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Journal of Approximation Theory 120 (2003) 1–19

Journal of Approximation Theory

http://www.elsevier.com/locate/jat

Construction of trivariate compactly supported biorthogonal box spline wavelets

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Received 19 April 1999; accepted in revised form 27 September 2002

Abstract

We give a formula for the duals of the masks associated with trivariate box spline functions. We show how to construct trivariate nonseparable compactly supported biorthogonal wavelets associated with box spline functions. The biorthogonal wavelets may have arbitrarily high regularities.

Keywords: Trivariate; Box splines; Biorthogonal wavelets

1. Introduction

In [8], Cohen, Daubechies, and Feauveau constructed biorthogonal dual functions associated with univariate B-spline functions B_n and compactly supported biorthogonal wavelets associated with B_n. Since then, the theory of multivariate biorthogonal wavelets has been developed rapidly (cf., e.g., [6]). Since box spline functions are a natural generalization of the well-known B-spline functions, several researches have been done to construct bivariate compactly supported biorthogonal wavelets associated with box spline functions (cf. e.g., [7,10,14,17,27–29]). Let $B_{\ell,m,n}$ be the bivariate box spline whose Fourier transform is

$$\hat{B}_{l,m,n}(\omega) = \left(\frac{1-e^{i\omega_1}}{i\omega_1}\right)^l \left(\frac{1-e^{i\omega_2}}{i\omega_2}\right)^m \left(\frac{1-e^{i(\omega_1+\omega_2)}}{i(\omega_1+\omega_2)}\right)^n$$

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[1] Supported by the National Science Foundation under Grant DMS-9870187.

0021-9045/02/$ - see front matter
PII: S0021-9045(02)00009-6

for any positive integers ℓ, m, n and $\omega = (\omega_1, \omega_2) \in \mathbf{R}^2$. (For properties of bivariate box spline functions, see [2,4]. For computation of these bivariate box spline functions, see [5,22].) It is known that the integer translates and their dilations of a box spline function $B_{\ell,m,n}$ form a multi-resolution approximation of $L_2(\mathbf{R}^2)$ (cf. [2] or [26]). For small integers l, m, n, several different constructions of those biorthogonal wavelets were given in [7,10,28,29]. In a recent paper [17], a general construction of dual refinable functions of box splines and bi-orthogonal wavelets based on an arbitrary order of box splines in any number of dimensions was given. Its duals can have an arbitrary high order of the regularity. In another recent paper [14], He and Lai gave an explicit formula of the dual function $\tilde{B}_{\ell,m,n}$ associated with box spline function $B_{\ell,m,n}$ for any integers ℓ, m, n and compactly supported biorthogonal wavelets associated with box spline function $B_{l,m,n}$ were constructed. Those biorthogonal wavelets can be constructed to have arbitrarily high regularities.

In this paper, we are interested in generalizing the explicit formula for the dual box spline functions and construction of biorthogonal box spline wavelets in [14] to the trivariate setting. That is, we shall construct the compactly supported biorthogonal wavelets associated with trivariate box spline functions. Let $B_{l,m,n,p,q,r}$ be the trivariate box spline function whose Fourier transform is

$$\hat{B}_{l,m,n,p,q,r}(\omega) = \left(\frac{1 - e^{i\omega_1}}{i\omega_1}\right)^l \left(\frac{1 - e^{i\omega_2}}{i\omega_2}\right)^m \left(\frac{1 - e^{i\omega_3}}{i\omega_2}\right)^n \times \left(\frac{1 - e^{i(\omega_1+\omega_2+\omega_3)}}{i(\omega_1 + \omega_2 + \omega_3)}\right)^p \left(\frac{1 - e^{i(\omega_2+\omega_3)}}{i(\omega_2 + \omega_3)}\right)^q \left(\frac{1 - e^{i(\omega_1+\omega_3)}}{i(\omega_1 + \omega_3)}\right)^r$$

for any nonnegative integers ℓ, m, n, p, q, r and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbf{R}^3$. (For this choice of the direction set and other properties of trivariate box spline functions, see [2].) Without loss of generality, we may assume that all $l, m,$ and n are positive. Since the tensor product case is not of interest here, we assume that at least one of p, q, r is not zero. It is known that $B_{l,m,n,p,q,r}$ generates a multiresolution approximation of $L_2(\mathbf{R}^3)$ (cf. [2, p. 90]). Our first step is to construct a compactly supported function $\tilde{B}_{l,m,n,p,q,r}$ generating a multiresolution approximation of $L_2(\mathbf{R}^3)$ which is a biorthogonal dual to $B_{l,m,n,p,q,r}$ in the following sense:

$$\int_{\mathbf{R}^3} B_{l,m,n,p,q,r}(\mathbf{x} - \mathbf{k}) \tilde{B}_{l,m,n,p,q,r}(\mathbf{x} - \mathbf{k})\, d\mathbf{x} = \delta_{\mathbf{k},\mathbf{k}'} \tag{1.1}$$

for all 3D-integers $\mathbf{k}, \mathbf{k}' \in \mathbf{Z}^3$, where $\delta_{\mathbf{j},\mathbf{k}}$ is the standard Kronecker notation defined by $\delta_{\mathbf{j},\mathbf{k}} = 0$ if $\mathbf{j} \neq \mathbf{k}$ and $\delta_{\mathbf{j},\mathbf{k}} = 1$ if $\mathbf{j} = \mathbf{k}$ and $\mathbf{Z}$ is the collection of all integers. Our second step is to construct compactly supported biorthogonal wavelets ψ_j and $\tilde{\psi}_j$ for $j = 1, \ldots, 7$ and two families of FIR filters $\{M_j, i = 1, \ldots, 7\}$ and $\{\tilde{M}_j, j = 1, \ldots, 7\}$ with

$$\hat{\psi}_j(\omega) = M_j(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{i\frac{\omega_3}{2}}) \hat{B}_{l,m,n,p,q}\left(\frac{\omega}{2}\right), \quad j = 1, \ldots, 7 \tag{1.2}$$

and

$$\hat{\tilde{\psi}}_j(\omega) = \tilde{M}_j(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{is\frac{\omega_3}{2}})\hat{\tilde{B}}_{l,m,n}\left(\frac{\omega}{2}\right), \quad j = 1, \ldots, 7 \tag{1.3}$$

such that the integer translates and their dilations of the ψ_j's and $\tilde{\psi}_j$'s form two dual Riesz bases for $L_2(\mathbf{R}^3)$ (cf. [8] for the univariate setting or [6,21] for the multivariate setting) and the two families of masks form an exact reconstruction of synthesis/analysis filter bank which may be possibly used in data compression for 3D seismic data files.

It should be pointed out that the study of constructing compactly supported biorthogonal wavelets associated with trivariate box spline functions is not a simple generalization of the counterpart in the bivariate setting. We are only able to extend our method in [14] to the case that either $q = 0$ or $r = 0$. In this paper, we first consider trivariate box spline $B_{\ell,m,n,p,q,r}$ with $r = 0$. The case associated with $B_{\ell,m,n,p,q,r}$ with $q = 0$ and $r > 0$ follows from the case $r = 0$ and $q > 0$ immediately by the box spline symmetry

$$B_{l,m,n,p,q,0}(x_3, x_2, x_1) = B_{n,m,l,p,0,q}(x_1, x_2, x_3).$$

However, the study of the construction of biorthogonal compactly supported wavelets associated with $B_{\ell,m,n,p,q,r}$ with $q > 0, r > 0$ has to be delayed. From now on, we shall use

$$B_{\ell,m,n,p,q} := B_{\ell,m,n,p,q,0}.$$

We shall give an explicit formula for $\hat{\tilde{B}}_{l,m,n,p,q}$ for any given positive integers l, m, n, p and q in Section 2. The formula is a generalization of the counterpart in the bivariate setting in [14]. The regularities of these biorthogonal dual functions are studied in Section 2.2 which is based on the theory developed in [13]. General results on the regularity can be found in [9,18]. Although there exist some general schemes on how to find matrix extension for constructing biorthogonal wavelets (cf. [3,17,28,29]), we will give a new matrix extension scheme, which is easier to implement, for constructing M_j's and $\tilde{M}_j$'s that lead to compactly supported biorthogonal wavelets with arbitrarily high regularities in Section 3. The proof of the fact that these ψ_j's and $\tilde{\psi}_j$'s generate two dual Riesz bases may be based on a straightforward generalization of the arguments for the univariate setting in [8] or based on the multivariate theory in [6,11,30]. Finally, we give several examples for small integers ℓ, m, n, p, q in Section 4.

2. Construction of compactly supported biorthogonal dual functions

2.1. Construction of biorthogonal dual masks

In the following discussion, we assume that $z = (z_1, z_2, z_3) \in \mathbf{C}^3$. We know that

$$M_0(z) = \left(\frac{1+z_1}{2}\right)^{\ell}\left(\frac{1+z_2}{2}\right)^{m}\left(\frac{1+z_3}{2}\right)^{n}\left(\frac{1+z_1z_2z_3}{2}\right)^{p}\left(\frac{1+z_2z_3}{2}\right)^{q}$$

is the refinement mask of the box spline function $B_{\ell,m,n,p,q}$. For any positive integer N, we define a bivariate polynomial

$$\mathscr{L}_N(x,y) := \sum_{k=0}^{N-1} \binom{2N-1}{k} y^k x^{N-1-k}, \tag{2.1}$$

which satisfies

$$x^N \mathscr{L}_N(x,y) + y^N \mathscr{L}_N(y,x) = (x+y)^{2N-1}. \tag{2.2}$$

Define

$$H_\tau(x,y) := \mathscr{L}_\tau\left(\frac{1+x}{2}\frac{1+y}{2}, \frac{1-x}{2}\frac{1-y}{2}\right) \tag{2.3}$$

for any positive integer τ. It follows immediately from (2.2) that

$$\begin{aligned}&\left(\frac{1+x}{2}\right)^\tau \left(\frac{1+y}{2}\right)^\tau H_\tau(x,y) + \left(\frac{1-x}{2}\right)^\tau \left(\frac{1-y}{2}\right)^\tau H_\tau(-x,-y)\\ &\quad = \left(\frac{1+xy}{2}\right)^{2\tau-1}.\end{aligned} \tag{2.4}$$

Let

$$P_N(y) := \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k.$$

It is well known (see [12]) that

$$(1-y)^N P_N(y) + y^N P_N(1-y) = 1. \tag{2.5}$$

For $z = (z_1, z_2, z_3)$, we define

$$D_N(z) := (z_1z_2z_3)^{-N} \sum_{k=0}^{N-1} \binom{N-1+k}{k} (-1)^k (z_1z_2z_3)^{-k} \left(\frac{1-z_1z_2z_3}{2}\right)^{2k}.$$

Note that since each term in the summation is nonnegative and $(z_1z_2z_3)^N D_N(z) \geqslant 1$ for $|z_1| = |z_2| = |z_3| = 1$. If we take $z_j = e^{i\omega_j},\ j = 1,2,3$, and let $y = \sin^2 \frac{\omega_1+\omega_2+\omega_3}{2}$ in (2.5), we get

$$\begin{aligned}&\left(\frac{1+z_1z_2z_3}{2}\right)^{2N} D_N(z) + \left(\frac{1-z_1z_2z_3}{2}\right)^{2N} D_N(-z) = 1,\\ &\quad |z_j| = 1,\ \ j = 1,2,3\end{aligned} \tag{2.6}$$

for any positive integer N. Now we can define the refinement mask $\tilde{M}_0(z)$ for $\tilde{B}_{\ell,m,n,p,q}$ as follows:

$$\begin{aligned}\tilde{M}_0(z) \;:=&\; \left(\frac{1+z_1^{-1}}{2}\right)^{\sigma-\ell} \left(\frac{1+z_2^{-1}}{2}\right)^{\sigma-m} \left(\frac{1+z_3^{-1}}{2}\right)^{\sigma-n}\\ &\times \left(\frac{1+z_1^{-1}z_2^{-1}z_3^{-1}}{2}\right)^{\rho-p} \left(\frac{1+z_2^{-1}z_3^{-1}}{2}\right)^{\sigma-q} H_{\sigma,L}(z^{-1}) D_{L+\eta}(z^{-1})\end{aligned}$$

with $z^{-1} := (z_1^{-1}, z_2^{-1}, z_3^{-1})$, where

$$H_{\sigma,L}(z) := \left(\frac{1+z_1}{2}\right)^{L-\sigma}\left(\frac{1+z_2z_3}{2}\right)^{L-3\sigma+1} H_\sigma(z_2,z_3)H_L(z_1,z_2z_3)$$

and the positive integers σ, ρ, L, η are so chosen that $\sigma > \max(\ell, m, n, q)$, $\eta > (p-1)/2$, $\rho = 2\eta + 1$ and $L \geqslant 3\sigma - 1$.

We are ready to present the main result of this subsection.

Theorem 2.1. *$\tilde{M}_0(z)$, defined above, is a dual mask of M_0 satisfying*

$$\sum_{\ell_1,\ell_2,\ell_3 \in \{0,1\}} M_0\overline{\tilde{M}_0}((-1)^{\ell_1}z_1, (-1)^{\ell_2}z_2, (-1)^{\ell_3}z_3) = 1,$$
$$|z_1| = |z_2| = |z_3| = 1. \tag{2.7}$$

Proof. First we claim that

$$\sum_{\substack{\ell_1,\ell_2,\ell_3 \in \{0,1\} \\ (-1)^{\ell_1+\ell_2+\ell_3}=1}} M_0\overline{\tilde{M}_0}((-1)^{\ell_1}z_1, (-1)^{\ell_2}z_2, (-1)^{\ell_3}z_3)$$
$$= D_{L+\eta}(z)\left(\frac{1+z_1z_2z_3}{2}\right)^{2(L+\eta)}. \tag{2.8}$$

Indeed, the left-hand side of (2.8) can be written as

$$\begin{aligned}
&D_{L+\eta}(z)\left(\frac{1+z_1z_2z_3}{2}\right)^{\rho}\left[\left(\frac{1+z_1}{2}\right)^{L}\left(\frac{1+z_2z_3}{2}\right)^{L-2\sigma+1} H_L(z_1,z_2z_3)\right.\\
&\times\left(\left(\frac{1+z_2}{2}\right)^{\sigma}\left(\frac{1+z_3}{2}\right)^{\sigma}H_\sigma(z_2,z_3) + \left(\frac{1-z_2}{2}\right)^{\sigma}\left(\frac{1-z_3}{2}\right)^{\sigma}\right.\\
&\left.\times H_\sigma(-z_2,-z_3)\right) + \left(\frac{1-z_1}{2}\right)^{L}\left(\frac{1-z_2z_3}{2}\right)^{L-2\sigma+1} H_L(-z_1,-z_2z_3)\\
&\times\left(\left(\frac{1-z_2}{2}\right)^{\sigma}\left(\frac{1+z_3}{2}\right)^{\sigma}H_\sigma(-z_2,z_3) + \left(\frac{1+z_2}{2}\right)^{\sigma}\right.\\
&\left.\left.\times\left(\frac{1-z_3}{2}\right)^{\sigma}H_\sigma(z_2,-z_3)\right)\right].
\end{aligned}$$

Then (2.8) follows by using (2.4) twice for $\tau = \sigma$ and L, respectively. It is easy to see (2.7) by (2.8) and (2.6). □

We are now able to define the dual $\tilde{B}_{\ell,m,n,p,q}$ associated with box spline function $B_{\ell,m,n,p,q}$ by

$$\hat{\tilde{B}}_{\ell,m,n,p,q}(\omega_1,\omega_2,\omega_3) = \prod_{k=1}^{\infty} \tilde{M}_0(e^{i\omega_1/2^k}, e^{i\omega_2/2^k}, e^{i\omega_3/2^k}). \tag{2.9}$$

We first note that $\tilde{M}_0(1,1,1)=1$ and hence $\hat{\tilde{B}}_{\ell,m,n,p,q}$ is well defined for each $\omega \in \mathbf{R}^3$. We shall show in the next subsection $\tilde{B}_{\ell,m,n,p,q}$ can have arbitrarily high regularity by choosing integers σ, $\rho(=2\eta+1)$ and L sufficiently large. We will show that $\tilde{B}_{\ell,m,n,p,q}$ is a dual to box spline $B_{\ell,m,n,p,q}$ in the sense of (1.1) in Section 2.3.

2.2. Smoothness of the dual $\tilde{B}_{\ell,m,n,p,q}$

To make $\tilde{B}_{\ell,m,n,p,q} \in C^\alpha(\mathbf{R}^3)$ for $\alpha \geqslant 0$, we need to estimate the infinite product in (2.9). Note that

$$\begin{aligned}
|H_\tau(e^{i\xi_1}, e^{i\xi_2})| &\leqslant \sum_{k=0}^{\tau-1} \binom{2\tau-1}{k} \left|\sin\frac{\xi_1}{2}\sin\frac{\xi_2}{2}\right|^k \left|\cos\frac{\xi_1}{2}\cos\frac{\xi_2}{2}\right|^{\tau-1-k} \\
&\leqslant \left(\sum_{k=0}^{\tau-1} \binom{2\tau-1}{k} \left|\sin^2\frac{\xi_1}{2}\right|^k \left|\cos^2\frac{\xi_1}{2}\right|^{\tau-1-k}\right)^{1/2} \\
&\quad \times \left(\sum_{k=0}^{\tau-1} \binom{2\tau-1}{k} \left|\sin^2\frac{\xi_2}{2}\right|^k \left|\cos^2\frac{\xi_2}{2}\right|^{\tau-1-k}\right)^{1/2} \\
&= P_\tau\left(\sin^2\frac{\xi_1}{2}\right)^{1/2} P_\tau\left(\sin^2\frac{\xi_2}{2}\right)^{1/2}.
\end{aligned}$$

The last equality can be seen in [14]. Now we need a result from [13],

$$\prod_{j=1}^\infty P_\tau\left(\sin^2\frac{\xi}{2^{j+1}}\right) \leqslant c_0(1+|\xi|)^{2\mu\tau},$$

where $\mu := \frac{\log 3}{2\log 2} < 1$. Also notice that $|D_{L+\eta}(z)| = P_{L+\eta}(\sin^2\frac{\omega_1+\omega_2+\omega_3}{2})$, we get

$$\begin{aligned}
&\prod_{k=1}^\infty |\tilde{M}_0(e^{i\frac{\omega_1}{2^k}}, e^{i\frac{\omega_2}{2^k}}, e^{i\frac{\omega_3}{2^k}})| \\
&\quad \leqslant \left|\operatorname{sinc}\frac{\omega_1}{2}\right|^{L-\ell} \left|\operatorname{sinc}\frac{\omega_2}{2}\right|^{\sigma-m} \left|\operatorname{sinc}\frac{\omega_3}{2}\right|^{\sigma-n} \\
&\qquad \times \left|\operatorname{sinc}\frac{\omega_2+\omega_3}{2}\right|^{L-2\sigma-q+1} \left|\operatorname{sinc}\frac{\omega_1+\omega_2+\omega_3}{2}\right|^{\rho-p} \\
&\qquad \times C(1+|\omega_2|)^{\mu\sigma}(1+|\omega_3|)^{\mu\sigma}(1+|\omega_1|)^{\mu L}(1+|\omega_2+\omega_3|)^{\mu L} \\
&\qquad \times (1+|\omega_1+\omega_2+\omega_3|)^{2\mu(L+\eta)} \\
&\quad \leqslant C(1+|\omega_1|)^{(\mu-1)L+\ell}(1+|\omega_2|)^{(\mu-1)\sigma+m}(1+|\omega_3|)^{(\mu-1)\sigma+n} \\
&\qquad \times (1+|\omega_2+\omega_3|)^{(\mu-1)L+2\sigma+q-1}(1+|\omega_1+\omega_2+\omega_3|)^{2\mu(L+\eta)-\rho+p},
\end{aligned}$$

where $\operatorname{sinc}\xi := \frac{\sin\xi}{\xi}$ is the well-known sinc function.

For fixed ℓ, m, n, p, q and for any $\alpha \geqslant 0$, we choose σ, η, L and $\rho = 2\eta+1$, such that

$$\max((\mu-1)L+\ell, (\mu-1)\sigma+m, (\mu-1)\sigma+n) < -(\alpha+1)$$

and

$$(\mu - 1)L + 2\sigma + q - 1 \leqslant 0, \quad 2\mu(L + \eta) - \rho + p \leqslant 0.$$

That is

$$\sigma > (\max(m, n) + \alpha + 1)/(1 - \mu), \tag{2.10}$$

$$L > \max(\ell + \alpha + 1, 2\sigma + q - 1)/(1 - \mu), \tag{2.11}$$

$$\rho = 2\eta + 1 \quad \text{with } \eta \geqslant \frac{2\mu L + p - 1}{2(1 - \mu)}. \tag{2.12}$$

Therefore, we have established the following.

Theorem 2.2. *Let σ, L, ρ and η be integers satisfying* (2.10), (2.11) *and* (2.12). *Then $\tilde{B}_{\ell,m,n,p,q}$ defined in* (2.9) *is in $C^{\alpha}(\mathbf{R}^3)$*

2.3. Biorthogonality and Riesz basis property

We next show that $\tilde{B}_{\ell,m,n,p,q}$ defined in (2.9) is a biorthogonal dual to box spline function $B_{\ell,m,n,p,q}$ in the sense of (1.1). Indeed we have

Theorem 2.3. *For σ, L and $\rho (= 2\eta + 1)$ sufficiently large, the integer translates of $\tilde{B}_{\ell,m,n,p,q}$ form a Riesz basis for $\overline{\mathrm{span}_{L_2(\mathbf{R}^3)}\{\tilde{B}_{\ell,m,n,p,q}(\mathbf{x} - \mathbf{k}), \mathbf{k} \in \mathbf{Z}^3\}}$.*

Proof. Mainly, we need to prove the following inequality (see e.g. [25, Chapter 2]):

$$0 < A \leqslant \sum_{\mathbf{k} \in Z^3} |\hat{\tilde{B}}_{\ell,m,n,p,q}(\omega + 2\pi\mathbf{k})|^2 \leqslant B < +\infty.$$

The second inequality follows easily from the proof of Theorem 2.2 by choosing $\alpha = 0$. The first inequality is an immediate result of Lemma 2.5, which may be proved by an extended argument in [14]. □

Remark 2.4. We note that the choice of $\alpha = 0$ in the proof of Theorem 2.3 is a little stronger than necessary to make $\tilde{B}_{\ell,m,n,p,q}$ to generate a Riesz basis. For specific ℓ, m, n, p and q, one may use the methods like spectral radius (cf. [11,18]) to get better estimate of decay of $\hat{\tilde{B}}_{\ell,m,n,p,q}(\omega)$.

Lemma 2.5. *For σ, L and $\rho (= 2\eta + 1)$ sufficiently large,*

$$\sum_{\mathbf{k} \in Z^3} |\hat{B}_{\ell,m,n,p,q} \hat{\tilde{B}}_{\ell,m,n,p,q}(\omega + 2\pi\mathbf{k})|^2 \geqslant A > 0. \tag{2.13}$$

Proof. During the reviewing process of this paper, Prof. Rong-Qing Jia suggested another approach to prove Theorem 2.6. The new proof is much simpler than the original proof of Lemma 2.5 which is a straightforward, but long and tedious calculation of the left-hand side of (2.13) for different regions of ω. See [15] for the original proof. Thus, for the convenience of the reader, we present the simpler proof here. The authors would like to thank Prof. Jia for his suggestion and help in the reading of the proof. $\square$

By noting that $\hat{\tilde{B}}_{\ell,m,n,p,q}(0,0,0) = 1$ and $\hat{\tilde{B}}_{\ell,m,n,p,q}$ is continuous, we can use a result in [6, Theorem 3.3] to get the following Theorem 2.6.

Theorem 2.6. *For σ, L and $\rho(= 2\eta + 1)$ sufficiently large, $\tilde{B}_{\ell,m,n,p,q}$ generates a multiresolution approximation of $L_2(\mathbf{R}^3)$, and $\tilde{B}_{\ell,m,n,p,q}$ is a biorthogonal dual to box spline $B_{\ell,m,n,p,q}$.*

Proof. The proof presented here is suggested by Prof. Rong-Qing Jia. It follows the ideas in [19] which established some similar results in the univariate setting. Mainly, we need to prove that $\tilde{B}_{\ell,m,n,p,q}$ is a biorthogonal dual to box spline $B_{\ell,m,n,p,q}$. Note that the mask M_0 for $B_{\ell,m,n,p,q}$ and the mask $\tilde{M}_0$ for $\tilde{B}_{\ell,m,n,p,q}$ satisfy the discrete biorthogonal condition (2.7). We only need to prove the convergence of the cascade algorithms associated with the masks M_0 and $\tilde{M}_0$ in the L_2 norm. It is clear that the cascade algorithm associated with M_0 converges to the well-known box spline function $B_{\ell,m,n,p,q}$ in the L_2 norm. We thus need to show that the cascade algorithm associated with $\tilde{M}_0$ for a stable initial function $\phi_0 \in L_2(\mathbf{R}^3)$ converges in the L_2 norm. We choose

$$\phi_0 = B_{L-\ell,\sigma-m,\sigma-n,2\eta+1-p,L-2\sigma-q+1}.$$

With $\tilde{M}_0(z) = \frac{1}{8}\sum_{j\in\mathbf{Z}^3} c_j z^j$, the cascade algorithm

$$\phi_{k+1}(x) = \sum_j c_j \phi_k(2x - j)$$

for $k = 0, 1, 2, \ldots$. By the Fourier transform, we have

$$\begin{aligned}
&\hat{\phi}_k(\omega_1,\omega_2,\omega_3)\\
&\quad= \prod_{j=1}^{k} \tilde{M}_0(e^{i\omega_1/2^j}, e^{i\omega_3/2^j}, e^{i\omega_3/2^j})\hat{B}_{L-\ell,\sigma-m,\sigma-n,2\eta+1-p,L-2\sigma-q+1}(\omega/2^k)\\
&\quad= \hat{B}_{L-\ell,\sigma-m,\sigma-n,2\eta+1-p,L-2\sigma-q+1}(\omega)\\
&\qquad\times \prod_{j=1}^{k} H_\sigma(e^{i\omega_2/2^j}, e^{i\omega_3/2^j})H_L(e^{i\omega_1/2^j}, e^{i(\omega_2+\omega_3)/2^j})D_{L+\eta}(e^{i\omega/2^j}).
\end{aligned}$$

It follows from

$$|H_\sigma(e^{i\omega_2}, e^{i\omega_3})| \leqslant P_\sigma(\sin^2(\omega_2/2))^{1/2} P_\sigma(\sin^2(\omega_3/2))^{1/2},$$

$$|D_{L+\eta}(\omega)| = P_{L+\eta}(\sin^2((\omega_1+\omega_2+\omega_3)/2))$$

and the fact that $|P_\tau(\sin^2(\omega))| \geqslant 1$ for any positive integer τ that

$$\begin{aligned}
&|\hat{\phi}_k(\omega_1, \omega_2, \omega_3)| \\
&\quad \leqslant |\hat{B}_{L-\ell,\sigma-m,\sigma-n,2\eta+1-p,L-2\sigma-q+1}(\omega)| \\
&\qquad \times \prod_{j=1}^{\infty} P_L(e^{i\omega_1/2^j})^{1/2} P_\sigma(e^{i\omega_2/2^j})^{1/2} P_\sigma(e^{i\omega_3/2^j})^{1/2} \\
&\qquad \times \prod_{j=1}^{\infty} P_L(e^{i(\omega_2+\omega_3)/2^j})^{1/2} P_{L+\eta}(e^{i(\omega_1+\omega_2+\omega_3)/2^j}) \\
&\quad \leqslant C(1+|\omega_1|)^{(\mu-1)L+\ell}(1+|\omega_2|)^{(\mu-1)\sigma+m}(1+|\omega_3|)^{(\mu-1)\sigma+n} \\
&\qquad \times (1+|\omega_2+\omega_3|)^{(\mu-1)L+2\sigma+q-1}(1+|\omega_1+\omega_2+\omega_3|)^{2\mu(L+\eta)-\rho+p},
\end{aligned}$$

where $\mu = \frac{\log 3}{2\log 2}$. Thus,

$$|\hat{\phi}_k(\omega)| \leqslant C((1+|\omega_1|)(1+|\omega_2|)(1+\omega_3|))^{-(\alpha+1)}$$

for some $\alpha > 0$. For any $\varepsilon > 0$, we have

$$\int_{|\omega_1|>\frac{1}{\varepsilon}, |\omega_2|>\frac{1}{\varepsilon}, |\omega_3|>\frac{1}{\varepsilon}} |\hat{\phi}_{k+1}(\omega) - \hat{\phi}_k(\omega)|^2 \, d\omega \leqslant C\varepsilon^{3+6\alpha}.$$

Since $\hat{\phi}_k$ converges pointwise uniformly on any compact set, there exists an integer k_0 big enough such that for $k > k_0$,

$$\int_{|\omega_1|\leqslant\frac{1}{\varepsilon}, |\omega_2|\leqslant\frac{1}{\varepsilon}, |\omega_3|\leqslant\frac{1}{\varepsilon}} |\hat{\phi}_{k+1}(\omega) - \hat{\phi}_k(\omega)|^2 \, d\omega \leqslant \varepsilon.$$

Hence, $\hat{\phi}_k$ is a Cauchy sequence and hence converges in the L_2 norm. That is, the cascade algorithm converges. Therefore, $\hat{B}_{\ell,m,n,p,q}$ is biorthogonal dual to box spline $B_{\ell,m,n,p,q}$. $\square$

3. Construction of compactly supported biorthogonal wavelets

First, we introduce a notation $A(P_0, \ldots, P_7)$ for any 8 Laurent polynomials $P_j(z), j = 0, \ldots, 7$ with $z = (z_1, z_2, z_3)$. $A(P_0, \ldots, P_7)$ is defined as an 8×8 matrix with columns

$$\begin{aligned}
&[P_j(z), P_j(-z_1, z_2, z_3), P_j(z_1, -z_2, z_3), P_j(z_1, z_2, -z_3), P_j(-z_1, -z_2, z_3), \\
&\quad P_j(z_1, -z_2, -z_3), P_j(-z_1, z_2, -z_3), P_j(-z)]^T, \quad j = 0, \ldots, 7.
\end{aligned}$$

To construct biorthogonal wavelets associated with a trivariate box spline function, we need to start from the mask M_0 for the box spline function $B_{\ell,m,n,p,q}$ and the mask $\tilde{M}_0$ for its dual function $\tilde{B}_{\ell,m,n,p,q}$ to find masks $M_1, \ldots, M_7$ and $\tilde{M}_1, \ldots, \tilde{M}_7$

such that

$$A(M_0, \dots, M_7)^T A(\overline{\tilde{M}_0}, \dots, \overline{\tilde{M}_7}) = I_8, \quad |z_1| = |z_2| = |z_3| = 1, \tag{3.1}$$

where I_8 denotes the 8×8 identity matrix. Then we can define biorthogonal wavelets ψ_j and $\tilde{\psi}_j$ for $j = 1, \dots, 7$ by, in terms of their Fourier transforms,

$$\hat{\psi}_j(\omega_1, \omega_2) = M_j(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{i\frac{\omega_3}{2}})\hat{B}_{l,m,n,p,q}\left(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\right), \quad j = 1, \dots, 7 \tag{3.2}$$

and

$$\hat{\tilde{\psi}}_j(\omega_1, \omega_2) = \tilde{M}_j(e^{i\frac{\omega_1}{2}}, e^{i\frac{\omega_2}{2}}, e^{i\frac{\omega_3}{2}})\hat{\tilde{B}}_{l,m,n,p,q}\left(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\right), \quad j = 1, \dots, 7. \tag{3.3}$$

By a result in literature (cf. [21] or [6]), these ψ_j's and $\tilde{\psi}_j$'s generate biorthogonal wavelets. That is, $\{2^\ell \psi_j(2^\ell \mathbf{x} - \mathbf{k}); \ell \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^3,\ j = 0, \dots, 7\}$ and $\{2^{\ell'} \psi_{j'}(2^{\ell'} \mathbf{x} - \mathbf{k}'); \ell' \in \mathbf{Z}, \mathbf{k}' \in \mathbf{Z}^3,\ j' = 0, \dots, 7\}$ constitute two dual Riesz bases for $L_2(\mathbf{R}^3)$, and

$$\int_{\mathbf{R}^3} 2^\ell \psi_j(2^\ell \mathbf{x} - \mathbf{k}) 2^{\ell'} \tilde{\psi}_{j'}(2^{\ell'} \mathbf{x} - \mathbf{k}')\, d\mathbf{x} = \delta_{\ell,\ell'} \delta_{j,j'} \delta_{\mathbf{k},\mathbf{k}'}.$$

There is a matrix extension method available in the literature (cf. [20,28,29]) to find such $M_j, \tilde{M}_j,\ j = 1, \dots, 7$. However, we would like to generalize the extension method in [14] to deal with these $M_j, \tilde{M}_j$'s. Our method does not rely on the Quillen–Suslin Theorem and does not need an orthogonal procedure as the extension method given in [28,29].

Our method for the construction of $M_j, \tilde{M}_j, j = 1, \dots, 7$ satisfying (3.1) may be divided into three steps:

Step I: Find Laurent polynomials $J_j, j = 1, \dots, 7$, such that the determinant of the matrix $A(M_0, J_1, \dots, J_7)$ is a non-trivial monomial. Since $M_0(z)$, $M_0(-z_1, z_2, z_3), M_0(z_1, -z_2, z_3)$, $M_0(z_1, z_2, -z_3), M_0(-z_1, -z_2, z_3), M_0(z_1, -z_2, -z_3), M_0(-z_1, z_2, -z_3)$, and $M_0(-z)$ have no common zeros on $(\mathbf{C} \setminus \{0\})^3$, the existence of $J_1, \dots, J_7$ is ensured by the well-known Quillen–Suslin Theorem (cf. [23] or [31]). A computation of $J_1, \dots, J_7$ may be performed based on a general algorithm given in [24]. However, by taking advantage of the special properties of box spline functions, we shall give a concrete and elementary construction for those $J_1, \dots, J_7$.

Step II: Compute the inverse of $A(M_0, J_1, \dots, J_7)^T$. The inverse matrix also has the form of $A(\overline{p_0}, \overline{\tilde{M}_1}, \dots, \overline{\tilde{M}_7})$ for Laurent polynomials $p_0, \tilde{M}_1, \dots, \tilde{M}_7$.

Step III: Replace p_0 by $\tilde{M}_0$ in $A(\overline{p_0}, \overline{\tilde{M}_1}, \dots, \overline{\tilde{M}_7})$. The inverse of $A(\overline{\tilde{M}_0}, \overline{\tilde{M}_1}, \dots, \overline{\tilde{M}_7})$ will be the form of $A(M_0, M_1, \dots, M_7)$. This will be clarified later.

First of all, let us give a detailed account for the first step. Let us write the mask $M_0(z)$ in the polyphase form

$$\begin{aligned} M_0(z) = &f_0(z^2) + z_1 f_1(z^2) + z_2 f_2(z^2) + z_3 f_3(z^2) \\ &+ z_1 z_2 f_4(z^2) + z_2 z_3 f_5(z^2) + z_1 z_3 f_6(z^2) + z_1 z_2 z_3 f_7(z^2), \end{aligned}$$

where $z^2 := (z_1^2, z_2^2, z_3^2)$. It follows that $f_0, f_1, \ldots, f_7$ have no common zeros since

$$\begin{aligned}
&[M_0(z), M_0(-z_1, z_2, z_3), M_0(z_1, -z_2, z_3), M_0(z_1, z_2, -z_3),\\
&M_0(-z_1, -z_2, z_3), M_0(z_1, -z_2, -z_3), M_0(-z_1, z_2, -z_3), M_0(-z)]^T\\
&\quad = U(z)[f_0(z^2), \ldots, f_7(z^2)]^T,
\end{aligned} \tag{3.4}$$

where

$$U(z) := \begin{bmatrix}
1 & z_1 & z_2 & z_3 & z_1z_2 & z_2z_3 & z_1z_3 & z_1z_2z_3\\
1 & -z_1 & z_2 & z_3 & -z_1z_2 & z_2z_3 & -z_1z_3 & -z_1z_2z_3\\
1 & z_1 & -z_2 & z_3 & -z_1z_2 & -z_2z_3 & z_1z_3 & -z_1z_2z_3\\
1 & z_1 & z_2 & -z_3 & z_1z_2 & -z_2z_3 & -z_1z_3 & -z_1z_2z_3\\
1 & -z_1 & -z_2 & z_3 & z_1z_2 & -z_2z_3 & -z_1z_3 & z_1z_2z_3\\
1 & z_1 & -z_2 & -z_3 & -z_1z_2 & z_2z_3 & -z_1z_3 & z_1z_2z_3\\
1 & -z_1 & z_2 & -z_3 & -z_1z_2 & -z_2z_3 & z_1z_3 & z_1z_2z_3\\
1 & -z_1 & -z_2 & -z_3 & z_1z_2 & z_2z_3 & z_1z_3 & -z_1z_2z_3
\end{bmatrix}, \tag{3.5}$$

whose determinant is $4096z_1^4z_2^4z_3^4$.

We have to treat the case $q = 0$ and $q > 0$ separately. We first show

Lemma 3.1. *Suppose that $q > 0$. Then the first seven polynomials $f_0, \ldots, f_6$ have no common zeros on $(\mathbf{C} \setminus \{0\})^3$.*

Proof. Suppose that $z^2 \in (\mathbf{C} \setminus \{0\})^3$ is one of the common zeros of these seven polynomials. It follows that

$$\begin{aligned}
M_0(z) &= M_0(-z_1, -z_2, z_3) = M_0(-z_1, z_2, -z_3) = M_0(z_1, -z_2, -z_3)\\
&= z_1z_2z_3f_7(z^2),\\
M_0(-z) &= M_0(-z_1, z_2, z_3) = M_0(z_1, -z_2, z_3) = M_0(z_1, z_2, -z_3)\\
&= -z_1z_2z_3f_7(z^2).
\end{aligned}$$

Thus, we have

$$(1+z_1)^\ell(1+z_2)^m(1+z_3)^n(1+z_1z_2z_3)^p(1+z_2z_3)^q, \tag{3.6}$$

$$= -(1-z_1)^\ell(1+z_2)^m(1+z_3)^n(1-z_1z_2z_3)^p(1+z_2z_3)^q, \tag{3.7}$$

$$= -(1+z_1)^\ell(1+z_2)^m(1-z_3)^n(1-z_1z_2z_3)^p(1-z_2z_3)^q, \tag{3.8}$$

$$= (1-z_1)^\ell(1+z_2)^m(1-z_3)^n(1+z_1z_2z_3)^p(1-z_2z_3)^q, \tag{3.9}$$

$$= (1-z_1)^\ell(1-z_2)^m(1+z_3)^n(1+z_1z_2z_3)^p(1-z_2z_3)^q, \tag{3.10}$$

$$= -(1+z_1)^\ell(1-z_2)^m(1+z_3)^n(1-z_1z_2z_3)^p(1-z_2z_3)^q, \tag{3.11}$$

$$= (1+z_1)^{\ell}(1-z_2)^m(1-z_3)^n(1+z_1z_2z_3)^p(1+z_2z_3)^q, \tag{3.12}$$

$$= -(1-z_1)^{\ell}(1-z_2)^m(1-z_3)^n(1-z_1z_2z_3)^p(1+z_2z_3)^q. \tag{3.13}$$

It is obvious that all those terms in (3.6)–(3.13) above cannot be zero simultaneously. Otherwise all polynomials $f_0, \dots, f_7$ would have a common zero $z^2 \in (\mathbf{C} \setminus \{0\})^3$.

From (3.6) and (3.12), and (3.9) and (3.10), respectively, we have

$$(1+z_2)^m(1+z_3)^n = (1-z_2)^m(1-z_3)^n \text{ and } (1-z_2)^m(1+z_3)^n$$
$$= (1+z_2)^m(1-z_3)^n.$$

Thus, $|1+z_2|^{2m} = |1-z_2|^{2m}$ and $|1-z_3|^{2n} = |1+z_3|^{2n}$. That is, z_2 and z_3 have to be purely imaginary numbers. Let us write $z_2 = bi$ and $z_3 = ci$ with $b, c \in \mathbf{R}$.

Again from (3.6) and (3.10), and (3.7) and (3.11), respectively, we have

$$(1+z_1)^{\ell}(1+z_2)^m(1+z_2z_3)^q = (1-z_1)^{\ell}(1-z_2)^m(1-z_2z_3)^q,$$

$$(1-z_1)^{\ell}(1+z_2)^m(1+z_2z_3)^q = (1+z_1)^{\ell}(1-z_2)^m(1-z_2z_3)^q.$$

It is easy to see that z_1 is a purely imaginary number. Let $z_1 = ai$ with $a \in \mathbf{R}$. By (3.8) and (3.13), we have

$$(1+ai)^{\ell}(1+bi)^m(1+bc)^q = (1-ai)^{\ell}(1-bi)^m(1-bc)^q.$$

Taking the absolute value both sides, we get $|1-bc|^q = |1+bc|^q$ or $bc = 0$. That is, $b = 0$ or $c = 0$ which contradicts the assumption that $z \in (\mathbf{C} \setminus \{0\})^3$. This completes the proof. $\square$

Lemma 3.2. *Suppose that $q = 0$. Then the first six polynomials $f_0, \dots, f_5$ have at most finitely many common zeros on $(\mathbf{C} \setminus \{0\})^3$.*

Proof. Suppose that $z^2 \in (\mathbf{C} \setminus \{0\})^3$ is one of the common zeros of these six polynomials. It follows that

$$M_0(z) = -M_0(-z_1, z_2, z_3) = -M_0(z_1, z_2, -z_3)$$
$$= M_0(-z_1, z_2, -z_3) = z_1z_3f_6(z^2) + z_1z_2z_3f_7(z^2),$$

$$M_0(-z) = M_0(z_1, -z_2, z_3) = -M_0(-z_1, -z_2, z_3)$$
$$= -M_0(z_1, -z_2, -z_3) = z_1z_3f_6(z^2) - z_1z_2z_3f_7(z^2).$$

Thus, we have

$$(1+z_1)^{\ell}(1+z_2)^m(1+z_3)^n(1+z_1z_2z_3)^p$$
$$= -(1-z_1)^{\ell}(1+z_2)^m(1+z_3)^n(1-z_1z_2z_3)^p$$
$$= -(1+z_1)^{\ell}(1+z_2)^m(1-z_3)^n(1-z_1z_2z_3)^p$$
$$= (1-z_1)^{\ell}(1+z_2)^m(1-z_3)^n(1+z_1z_2z_3)^p \tag{3.14}$$

and

$$\begin{aligned}(1+z_1)^{\ell}(1-z_2)^m(1+z_3)^n(1-z_1z_2z_3)^p\\ &= -(1-z_1)^{\ell}(1-z_2)^m(1+z_3)^n(1+z_1z_2z_3)^p\\ &= -(1+z_1)^{\ell}(1-z_2)^m(1-z_3)^n(1+z_1z_2z_3)^p\\ &= (1-z_1)^{\ell}(1-z_2)^m(1-z_3)^n(1-z_1z_2z_3)^p.\end{aligned}$$

The above two groups of equations cannot be zero simultaneously. Without loss of generality, we assume the first group of equations is not zero. Then we can get

$$|1+z_1||1+z_3| = |1-z_1||1-z_3| \text{ and } |1-z_1||1+z_3| = |1+z_1||1-z_3|.$$

It follows that $z_1+\overline{z_1}=0$ and $z_3+\overline{z_3}=0$. That is, $z_1=ai$ and $z_3=ci$ with a and c real. By (3.14), we have

$$\begin{aligned}(1+ai)^{\ell}(1-acz_2)^p &= -(1-ai)^{\ell}(1+acz_2)^p,\\ -(1-ai)^{\ell}(1-acz_2)^p &= (1+ai)^{\ell}(1+acz_2)^p,\end{aligned}$$

which implies that

$$(1-acz_2)^{2p} = (1+acz_2)^{2p}. \tag{3.15}$$

It is easy to see that z_2 is a purely imaginary number. Let $z_2=bi$ for some $b\in\mathbf{R}$. It follows from (3.14) that

$$\begin{aligned}(1+ai)^{\ell}(1-abci)^p &= -(1-ai)^{\ell}(1+abci)^p,\\ -(1+ai)^{\ell}(1+abci)^p &= (1-ai)^{\ell}(1-abci)^p.\end{aligned} \tag{3.16}$$

Look at the complex conjugate of both sides of (3.16), one can see that $(1+ai)^{\ell}(1-abci)^p$ is a purely imaginary number and so is $(1+ai)^{\ell}(1+abci)^p$. Thus, $(1+ai)^{2\ell}(1+a^2b^2c^2)^p$ is a real number or $(1+ai)^{2\ell}$ is a real number. Consequently, $\sum_{k=0}^{\ell-1}\binom{2\ell}{2k+1}a^{2k}(-1)^k=0$, which has only finitely many real solutions for a. Similarly there are only finitely many real solutions for c. Eq. (3.15) becomes $(1+abci)^{2p}=(1-abci)^{2p}$, which implies that $(1+abci)^{2p}$ is a real number. Obviously there are finitely many b's to make $(1+abci)^{2p}$ real. Hence, at most finitely many z's satisfy (3.14). This completes the proof of the Lemma 3.3. □

Lemma 3.3. *There exists an 8×8 Laurent polynomial matrix $\mathscr{B}(z)$ with real coefficients such that the first column of $\mathscr{B}$ is $[f_0,f_1,\dots,f_7]^T$ and the determinant of $\mathscr{B}$ is* 1.

Proof. We first consider the case that $q=0$. By Lemma 3.2, we may assume that $f_0,\dots,f_5$ have r common zeros in $(\mathbf{C}\setminus\{0\})^3$ for $r\geqslant 1$ (if $r=0$, then it is trivial), which are $w_j, j=1,\dots,r$. Now we consider f_6+kf_7 for some real number k. Since $f_0,\dots,f_7$ have no common zero, $f_6(w_j)$ and $f_7(w_j)$ cannot be equal to zero simultaneously for any $j = 1, \dots, r$. Thus, there exists a $k_0 \neq 0$ such that $\tilde{f_6} = f_6 + k_0 f_7$ does not vanish on all the w_j's. It follows that $f_0, \dots, f_5, \tilde{f_6}$ have no common zero in $(\mathbf{C} \diagdown \{0\})^3$. By Hilbert's Nullstellensatz Theorem, (cf. [16]), there exist polynomials $p_0, \dots, p_6$ with real coefficients such that

$$\sum_{j=0}^{5} f_j(z) p_j(z) + \tilde{f_6}(z) p_6(z) = 1.$$

Note that

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & -k_0 \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \tilde{f_6} \\ f_7 \end{bmatrix},$$

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \tilde{f_6} \\ f_7 \end{bmatrix} = \begin{bmatrix} & & & & & & & 1 \\ & & & & & & 1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ 1 & p_6(f_7-1) & p_5(f_7-1) & \cdots & \cdots & \cdots & p_1(f_7-1) & p_0(f_7-1) \end{bmatrix} \times \begin{bmatrix} 1 \\ \tilde{f_6} \\ f_5 \\ f_4 \\ f_3 \\ f_2 \\ f_1 \\ f_0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ \tilde{f}_6 \\ f_5 \\ f_4 \\ f_3 \\ f_2 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 & & & & & & & \\ \tilde{f}_6 & 1 & & & & & & \\ f_5 & & 1 & & & & & \\ f_4 & & & 1 & & & & \\ f_3 & & & & 1 & & & \\ f_2 & & & & & 1 & & \\ f_1 & & & & & & 1 & \\ f_0 & & & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The desirable matrix $\mathscr{B}$ is the product of the three matrices above whose determinant is equal to 1.

For the case that $q>0$, we use Lemma 3.1. In this case, we take $k=0$, that is, $\tilde{f}_6 = f_6$. The desirable matrix $\mathscr{B}$ is the product of the last two matrices above. This completes the proof of Lemma 3.3. $\square$

We now give the detail of Steps II and III. By Lemma 3.3 and (3.4), we can take $A(M_0, J_1, \ldots, J_7) = U(z)\mathscr{B}(z^2)$, where $U(z)$ is defined in (3.5). Since the determinant of $A(M_0, J_1, \ldots, J_7)$ is $4096 z_1^4 z_2^4 z_3^4$, it is invertible on the Laurent polynomial ring. Let $A(\overline{p_0}, \overline{\tilde{M}_1}, \ldots, \overline{\tilde{M}_7})$ be the inverse of $A(M_0, J_1, \ldots, J_7)^T$. Using a definition of the inverse of matrices, it is easy to see that

$$M_0(z) = \frac{1}{4096 z_1^4 z_2^4 z_3^4} \times \det \begin{bmatrix} \overline{\tilde{M}_1}(-z_1, z_2, z_3) & \overline{\tilde{M}_1}(z_1, -z_2, z_1) & \cdots & \overline{\tilde{M}_1}(-z) \\ \overline{\tilde{M}_2}(-z_1, z_2, z_3) & \overline{\tilde{M}_2}(z_1, -z_2, z_3) & \cdots & \overline{\tilde{M}_2}(-z) \\ \vdots & \vdots & & \vdots \\ \overline{\tilde{M}_7}(-z_1, z_2, z_3) & \overline{\tilde{M}_7}(z_1, -z_2, z_3) & \cdots & \overline{\tilde{M}_7}(-z) \end{bmatrix}. \tag{3.17}$$

Replacing p_0 in $A(\overline{p_0}, \overline{\tilde{M}_1}, \ldots, \overline{\tilde{M}_7})$ by the dual mask $\tilde{M}_0$ which is given in Theorem 2.1, we notice that $\det(A(\overline{\tilde{M}_0}, \overline{\tilde{M}_1}, \ldots, \overline{\tilde{M}_7})) = 4096 z_1^4 z_2^4 z_3^4$ by the co-factor expansion of the first column, (3.17) and (2.7). Let $A(q_0, M_1, \ldots, M_7)$ be the inverse of $A(\overline{\tilde{M}_0}, \overline{\tilde{M}_1}, \ldots, \overline{\tilde{M}_7})$. One can see that q_0 in $A(q_0, M_1, \ldots, M_7)$ is exactly the same as M_0 by observing that they both have the same expression of the right-hand side of

(3.17). Therefore,

$$A(M_0, M_1, \ldots, M_7)A(\overline{\tilde{M}_0}, \overline{\tilde{M}_1}, \ldots, \overline{\tilde{M}_7})^T = I_8.$$

We remark here that the method used in Steps II and III can be generalized to any multivariate settings.

4. Examples

In the following, let us give some examples associated with box spline functions for small integers (ℓ, m, n, p). Based on the construction in the previous section for case that $q \neq 0$, we only need to find polynomials $p_0, \ldots, p_6$ such that

$$p_0 f_0 + \cdots + p_6 f_6 = 1, \tag{4.1}$$

where $f_0, \ldots, f_6$ are the first 7 polyphase components of the mask for box spline function $B_{\ell,m,n,p,q,0}$. For $q = 0$, for the small integers ℓ, m, n, p, we can verify that $f_0, \ldots, f_6$ have no common zeros on $(\mathbf{C} \setminus \{0\})^3$. Thus, we can use the same method as $q \neq 0$ to construct the masks $M_1, \ldots, M_7$ and $J_2, \ldots, J_7$.

We may use the Gröbner basis method as described in [1] to compute the polynomials $p_0, \ldots, p_6$ satisfying (4.1) for polynomials $f_0, \ldots, f_6$ associated with box spline functions. (The authors wish to thank Dr. Lingyun Ma for her MATHEMATICA programs for computing $p_0, \ldots, p_6$ based on Buchberger's algorithm using the Gröbner basis.) Some outputs of those programs are given below.

Example 1. For the box spline $B_{1,1,1,1}$, we have

$$p_0 = 1/2, \;\; p_1 = -z_1^2/2, \;\; p_2 = p_3 = p_4 = 0, \;\; p_5 = 1/2, \;\; p_6 = 0.$$

Example 2. For the box spline $B_{2,2,1,1}$, we have

$$p_0 = 1/8, \;\; p_1 = -1/16, \;\; p_2 = 1/16, \;\; p_3 = -z_3^2, \;\; p_4 = 1/4,$$
$$p_5 = -1/16 - z_3^2/16, \;\; p_6 = 0.$$

Example 3. For the box spline $B_{2,2,2,1}$, we have

$$p_0 = \frac{1+3z_3^2}{16}, \; p_1 = \frac{1}{2} + \frac{25z_2^2}{128} - \frac{5z_3^2}{128}, \; p_2 = \frac{1}{2} + \frac{5z_3^2}{32},$$

$$p_3 = -\frac{17}{32} - \frac{25z_2^2}{128} - \frac{39z_3^2}{128}, \; p_4 = -\frac{75z_2^2}{128} - \frac{9z_3^2}{128},$$

$$p_5 = \frac{75z_2^2}{128} - \frac{63z_3^2}{128}, \; p_6 = \frac{3z_3^2}{32}.$$

Example 4. For box spline $B_{2,2,2,2}$, we have

$$\begin{aligned} p_0 = &-\frac{8779}{1742528} + \frac{61137z_1^2}{435632} + \frac{977555z_2^2}{3485056} + \frac{2906109z_3^2}{3485056} + \frac{61137z_1^2z_3^2}{435632} \\ &+ \frac{470475z_2^2z_3^2}{3485056} - \frac{54247z_1^2z_2^2z_3^2}{1742528} + \frac{3104437z_3^4}{3485056}, \end{aligned}$$

$$\begin{aligned} p_1 = &-\frac{6486213}{3485056} - \frac{674691z_1^2}{3485056} - \frac{61137z_1^4}{1742528} - \frac{623455z_2^2}{13940224} + \frac{926105z_1^2z_2^2}{13940224} \\ &- \frac{17725709z_3^2}{13940224} + \frac{4845395z_1^2z_3^2}{13940224} - \frac{61137z_1^4z_3^2}{1742528} - \frac{299335z_2^2z_3^2}{13940224} \\ &- \frac{82347z_1^2z_2^2z_3^2}{13940224} + \frac{54247z_1^4z_2^2z_3^2}{3485056} - \frac{3104437z_3^4}{13940224} - \frac{3104437z_1^2z_3^4}{13940224}, \end{aligned}$$

$$\begin{aligned} p_2 = &\frac{1915821}{871264} + \frac{172667z_1^2}{1742528} - \frac{2813863z_2^2}{13940224} - \frac{316429z_1^2z_2^2}{1742528} - \frac{193695z_2^4}{13940224} \\ &+ \frac{13809155z_3^2}{13940224} - \frac{61137z_1^2z_3^2}{1742528} - \frac{2815445z_2^2z_3^2}{3485056} - \frac{61137z_1^2z_2^2z_3^2}{1742528} \\ &- \frac{641615z_2^4z_3^2}{13940224} - \frac{3104437z_3^4}{13940224} - \frac{3104437z_2^2z_3^4}{13940224}, \end{aligned}$$

$$\begin{aligned} p_3 = &-\frac{16063}{108908} + \frac{63823z_1^2}{435632} - \frac{83985z_2^2}{435632} + \frac{23433z_3^2}{435632} - \frac{58451z_1^2z_3^2}{435632} \\ &- \frac{529635z_2^2z_3^2}{1742528} - \frac{3119483z_3^4}{1742528}. \end{aligned}$$

$$p_4 = \frac{-58451z_1^2}{217816} - \frac{14565z_2^2}{217816} + \frac{3191z_1^2z_2^2}{27227} - \frac{27739z_3^2}{871264} - \frac{42785z_2^2z_3^2}{871264},$$

$$\begin{aligned} p_5 = &-\frac{456687}{1742528} - \frac{61137z_1^2}{871264} + \frac{2493825z_2^2}{6970112} - \frac{19796117z_3^2}{6970112} + \frac{61137z_1^2z_3^2}{871264} \\ &+ \frac{656727z_2^2z_3^2}{6970112} + \frac{190301z_1^2z_2^2z_3^2}{1742528} + \frac{193695z_2^4z_3^2}{3485056} + \frac{3104437z_3^4}{6970112} \\ &+ \frac{3053665z_2^2z_3^4}{3485056}, \end{aligned}$$

$$p_6 = \frac{364375}{871264} - \frac{194155z_1^2}{871264} + \frac{193695z_2^2}{6970112} + \frac{18188357z_3^2}{6970112} + \frac{61137z_1^2z_3^2}{871264} + \frac{641615z_2^2z_3^2}{6970112} + \frac{3104437z_3^4}{6970112}.$$

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