

## The Structure of the Inverse to the Sylvester Resultant Matrix

Boris D. Lubachevsky  
AT&T Bell Laboratories  
Murray Hill, New Jersey 07974

Submitted by Richard A. Brualdi

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### ABSTRACT

Given polynomials  $a(\lambda)$  of degree  $m$  and  $b(\lambda)$  of degree  $n$ , we represent the inverse to the Sylvester resultant matrix of  $a$  and  $b$ , if this inverse exists, as a canonical sum of  $m + n$  dyadic matrices each of which is a rational function of zeros of  $a$  and  $b$ .

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A method to solve the equation

$$ax + by = c, \quad (1)$$

where  $a = a(\lambda)$ ,  $b = b(\lambda)$ , and  $c = c(\lambda)$  are given and  $x = x(\lambda)$  and  $y = y(\lambda)$  are unknown univariable polynomials of degrees  $m$ ,  $n$ ,  $m + n - 1$ ,  $n - 1$ , and  $m - 1$ , respectively, is to solve the system of  $m + n$  linear algebraic equations

$$zS = d, \quad (2)$$

where  $z = z(x, y)$  and  $d = d(c)$  are  $(m + n)$ -dimensional row vectors composed of the coefficients of the polynomials  $x$ ,  $y$ , and  $c$ , respectively, and  $S = S(a, b)$  is the  $(m + n) \times (m + n)$  Sylvester resultant matrix of polynomials  $a$  and  $b$ . Solving Equation (1) constitutes an important single-input, single-output case in the zero-placement procedure in control theory (see, e.g., [2]). In this paper we explicitly represent the matrix  $S^{-1}$  as a canonical sum of  $m + n$  dyadic matrices each of which is a rational function of zeros of  $a$  and  $b$ ; thus we give an explicit solution  $(x(\lambda), y(\lambda))$  for (1). This solution may have a practical application in certain situations of adaptive control.

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Although the formulation of this solution appears very simple (see Corollaries 1 and 3), the author is able to mention no other work with this solution.

In the case in which both  $a(\lambda)$  and  $b(\lambda)$  have no multiple zeros, the representation of  $S^{-1}$  is much simpler than in the general case. First, we formulate the results for this special case.

**THEOREM 1.** *Let  $a(\lambda) = a_0\lambda^m + a_1\lambda^{m-1} + \dots + a_m = a_0(\lambda - \alpha_1)(\lambda - \alpha_2) \dots (\lambda - \alpha_m)$  and  $b(\lambda) = b_0\lambda^n + b_1\lambda^{n-1} + \dots + b_n = b_0(\lambda - \beta_1)(\lambda - \beta_2) \dots (\lambda - \beta_n)$  be complex polynomials,  $a_0b_0 \neq 0$ , and  $S = S(a, b)$  be their Sylvester matrix,*

$$S(a, b) = \begin{matrix} \begin{matrix} \left[ \begin{array}{ccccccc} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & 0 \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \cdot & \cdot & \cdot & a_{m-1} & a_m & 0 \\ 0 & \cdot & \cdot & \cdot & a_{m-2} & a_{m-1} & a_m \\ b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & 0 \\ 0 & b_0 & b_1 & \cdot & \cdot & \cdot & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \cdot & \cdot & \cdot & b_{n-1} & b_n & 0 \\ 0 & \cdot & \cdot & \cdot & b_{n-2} & b_{n-1} & b_n \end{array} \right] \end{matrix} & \begin{matrix} \updownarrow \\ n \\ \text{rows} \\ \updownarrow \\ m \\ \text{rows} \\ \updownarrow \end{matrix} \end{matrix} \\ \leftarrow n \cdot m \text{ columns} \rightarrow \end{matrix}$$

If all zeros  $\alpha_i$  are simple (i.e. pairwise different), and all zeros  $\beta_i$  are simple, then the adjoint matrix  $\text{adj} S$  is

$$\begin{aligned} \text{adj} S(a, b) = & a_0^n \sum_{i=1}^m \begin{bmatrix} \alpha_i^{m+n-1} \\ \alpha_i^{m+n-2} \\ \vdots \\ 1 \end{bmatrix} \frac{\prod_{\substack{r=1 \\ r \neq i}}^m b(\alpha_r)}{a(\lambda) \Big|_{\lambda = \alpha_i}} \left[ 0, \dots, 0, \text{row}_m \left( \frac{a(\lambda)}{\lambda - \alpha_i} \right) \right] \\ & + (-1)^{mn} b_0^m \sum_{j=1}^n \begin{bmatrix} \beta_j^{m+n-1} \\ \beta_j^{m+n-2} \\ \vdots \\ 1 \end{bmatrix} \frac{\prod_{\substack{s=1 \\ s \neq j}}^n a(\beta_s)}{b(\lambda) \Big|_{\lambda = \beta_j}} \left[ \text{row}_n \left( \frac{b(\lambda)}{\lambda - \beta_j} \right), 0, \dots, 0 \right], \end{aligned} \tag{3}$$

where  $\text{row}_k(p(\lambda))$  denotes row  $[p_0, p_1, \dots, p_{k-1}]$  composed of the coefficients of the polynomial  $p(\lambda) = p_0\lambda^{k-1} + p_1\lambda^{k-2} + \dots + p_{k-1}$ .

**COROLLARY 1.** *In the assumptions of Theorem 1 if  $\det S(a, b) \neq 0$ , then*

$$\begin{aligned}
 S(a, b)^{-1} = & \sum_{i=1}^m \begin{bmatrix} \alpha_i^{m+n-1} \\ \alpha_i^{m+n-2} \\ \vdots \\ 1 \end{bmatrix} \frac{1}{b(\alpha_i) \frac{a(\lambda)}{\lambda - \alpha_i} \Big|_{\lambda = \alpha_i}} \left[ 0, \dots, 0, \text{row}_m \left( \frac{a(\lambda)}{\lambda - \alpha_i} \right) \right] \\
 & + \sum_{j=1}^n \begin{bmatrix} \beta_j^{m+n-1} \\ \beta_j^{m+n-2} \\ \vdots \\ 1 \end{bmatrix} \frac{1}{a(\beta_j) \frac{b(\lambda)}{\lambda - \beta_j} \Big|_{\lambda = \beta_j}} \left[ \text{row}_n \left( \frac{b(\lambda)}{\lambda - \beta_j} \right), 0, \dots, 0 \right],
 \end{aligned} \tag{4}$$

and the solution to Equation (1) is given by the formulas

$$x(\lambda) = \sum_{j=1}^n \frac{\frac{b(\lambda)}{\lambda - \beta_j}}{\frac{b(\lambda)}{\lambda - \beta_j} \Big|_{\lambda = \beta_j}} \frac{c(\beta_j)}{a(\beta_j)}, \quad y(\lambda) = \sum_{i=1}^m \frac{\frac{a(\lambda)}{\lambda - \alpha_i}}{\frac{a(\lambda)}{\lambda - \alpha_i} \Big|_{\lambda = \alpha_i}} \frac{c(\alpha_i)}{b(\alpha_i)}, \tag{5}$$

i.e.,  $x(\lambda)$  is the Lagrange interpolation polynomial for the function  $c(\lambda)/a(\lambda)$  over the set of zeros of  $b(\lambda)$ , and  $y(\lambda)$  is the one for the function  $c(\lambda)/b(\lambda)$  over the set of zeros of  $a(\lambda)$ .

The following corollary describes the asymptotic structure of  $S^{-1}$ ,  $x(\lambda)$ , and  $y(\lambda)$  when a zero of  $a(\lambda)$  approaches a zero of  $b(\lambda)$ .

**COROLLARY 2.** *Let the coefficients of the polynomials  $a(\lambda) = a_\tau(\lambda)$  and  $b(\lambda) = b_\tau(\lambda)$  depend on a parameter  $\tau$ ,  $\tau \in \{\tau\}$ . Denote by  $x_\tau(\lambda)$  and  $y_\tau(\lambda)$  the corresponding solutions of (1). Let  $\tau^*$  be an accumulation point in  $\{\tau\}$ ,*

and assume that for all  $\tau \neq \tau^*$  the zeros  $\alpha_i = \alpha_i(\tau)$  and  $\beta_j = \beta_j(\tau)$  are simple and that  $a_\tau(\lambda) \rightarrow a^*(\lambda)$ ,  $\alpha_i(\tau) \rightarrow \alpha_i^*$ ,  $i = 1, \dots, m$ ,  $b_\tau(\lambda) \rightarrow b^*(\lambda)$ ,  $\beta_j(\tau) \rightarrow \beta_j^*$ ,  $j = 1, \dots, n$ , for  $\tau \rightarrow \tau^*$ . If

$$\alpha_1^* = \beta_1^* \stackrel{\text{def}}{=} \theta$$

and  $\alpha_i^* \neq \beta_j^*$  for all pairs  $(i, j)$  different from  $(i = 1, j = 1)$  [zeros of  $a^*(\lambda)$  and  $b^*(\lambda)$  are not necessarily simple], then

$$\begin{aligned} \lim_{\tau \rightarrow \tau^*} [\alpha_1(\tau) - \beta_1(\tau)] S(a_\tau, b_\tau)^{-1} = \\ \times \begin{bmatrix} \theta^{m+n-1} \\ \theta^{m+n-2} \\ \vdots \\ 1 \end{bmatrix} \frac{1}{\left. \frac{a^*(\lambda)b^*(\lambda)}{(\lambda - \theta)^2} \right|_{\lambda = \theta}} \left[ -\text{row}_n \left( \frac{b^*(\lambda)}{\lambda - \theta} \right), \text{row}_m \left( \frac{a^*(\lambda)}{\lambda - \theta} \right) \right], \\ \lim_{\tau \rightarrow \tau^*} [\alpha_1(\tau) - \beta_1(\tau)] x_\tau(\lambda) = - \frac{c(\theta)}{\left. \frac{a^*(\lambda)b^*(\lambda)}{(\lambda - \theta)^2} \right|_{\lambda = \theta}} \frac{b^*(\lambda)}{\lambda - \theta}, \\ \lim_{\tau \rightarrow \tau^*} [\alpha_1(\tau) - \beta_1(\tau)] y_\tau(\lambda) = \frac{c(\theta)}{\left. \frac{a^*(\lambda)b^*(\lambda)}{(\lambda - \theta)^2} \right|_{\lambda = \theta}} \frac{a^*(\lambda)}{\lambda - \theta}. \end{aligned}$$

In the case of simple zeros the representation of  $S^{-1}$  relates to the Lagrange interpolation formula. In the general case the representation of  $S^{-1}$  shall relate to the general Hermite polynomial interpolation formula. A version of this formula is presented below for reference.

Let

$$a(\lambda) = a_0(\lambda - \alpha_1)^{m_1}(\lambda - \alpha_2)^{m_2} \cdots (\lambda - \alpha_s)^{m_s}, \quad (6)$$

be a complex polynomial of degree  $m = m_1 + m_2 + \dots + m_s$ , with zeros  $\alpha_1, \dots, \alpha_s$ ,  $m_k > 0$ ,  $k = 1, \dots, s$ ,  $a_0 \neq 0$ ,  $\alpha_k \neq \alpha_i$  for  $k \neq i$ . Let  $f(\lambda)$  be a complex function which is defined at  $\lambda = \alpha_k$  and has  $m_k - 1$  successive derivatives  $f^{[1]}(\alpha_k), f^{[2]}(\alpha_k), \dots, f^{[m_k - 1]}(\alpha_k)$ ,  $k = 1, 2, \dots, s$ . The polynomial  $p(\lambda)$  of degree no larger than  $m - 1$  for interpolation  $f(\lambda)$  is as follows:

$$p(\lambda) = \sum_{i=0}^{m_1-1} f^{[i]}(\alpha_1)u_{1,i}(\lambda) + \sum_{i=0}^{m_2-1} f^{[i]}(\alpha_2)u_{2,i}(\lambda) + \dots + \sum_{i=0}^{m_s-1} f^{[i]}(\alpha_s)u_{s,i}(\lambda),$$

where  $u_{k,i}(\lambda)$ ,  $i = 0, 1, \dots, m_k - 1$ ,  $k = 1, 2, \dots, s$ , are  $m$  polynomials of degrees no larger than  $m - 1$ . The values of the polynomial  $p(\lambda)$  and its  $m_k - 1$  successive derivatives are the same as those of the function  $f(\lambda)$  at  $\lambda = \alpha_k$ ,  $k = 1, \dots, s$ .

The polynomials  $u_{k,i}$  are independent of  $f$  and are uniquely defined given the zeros of the polynomial  $a$  with their multiplicities. They are called the *fundamental* polynomials. For any particular set  $\{m_1, m_2, \dots, m_s\}$  explicit expressions for the fundamental polynomials can be derived [3, 5]. In the statement of Theorem 2 below we assume availability of the fundamental polynomials  $u_{k,i}$  corresponding to the polynomial  $a$  as in (6).

Similarly,  $n$  fundamental polynomials  $v_{l,j}(\lambda)$  of degree no larger than  $n - 1$  correspond to a complex polynomial  $b(\lambda)$  of degree  $n$  as follows. If

$$b(\lambda) = b_0(\lambda - \beta_1)^{n_1}(\lambda - \beta_2)^{n_2} \dots (\lambda - \beta_t)^{n_t}, \tag{7}$$

the zeros of  $\beta(\lambda)$  are  $\beta_1, \dots, \beta_t$ ,  $n_l > 0$ ,  $l = 1, \dots, t$ ,  $\beta_l \neq \beta_j$  for  $l \neq j$ , then

$$p(\lambda) = \sum_{j=0}^{n_1-1} f^{[j]}(\beta_1)v_{1,j}(\lambda) + \sum_{j=0}^{n_2-1} f^{[j]}(\beta_2)v_{2,j}(\lambda) + \dots + \sum_{j=0}^{n_t-1} f^{[j]}(\beta_t)v_{t,j}(\lambda)$$

is the corresponding interpolation polynomial.

We introduce the notation  $\Lambda_k(\lambda)$  for the following  $\lambda$ -column of height  $k$ :

$$\Lambda_k(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} \lambda^{k-1} \\ \lambda^{k-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}.$$

If  $k = m + n$ , the subscript can be omitted:  $\Lambda(\lambda) \stackrel{\text{def}}{=} \Lambda_{m+n}(\lambda)$ . Theorem 1, in particular, implies that in the case of simple zeros each column of  $S^{-1}$  is a linear combination of the columns obtained as the values of  $\Lambda(\lambda)$  when  $\lambda$  takes on the values of the zeros of the polynomials  $a$  and  $b$ . The following Theorem 2, in particular, asserts that in the general case, each column of  $S^{-1}$  is a linear combination of the columns obtained as the values of  $\Lambda(\lambda)$  and its derivatives

$$\Lambda^{[1]}(\lambda) = \begin{bmatrix} (m+n-1)\lambda^{m+n-2} \\ (m+n-2)\lambda^{m+n-3} \\ \vdots \\ 2\lambda \\ 1 \\ 0 \end{bmatrix},$$

$$\Lambda^{[2]}(\lambda) = \begin{bmatrix} (m+n-1)(m+n-2)\lambda^{m+n-3} \\ (m+n-2)(m+n-3)\lambda^{m+n-4} \\ \vdots \\ 2 \\ 0 \\ 0 \end{bmatrix}, \dots$$

when  $\lambda$  takes on the values of the zeros of the polynomials  $a$  and  $b$ .

**THEOREM 2.** *Let  $a(\lambda)$  be a complex polynomial of degree  $m$  as in (6), and let  $u_{k,i}(\lambda)$ ,  $k = 1, \dots, s$ ,  $i = 1, \dots, m_k$ , be the  $m$  corresponding fundamental polynomials. Let  $b(\lambda)$  be a complex polynomial of degree  $n$  as in (7), and let  $v_{l,j}(\lambda)$ ,  $l = 1, \dots, t$ ,  $j = 1, \dots, n_l$ , be the  $n$  corresponding fundamental polynomials. Let  $S = S(a, b)$  be the  $(m+n) \times (m+n)$  Sylvester matrix defined as in Theorem 1.*

I. If  $\det S \neq 0$ , then there exist  $m$  polynomials  $U_{k,i}(\lambda)$ ,  $k = 1, \dots, s$ ,  $i = 1, \dots, m_k$ , of degree no larger than  $m - 1$ , and  $n$  polynomials  $V_{l,j}(\lambda)$ ,  $l = 1, \dots, t$ ,  $j = 1, \dots, n_l$ , of degree no larger than  $n - 1$ , such that

$$\begin{aligned}
 S(a, b)^{-1} &= \sum_{i=0}^{m_1-1} \Lambda^{[i]}(\alpha_1) [0, \dots, 0, \text{row}_m(U_{1,i}(\lambda))] + \dots \\
 &+ \sum_{i=0}^{m_s-1} \Lambda^{[i]}(\alpha_s) [0, \dots, 0, \text{row}_m(U_{s,i}(\lambda))] \\
 &+ \sum_{j=0}^{n_1-1} \Lambda^{[j]}(\beta_1) [\text{row}_n(V_{1,j}(\lambda)), 0, \dots, 0] + \dots \\
 &+ \sum_{j=0}^{n_t-1} \Lambda^{[j]}(\beta_t) [\text{row}_n(V_{t,j}(\lambda)), 0, \dots, 0]. \tag{8}
 \end{aligned}$$

The polynomials  $U_{k,i}$  and  $V_{l,j}$ , which satisfy (8) and the restrictions on their degrees, are unique.

II. For each fixed  $k = 1, 2, \dots, s$ , the  $m_k$  polynomials  $U_{k,i}$ ,  $i = 0, 1, \dots, m_k - 1$ , can be found successively, beginning with  $i = m_k - 1$ , by the following recurrence:

$$\begin{aligned}
 U_{k,i}(\lambda) &= \frac{1}{b(\alpha_k)} \left[ u_{k,i}(\lambda) - \sum_{i_1=i+1}^{m_k-1} \binom{i}{i_1} b^{[i_1-i]}(\alpha_k) U_{k,i_1}(\lambda) \right], \\
 i &= m_k - 1, m_k - 2, \dots, 1, 0. \tag{9}
 \end{aligned}$$

For each fixed  $l = 1, 2, \dots, t$ , the  $n_l$  polynomials  $V_{l,j}$ ,  $j = 0, 1, \dots, n_l - 1$ , can be found successively, beginning with  $j = n_l - 1$ , by the following recurrence:

$$\begin{aligned}
 V_{l,j}(\lambda) &= \frac{1}{a(\beta_l)} \left[ v_{l,j}(\lambda) - \sum_{j_1=j+1}^{n_l-1} \binom{j}{j_1} a^{[j_1-j]}(\beta_l) V_{l,j_1}(\lambda) \right], \\
 j &= n_l - 1, n_l - 2, \dots, 1, 0.
 \end{aligned}$$

EXAMPLE. Let  $\alpha_k$  be a zero of multiplicity  $m_k = 2$  of the polynomial  $a(\lambda)$ . In the decomposition (8) two dyadic summand matrices correspond to

this zero,  $\Lambda(\alpha_k)[0, \dots, 0, \text{row}_m(U_{k,0}(\lambda))]$  and  $\Lambda^{[1]}(\alpha_k)[0, \dots, 0, \text{row}_m(U_{k,1}(\lambda))]$ . Below we derive an explicit expression for  $U_{k,0}$  and  $U_{k,1}$ . Denote  $\bar{a}_k(\lambda) = a(\lambda)/(\lambda - \alpha_k)^2$ . The corresponding fundamental polynomials are [5]

$$u_{k,0}(\lambda) = \frac{\bar{a}_k(\lambda)}{\bar{a}_k(\alpha_k)} \left[ 1 - \frac{\bar{a}_k^{[1]}(\alpha_k)(\lambda - \alpha_k)}{\bar{a}_k(\alpha_k)} \right],$$

$$u_{k,1}(\lambda) = \frac{\bar{a}_k(\lambda)(\lambda - \alpha_k)}{\bar{a}_k(\alpha_k)}.$$

Following the procedure in Theorem 2, we begin with the value  $i = m_k - 1 = 1$  in (9) and find

$$U_{k,1}(\lambda) = \frac{1}{b(\alpha_k)} u_{k,1}(\lambda) = \frac{\bar{a}_k(\lambda)(\lambda - \alpha_k)}{\bar{a}_k(\alpha_k)b(\alpha_k)}.$$

Note that the value of the sum  $\sum_{i_1=i+1}^{m_k-1}$  in (9) is assumed to be 0 in the above calculation, since  $m_k - 1 < i + 1$ . Next, for  $i = m_k - 2 = 0$  we calculate

$$U_{k,0}(\lambda) = \frac{1}{b(\alpha_k)} \left[ u_{k,0}(\lambda) - b^{[1]}(\alpha_k)U_{k,1}(\lambda) \right]$$

$$= \frac{\bar{a}_k(\lambda)}{b(\alpha_k)\bar{a}_k(\alpha_k)} \left[ 1 - (\lambda - \alpha_k) \left( \frac{\bar{a}_k^{[1]}(\alpha_k)}{\bar{a}_k(\alpha_k)} - \frac{b^{[1]}(\alpha_k)}{b(\alpha_k)} \right) \right]. \quad (10)$$

**COROLLARY 3.** *If  $\det S(a, b) \neq 0$ , then Equation (1) has the following solution:  $x(\lambda)$  is the general Hermite interpolation polynomial for the function  $c(\lambda)/a(\lambda)$  over the set of zeros (with their multiplicities) of  $b(\lambda)$ , and similarly  $y(\lambda)$  is the interpolation polynomial for the function  $c(\lambda)/b(\lambda)$  over the set of zeros (with their multiplicities) of  $a(\lambda)$ .*

**DISCUSSION.** If the zeros of  $a$  and  $b$  are simple, Theorem 1 represents  $\text{adj} S$  in the form (3). This representation is valid for both cases,  $\det S = 0$  and  $\det S \neq 0$ . A representation for  $\text{adj} S$  can be obtained if zeros are not simple if we multiply both sides of Equation (8) by  $\det S$ . This representation is of a form similar to that in (8); it is a linear combination of dyadic matrices. The columns in the dyads are the same as in (8). The rows are produced by the operation "row" from certain polynomials, which are equal to  $U_{k,i}$  and  $V_{l,j}$



up to a proportionality constant. These “row” polynomials are unique given the limitation on their degree, since  $U_{k,i}$  and  $V_{l,j}$  are unique polynomials. However this general representation is not necessarily valid in the case  $\det S = 0$ . If zeros are simple, then the zero terms  $a(\beta_j)$  and  $b(\alpha_i)$  in the denominators are canceled by the corresponding terms in  $\det S = \prod_{i=1}^m b(\alpha_i) = \prod_{j=1}^n a(\beta_j)$  which appears in the numerators. If zeros are not simple,  $\det S$  does not necessarily cancel all zero terms in the denominators.

In the example above it may be that the coefficients of  $b(\lambda)$  depend on a parameter  $\tau$ ,  $\tau \in \{\tau\}$ ,  $b(\lambda) = b_\tau(\lambda)$ , the set  $\{\tau\}$  has an accumulation point  $\tau^*$ , and  $b_\tau(\alpha_k) \rightarrow 0$  when  $\tau \rightarrow \tau^*$ , but  $b_\tau(\alpha_k) \neq 0$  for all  $\tau \neq \tau^*$  and  $|b_\tau^{(1)}(\alpha_k)| \geq \text{const} > 0$  for all  $\tau \in \{\tau\}$ . Due to the presence of the term  $b^{(1)}(\alpha_k)/b(\alpha_k)$  in (10),  $U_{k,0}(\lambda)\det S$  retains  $b(\alpha_k)$  in the denominator. Because of that, when  $\tau \rightarrow \tau^*$  the summand with  $\Lambda(\alpha_k)$  tends to infinity even after multiplication by  $\det S$ .

Due to the uniqueness of  $U_{k,i}$  and  $V_{l,j}$  in (8), we can be assured that, in general, no expression for  $\text{adj} S$  of the form similar to (8) exists which is valid for both the cases  $\det S \neq 0$  and  $\det S = 0$ .

AN APPLICATION. Solving the system of linear algebraic equations (2) is easier than finding the zeros of the polynomials  $a(\lambda)$  and  $b(\lambda)$ . Therefore the formulas (5) are inefficient for solving an isolated instance of the zero placement equation (1).

However, in the case in which the coefficients of  $a(\lambda)$  and  $b(\lambda)$  are continuously changing with time  $\tau$ , the formulas (5) appear more attractive. Instead of solving many instances of the system (2), one may track zeros of  $a(\lambda)$  and  $b(\lambda)$  as they change and substitute these zeros into (5) or into its corresponding generalization. This situation may arise if one uses the zero-placement procedure (1) in the course of adaptive control with adjustable and time-variable  $a(\lambda) = a_\tau(\lambda)$  and  $b(\lambda) = b_\tau(\lambda)$ .

A proof is only required for Theorem 2 and Corollary 3. Whereas it is easy to establish Corollary 3, given the result of Theorem 2, one can also establish it independently of Theorem 2, if one computes  $f(\lambda) = a(\lambda)x(\lambda) + b(\lambda)y(\lambda)$  and the appropriate number of derivatives of  $f(\lambda)$  for each zero of  $a$  and  $b$ .

*Proof of Theorem 2.* The notation  $\text{row}_k(p(\lambda))$  can be extended in an obvious way to the case of a column polynomial  $p$  of degree no larger than  $k - 1$ , in which case  $\text{row}_k(p(\lambda))$  is a  $k \times k$  matrix. We have

$$A(\text{row}_k(p(\lambda))) = \text{row}_k(Ap(\lambda)), \tag{11}$$

$$\text{row}_k(p(\lambda)) + \text{row}_k(q(\lambda)) = \text{row}_k(p(\lambda) + q(\lambda)) \tag{12}$$

for a  $k \times k$  complex matrix  $A$  and column  $\lambda$ -polynomials  $p(\lambda)$  and  $q(\lambda)$  of degree no larger than  $k - 1$  and height  $k$ .

Let  $Q$  be the  $(m + n) \times (m + n)$  matrix on the right-hand side of Equation (8). Applying rules (11) and (12) we obtain

$$Q = \text{row}_{m+n}(q(\lambda)),$$

$$q(\lambda) = q_1(\lambda) + q_2(\lambda),$$

where  $q(\lambda), q_1(\lambda), q_2(\lambda)$  are column  $\lambda$ -polynomials of degree no larger than  $m + n - 1$  and height  $m + n$ , and the polynomials  $q_1$  and  $q_2$  are given by

$$q_1(\lambda) = \sum_{k=1}^s \sum_{i=0}^{m_k-1} \Lambda^{[i]}(\alpha_k) U_{k,i}^r(\lambda), \quad q_2(\lambda) = \lambda^m \sum_{l=1}^t \sum_{j=0}^{n_l-1} \Lambda^{[j]}(\beta_l) V_{l,j}(\lambda).$$

The  $(m + n) \times (m + n)$  identity matrix  $I_{m+n}$  can be represented as  $I_{m+n} = \text{row}_{m+n}(\Lambda_{m+n}(\lambda))$ . The condition  $SQ = I_{m+n}$  can be rewritten as

$$Sq(\lambda) = \Lambda(\lambda).$$

Using the specific structure of the Sylvester matrix  $S$ , we have

$$S\Lambda_{m+n}(\lambda) = \begin{bmatrix} \Lambda_n(\lambda)a(\lambda) \\ \Lambda_m(\lambda)b(\lambda) \end{bmatrix}.$$

We can also calculate the corresponding  $r$ th derivative

$$S\Lambda^{[r]}(\lambda) = (S\Lambda(\lambda))^{[r]} = \begin{bmatrix} \sum_{i=0}^r \binom{i}{r} \Lambda_n^{[i]}(\lambda) a^{[r-i]}(\lambda) \\ \sum_{i=0}^r \binom{i}{r} \Lambda_m^{[i]}(\lambda) b^{[r-i]}(\lambda) \end{bmatrix}.$$

Since  $a(\alpha_k) = a^{[1]}(\alpha_k) = \dots = a^{[m_k-1]}(\alpha_k) = 0, k = 1, \dots, s$ , we have

$$S\Lambda^{[r]}(\alpha_k) = \begin{bmatrix} 0 \\ \sum_{i=0}^r \binom{i}{r} \Lambda_m^{[i]}(\alpha_k) b^{[r-i]}(\alpha_k) \end{bmatrix}, \quad r = 0, 1, \dots, m_k - 1.$$

Summing all terms  $S\Lambda^{[r]}(\alpha_k)U_{k,r}^r(\lambda)$ ,  $r = 0, 1, \dots, m_k - 1$ ,  $k = 1, \dots, s$ , we obtain

$$Sq_1(\lambda) = \begin{bmatrix} 0 \\ h(\lambda) \end{bmatrix},$$

where

$$\begin{aligned} h(\lambda) &= \sum_{k=1}^s \sum_{r=0}^{m_k-1} \sum_{i=0}^r \binom{i}{r} \Lambda_m^{[i]}(\alpha_k) b^{[r-i]}(\alpha_k) U_{k,r}^r(\lambda) \\ &= \sum_{k=1}^s \sum_{i=0}^{m_k-1} \Lambda_m^{[i]}(\alpha_k) \sum_{r=i}^{m_k-1} \binom{i}{r} b^{[r-i]}(\alpha_k) U_{k,r}^r(\lambda). \end{aligned} \tag{13}$$

According to (9) the following identity holds.

$$u_{k,i}(\lambda) = \sum_{i_1=i}^{m_k-1} \binom{i}{i_1} b^{[i_1-i]}(\alpha_k) U_{k,i_1}^{i_1}(\lambda). \tag{14}$$

The sum on the right-hand side of Equation (14) is the same as the internal sum in (13). Thus,  $h(\lambda) = \Lambda_m(\lambda)$ , according to the general polynomial interpolation formula, and

$$Sq_1(\lambda) = \begin{bmatrix} 0 \\ \Lambda_m(\lambda) \end{bmatrix}.$$

Similarly the identity

$$Sq_2(\lambda) = \begin{bmatrix} \lambda^m \Lambda_n(\lambda) \\ 0 \end{bmatrix}$$

can be established. Hence the representation (8) is proven.

The uniqueness of polynomials  $U_{k,i}$  and  $V_{l,j}$  easily follows from the nonsingularity of the  $(n + m) \times (n + m)$  generalized Vandermonde matrix [3]

$$\begin{bmatrix} \Lambda(\alpha_1), \dots, \Lambda^{[m_1-1]}(\alpha_1), \dots, \Lambda(\alpha_s), \dots, \Lambda^{[m_s-1]}(\alpha_s), \\ \Lambda(\beta_1), \dots, \Lambda^{[n_1-1]}(\beta_1), \dots, \Lambda(\beta_t), \dots, \Lambda^{[n_t-1]}(\beta_t) \end{bmatrix}$$

when  $\alpha_k \neq \beta_l$  for all pairs  $k, l$ ,  $k = 1, \dots, s$ ,  $l = 1, \dots, t$ . ■

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