynomials are greater than $-1$.

The same argument can be applied to the Krall-Legendre polynomials to show that the roots of these polynomials also lie in $(-1, 1)$. Note that for the Krall-Legendre polynomials

$$P_r(1) = A, \quad P_r(-1) = (-1)^r A, \quad \text{all } r.$$  \hspace{1cm} (15)
3. APPROXIMATION TO THE LARGEST ROOT

Denote the largest root of the Legendre polynomial equation $P_n(x) = 0$ by $\lambda_1$. We have already established that $\lambda_1 < 1$, but it is convenient to take $\lambda = 1$ as a first approximation to $\lambda_1$ and use the method of inverse iteration (Wilkinson [2]) to improve the approximation. First we form $A_1 - I$ and work out the decomposition

$$A_1 - I = LU,$$  \hspace{1cm} (16)

where $L$ is the lower triangular matrix

$$L = \begin{bmatrix} 1 & & & \\
 l_2 & 1 & & \\
 & l_3 & 1 & \\
 & & \ddots & \ddots \\
 & & & l_n & 1
\end{bmatrix}$$

and $U$ is the upper triangular matrix

$$U = \begin{bmatrix} u_1 & b_2 & & & \\
 & u_2 & b_3 & & \\
 & & \ddots & \ddots & \\
 & & & u_{n-1} & b_n \\
 & & & & u_n
\end{bmatrix}, \quad b_r = \frac{r - 1}{\sqrt{4(r - 1)^2 - 1}}.$$  \hspace{1cm} (17)

Once $L$ and $U$ have been determined, we solve the equations

$$UY_1 = (1, 1, \ldots, 1)^T$$  \hspace{1cm} (17)

and

$$LUy_2 = y_1.$$  \hspace{1cm} (18)

Then $y_2$ is a good first-order approximation to the eigenvector $x_1$ which corresponds to the eigenvalue $\lambda_1$. A better approximation to the eigenvalue is
then given by Rayleigh’s quotient

$$\lambda_n = y_2^T A y_2 / y_2^T y_2,$$  \hspace{1cm} (19)

which will be a second-order approximation.

The following algorithm gives the details of the calculation. The elements of $L$ and $U$ are respectively given by the continued fractions

$$l_k = - \frac{b_k}{1 - \frac{b_{k-1}^2}{1 - \frac{b_{k-2}^2}{\ddots}}},$$

$$u_k = - 1 - \frac{b_k^2}{(-1) - \frac{b_{k-1}^2}{\ddots}}.$$

The elements of $y_1$ are determined by back substitution:

$$y_n = \frac{1}{u_n},$$

$$y_k = \frac{1 - b_{k+1}y_{k+1}}{u_k}, \hspace{1cm} k = n - 1, \ldots, 1.$$

The elements of $y_2$ are determined in two stages:

1. $L v = y_1,$
2. $U y_2 = v.$

The elements of $v$ are obtained by forward substitution:

$$v_1 = y_1,$$

$$v_k = y_k - l_k v_{k-1}, \hspace{1cm} k = 2, \ldots, n.$$

If $y_2 = (t_1, t_2, \ldots, t_n)^T$ then

$$t_n = \frac{v_n}{u_n},$$

$$t_k = \frac{v_k - b_{k+1} t_{k+1}}{u_k}, \hspace{1cm} k = n - 1, \ldots, 1.$$
Finally, we have the improved estimate for $\lambda$ given by the Rayleigh quotient

$$\lambda_R = \frac{2\sum b_k t_k t_{k-1}}{\sum t_k^2}.$$  

A comparison of numerical results with the asymptotic formula (see Abramowitz and Stegun [5])

$$x_1 \approx 1 - \frac{j_{01}^2}{2n^2}, \quad (20)$$

where $j_{01}$, the first zero of the Bessel function $J_0(x)$, is given in the table below (note that $j_{01} \approx 2.4048\ldots$):

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda_R$</th>
<th>$x_1$</th>
<th>Largest root of $P_n(x) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.93225</td>
<td>0.91968</td>
<td>0.93246 95142</td>
</tr>
<tr>
<td>12</td>
<td>0.98149</td>
<td>0.97992</td>
<td>0.98156 06342</td>
</tr>
<tr>
<td>24</td>
<td>0.995168</td>
<td>0.994980</td>
<td>0.99518 72200</td>
</tr>
<tr>
<td>48</td>
<td>0.998766</td>
<td>0.998745</td>
<td>0.99877 10073</td>
</tr>
<tr>
<td>96</td>
<td>0.9996882</td>
<td>0.9996862</td>
<td>0.99968 95039</td>
</tr>
</tbody>
</table>

4. THE LAGUERRE POLYNOMIALS $\{L_n^{(-k)}(x)\}$, $k > 0$

Kwon and Littlejohn [6] have shown that, for example, the orthogonalizing inner product $\phi_2(\cdot, \cdot)$ for the Laguerre polynomials $\{L_n^{(-2)}(x)\}$ is

$$\phi_2(p, q) = p(0)q(0) - [p'(0)q(0) + p(0)q'(0)] + 2p'(0)q'(0) + \int_0^\infty p''(x)q''(x)e^{-x}dx. \quad (21)$$

Thus with $L_0^{(-2)}(x) = 1$ we find

$$L_1^{(-2)}(x) = -x - 1$$

satisfies (21). From Theorem 5.1 of Kwon and Littlejohn [6]

$$L_n^{(-\ell)}(x) = (-1)^k \frac{(n-k)!}{n!} x^k L_n^{(k)}(x), \quad n \geq k, \quad (22)$$
and so

\[ L_2^{(-3)}(x) = \frac{1}{2} x^2. \]

Carrying out a similar exercise for \( L_n^{(-3)}(x) \), where

\[
\phi_3(p, q) = p(0)q(0) - 2\left[ p'(0)q(0) + p(0)q'(0) \right] + 5p'(0)q'(0)
+ [ p''(0)q(0) + p(0)q''(0) ] - 3\left[ p''(0)q'(0) + p'(0)q''(0) \right]
+ 3p''(0)q''(0) + \int_0^{\infty} p^{(3)}(x)q^{(3)}(x)e^{-x} dx, \tag{23}
\]

we find

\[
L_0^{(-3)}(x) = 1, \quad L_1^{(-3)}(x) = -x - 2,
\]

\[
L_2^{(-3)}(x) = \frac{1}{2}(x^2 + 2x + 2), \quad L_3^{(-3)}(x) = -\frac{1}{6}x^3.
\]

The interesting thing about these polynomials is that they satisfy the recurrence relation given in Szego [7], namely

\[
nL_n^{(\alpha)}(x) = (-x + 2n + \alpha - 1)L_n^{(\alpha)}(x) - (n + \alpha - 1)L_{n-2}^{(\alpha)}(x),
\]

\[
L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = -x + \alpha + 1, \tag{24}
\]

where it was assumed that \( \alpha > -1 \), but now we assume the range of \( \alpha \) extended to integers \( \leq -1 \).

Consider \( \alpha = -k \); then (24) is satisfied by the polynomials \( L_{n}^{(-k)}(x) \), i.e.,

\[
L_0^{(-k)}(x) = 1, \quad L_1^{(-k)}(x) = -x - k + 1,
\]

\[
2L_2^{(-k)}(x) = (-x - k + 3)L_1^{(-k)}(x) - (-k + 1)L_0^{(-k)}(x),
\]

\[
3L_3^{(-k)}(x) = (-x - k + 5)L_2^{(-k)}(x) - (-k + 2)L_1^{(-k)}(x),
\]

\[
\vdots
\]

\[
kL_k^{(-k)}(x) = (-x + k - 1)L_{k-1}^{(-k)}(x) - ( -1 )L_{k-2}^{(-k)}(x). \tag{25}
\]
If we write

\[ p_r(x) = r!L_r^{(-k)}(x), \]

we find that the polynomials \( p_r(x) \) satisfy the three-term recurrence relation

\[
\begin{align*}
    p_0(x) &= 1, \\
    p_1(x) &= (-x - k + 1), \\
    p_r(x) &= (-x + 2r - k - 1)p_{r-1}(x) \\
    &- (r - 1)(-k + r - 1)p_{r-2}(x), \quad r \geq 2.
\end{align*}
\]

It follows from (27) that the roots of \( p_r(x) = 0 \) are exactly the eigenvalues of the unsymmetric tridiagonal matrix \( T_r \) of order \( r \),

\[
T_r = \begin{bmatrix}
    -k + 1 & \sqrt{k - 1} \\
    -\sqrt{k - 1} & -k + 3 & \sqrt{2(k - 2)} \\
    & -\sqrt{2(k - 2)} & -k + 5 & \sqrt{3(k - 3)} \\
    & & \ddots & \ddots & \ddots \\
    & & & -k + 2r - 3 & \sqrt{(r - 1)(k - r + 1)} \\
    & & & & -\sqrt{(r - 1)(k - r + 1)} & -k + 2r - 1
\end{bmatrix}
\]

It is interesting to note that for all \( r \) for which \( 2r - 1 < k \) the roots of \( p_r(x) = 0 \) have negative real part. This follows from Lyapunov's stability theorem (Barnett [8]), since we can find a negative definite matrix \( P \) such that \( A^T P + PA \) is positive definite. In fact, \( P = \text{diag}(-1, -1, \ldots, -1) \). When \( k > r > \frac{1}{2}(k + 1) \), the eigenvalues of \( T_r \) can have positive real part. For instance, when \( k = 6, r = 5 \), the matrix \( T_5 \) has the pair of eigenvalues \( 0.23981 \pm 3.12834i \), and these correspond with the roots of

\[ L_5^{(-6)}(x) = -\frac{1}{120}(x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120) = 0. \]

Of particular interest is the matrix \( T_k \), which has the special property that it is skew cross-symmetric, i.e.,

\[ a_{ij} = -a_{n+1-i, n+1-j}. \]

Skew cross-symmetric matrices have the property that if \( \lambda \) is an eigenvalue, so is \(-\lambda\). However, what is especially interesting about \( T_k \) is that all its
eigenvalues are zero. This is equivalent to Kwon and Littlejohn's result

\[ L_k^{(x)} = (-1)^k x^k / k! . \]  

(30)

From (30) and (24) the result (22) then follows.

The result about the eigenvalues of \( T_k \) is given in Theorem 3.

**Theorem 3.** The eigenvalues of \( T_k \) are zero.

**Proof.** The matrix \( T_k \) is similar to the tridiagonal matrix

\[
S_k = \begin{bmatrix}
-k + 1 & k - 1 & 0 \\
-1 & -k + 3 & 2(k - 2) & 0 \\
-1 & -k + 5 & 3(k - 3) \\
& & \ddots & \ddots \\
& & & \ddots & 2(k - 2) & 0 \\
& & & & -1 & k - 3 & k - 1 \\
& & & & -1 & k - 1 & 1
\end{bmatrix}
\]

(31)

\( S_k \) admits the decomposition

\[ S_k = LU , \]

where \( L \) is the lower triangular matrix

\[
\begin{bmatrix}
1 & & & & \\
(k - 1)^{-1} & 1 & & & \\
& (k - 2)^{-1} & 1 & & \\
& & \ddots & \ddots & \\
& & & \frac{1}{2} & 1 \\
& & & 1 & 1
\end{bmatrix}
\]
and $U$ is the upper triangular matrix

$$
\begin{bmatrix}
-k+1 & k-1 \\
-k+2 & 2(k-2) \\
-k+3 & 3(k-3) \\
\vdots & \vdots \\
-1 & k-1 \\
0 & 
\end{bmatrix}
$$

Since

$$\det S_k = \det L \det U,$$

it follows that

$$\det S_k = 0. \quad (32)$$

Consequently, $S_k$ (and hence $T_k$) has one zero eigenvalue (at least). The corresponding eigenvector is

$$x^{(t)} = \left[1, 1, \frac{1}{2!}, \ldots, \frac{1}{(k-1)!}\right]^T. \quad (33)$$

This follows from the fact that

$$(-k+1) + (k-1) = 0$$

(row 1),

$$-1 \cdot \frac{1}{(r-2)!} + (-k+2r-1) \frac{1}{(r-1)!} + r(k-r) \frac{1}{r!} = 0$$

(rows 2, 3, \ldots, $k-1$),

$$-1 \cdot \frac{1}{(k-2)!} + (k-1) \frac{1}{(k-1)!} = 0$$

(row $k$).
We now make use of the following lemma.

**Lemma.** If $A$ is a $n \times n$ matrix with eigenvalue $\lambda_1$ and corresponding eigenvector $x_1$ which is normalized so that its first component is unity, then the matrix

$$B = A - x_1 r_1(A) \quad (r_1(A) = \text{row 1 of } A)$$

has the same eigenvalues as $A$ with the exception of $\lambda_1$, which has been replaced by zero. If the eigenvectors of $A$ are $x_1, \ldots, x_n$, the eigenvectors of $B$ are $x_1, x_1 - x_i (i = 2, \ldots, n)$.

We construct the matrix

$$A_k^{(1)} = S_k - x^{(1)} r_1(S_k). \quad (34)$$

By the lemma, $A_k^{(1)}$ has the same eigenvalues as $S_k$, and further, because its top row has all its elements zero by the construction, it follows that the reduced matrix $[A_k^{(1)}]_\delta$ formed by deleting the first row and column of $A_k^{(1)}$ has the same eigenvalues as the remaining eigenvalues of $S_k$. Let

$$A_k^{(2)} = [A_k^{(1)}]_\delta. \quad (35)$$

$A_k^{(2)}$ has the same eigenvalues as $A_k^{(1)}$ with the exception of $\lambda_1$. If $a_{ij}^{(1)} (i, j \leq k)$ are the elements of $A_k^{(1)}$ and $a_{ij}^{(2)} (i, j \leq k - 1)$ are the elements of $A_k^{(2)}$, then

$$a_{i-1,j-1}^{(2)} = a_{ij}^{(1)}, \quad i = 2, \ldots, k, \quad j = 3, \ldots, k,$$

$$a_{i-1,1}^{(2)} = a_{i,2}^{(1)} - 1(k - 1) \frac{1}{(t - 1)!}, \quad i = 2, \ldots, k.$$

Denoting the columns of $A_k^{(2)}$ by $e_1, e_2, \ldots, e_{k-1}$, we have

$$e_1 + e_2 + \frac{1}{2!} e_3 + \cdots + \frac{1}{(k - 2)!} e_{k-1} = 0. \quad (36)$$
showing that $\det A_k^{(2)} = 0$ and hence that $A_k^{(2)}$ has a zero eigenvalue $\lambda_2$. Equation (36) indicates that the corresponding eigenvector is

$$x^{(2)} = \left[1, 1, \frac{1}{2!}, \ldots, \frac{1}{(k-2)!}\right]^T.$$  \hspace{1cm} (37)

Similarly we can construct the sequence of matrices $\{A_k^{(r)}\}$ of order $(k - r + 1) \times (k - r + 1)$, $r = 3, \ldots, k$, by the formula

$$A_k^{(r+1)} = \left[ A_k^{(r)} - x^{(r)} r_1(A_k^{(r)}) \right] \delta,$$  \hspace{1cm} (38)

where

$$x^{(r)} = \left[1, 1, \frac{1}{2!}, \ldots, \frac{1}{(k-r)!}\right]^T.$$  \hspace{1cm} (39)

From the lemma it follows that $A_k^{(r+1)}$ has $k - r + 1$ eigenvalues which are identical with the eigenvalues of $S_k$ which are not already determined. The same argument as above for $A_k^{(2)}$ shows that there is an additional zero eigenvalue with eigenvector given by $x^{(r+1)}$ defined by (39). This establishes that all the eigenvalues of $S_k$ and hence of $T_k$ are zero. This means that the characteristic equation of $T_k$ is

$$(-\lambda)^k = 0,$$

which means

$$p_k(x) = (-x)^k,$$

and this corresponds to (30).

REFERENCES

6 K. H. Kwon and L. L. Littlejohn, The orthogonality of the Laguerre polynomials \( \{L_n^\alpha(x)\} \) for all real \( \alpha \), submitted for publication.

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