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Liftings of dissident maps

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Abstract

We study dissident maps η on $\mathbb{R}^m$ for $m \in \{3, 7\}$ by investigating liftings $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ of the selfbijection $\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^m) \to \mathbb{P}(\mathbb{R}^m)$, $\eta_{\mathbb{P}}[v] = (\eta(v \wedge \mathbb{R}^m))^{\perp}$ induced by η. Our main result (Theorem 2.4) asserts the existence and uniqueness, up to a non-zero scalar multiple, of a lifting Φ whose component functions are homogeneous polynomials of degree d, relatively prime and without non-trivial common zero. We prove that $1 \leqslant d \leqslant m-2$.

We achieve a complete description of all dissident maps of degree one and we solve their isomorphism problem (Theorems 4.8 and 4.13). As a consequence, we achieve a complete description of all real quadratic division algebras of degree one and we solve their isomorphism problem (Theorems 5.1 and 5.3). Moreover we present examples of eight-dimensional real quadratic division algebras of degree 3 and 5 (Proposition 6.3). This extends earlier results of Osborn [Trans. Amer. Math. Soc. 105 (1962) 202–221], Hefendehl [Geometriae Dedicata 9 (1980) 129–152], Hefendehl-Hebeker [Arch. Math. 40 (1983) 50–60], Cuenca Mira et al. [Lin. Alg. Appl. 290 (1999) 1–22], Dieterich [Proc. Amer. Math. Soc. 128 (2000) 3159–3166] and Dieterich and Lindberg [Colloq. Math. 97 (2003) 251–276] on the classification of real quadratic division algebras.
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1. Preliminaries

In accordance with Bourbaki we view 0 as the least natural number. For each $n \in \mathbb{N}$ we set $\underline{n} = \{i \in \mathbb{N} \mid 1 \leqslant i \leqslant n\}$. By $\mathbb{R}_{>0}$, $\mathbb{R}_{\geqslant 0}$ we denote the rays of all positive, non-negative real numbers respectively. By $\mathbb{R}^{m\times n}$ we mean the vector space of all real matrices of size $m \times n$. The identity matrix in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{I}_n$. We write $\mathbb{R}^{n\times n}_{\mathrm{pd}}$, $\mathbb{R}^{n\times n}_{\mathrm{pds}}$, $\mathbb{R}^{n\times n}_{\mathrm{ant}}$ for the subsets of $\mathbb{R}^{n\times n}$ which consist of all positive definite, positive definite symmetric, antisymmetric matrices respectively. Moreover, we set $\mathbb{R}^{n\times n}_{spds} = \mathrm{SL}_n(\mathbb{R}) \cap \mathbb{R}^{n\times n}_{pds}$. We also set $\mathbb{R}^m = \mathbb{R}^{m\times 1}$ and write $\mathbf{e} = (e_1, \ldots, e_m)$ for the standard basis in $\mathbb{R}^m$. If $M \in \mathbb{R}^{m\times n}$, then M^T is the transpose of M, and $\underline{M}$ denotes the linear map $\underline{M} : \mathbb{R}^n \to \mathbb{R}^m$, $\underline{M}(x) = Mx$. If V is a vector space, then $[v_1, \ldots, v_\ell]$ denotes the linear hull of vectors $v_1, \ldots, v_\ell$ in V, and V^* denotes the dual space of V. Elements in the projective space $\mathbb{P}(V)$ are denoted by $[v]$, where $v \in V \setminus \{0\}$.

By a *Euclidean space* we mean a finite-dimensional Euclidean vector space $V = (V, \langle\ \rangle)$. Morphisms between Euclidean spaces, also called *orthogonal maps*, are linear maps respecting the scalar products. By $E = \mathbb{E}^m$ we denote the m-dimensional *Euclidean column space* $(\mathbb{R}^m, \langle\ \rangle)$, where $\langle v, w\rangle = v^T w$ is the standard scalar product. We write $\mathrm{Pd}(E)$, $\mathrm{Pds}(E)$, $\mathrm{Ant}(E)$, $\mathrm{Def}(E)$ for the subsets of $\mathrm{Hom}_{\mathbb{R}}(E, E)$ which consist of all positive definite, positive definite symmetric, antisymmetric, definite linear endomorphisms of E respectively. Moreover, we set $\mathrm{Spds}(E) = \mathrm{SL}(E) \cap \mathrm{Pds}(E)$.

A *dissident map* on a finite-dimensional Euclidean vector space V is an $\mathbb{R}$-linear map $\eta : V \wedge V \to V$ such that $v, w, \eta(v \wedge w)$ are linearly independent whenever $v, w \in V$ are. The interest in dissident maps originates in their relevance for the classification problem of real division algebras. Dissident maps are known to exist in the dimensions 0, 1, 3 and 7 only. They are trivial in dimensions 0 and 1, but still very little understood in dimensions 3 and 7.

The object class $\mathscr{E}$ of all dissident maps $\eta : V \wedge V \to V$ on a Euclidean space V, briefly denoted by $\eta = (V, \eta)$, becomes a category by viewing as morphisms $\sigma : (V, \eta) \to (V', \eta')$ the orthogonal maps $\sigma : V \to V'$ which satisfy $\sigma\eta = \eta'(\sigma \wedge \sigma)$. Note that a morphism $\sigma : (V, \eta) \to (V', \eta')$ is an isomorphism in the category $\mathscr{E}$ if and only if $\dim V = \dim V'$.

The polynomial ring $R = \mathbb{R}[X_1, \ldots, X_m]$ has dth homogeneous summand R_d. Recall that R is a factorial noetherian domain. The *greatest common divisor* of a sequence $p_1, \ldots, p_\ell$ in R, denoted by $\gcd(p_1, \ldots, p_\ell) = (d)$, is the unique minimal principal ideal (d) in R containing the ideal $(p_1, \ldots, p_\ell)$ generated by $p_1, \ldots, p_\ell$.

Let $\mathscr{K}$ be a category. Whenever a function $\dim : \mathrm{Ob}(\mathscr{K}) \to \mathbb{N}$ is defined, $\mathscr{K}_n$ denotes the full subcategory of $\mathscr{K}$ formed by $\dim^{-1}(n)$. For each $X \in \mathrm{Ob}(\mathscr{K})$, $\mathbb{I}_X$ and $[X]$ denote the identity morphism on X and the isoclass of X respectively. All categories considered in this article will be *svelte*, i.e. their isoclasses form a set, denoted by $\mathrm{Ob}(\mathscr{K})/\tilde{\to}$. A subset $\mathscr{C} \subset \mathrm{Ob}(\mathscr{K})$ is said to *exhaust* $\mathrm{Ob}(\mathscr{K})$, to be *irredundant*, to be a *cross-section* for $\mathrm{Ob}(\mathscr{K})/\tilde{\to}$ if and only if the canonical map $[?] : \mathscr{C} \to \mathrm{Ob}(\mathscr{K})/\tilde{\to}$, $C \mapsto [C]$ is surjective, injective, bijective respectively. A cross-section for $\mathrm{Ob}(\mathscr{K})/\tilde{\to}$ is also said to *classify* $\mathscr{K}$. By a *classification* of $\mathscr{K}$ (up to isomorphism) we mean the explicit display of a cross-section for $\mathrm{Ob}(\mathscr{K})/\tilde{\to}$.

A map $f : X \to Y$ is called *G-equivariant* if and only if it respects underlying actions of a group G on X and Y in the sense that $f(gx) = gf(x)$ holds for all $g \in G$ and $x \in X$.

2. Existence and uniqueness of liftings

Let $\eta : E \wedge E \to E$ be a dissident map on the Euclidean column space $E = \mathbb{E}^m$, where $3 \leqslant m < \infty$.[1] Then η induces a selfmap $\eta_{\mathbb{P}} : \mathbb{P}(E) \to \mathbb{P}(E)$ of the $(m-1)$-dimensional real projective space, given by $\eta_{\mathbb{P}}[v] = (\eta(v \wedge E))^{\perp}$. According to [7, Proposition 2.2], $\eta_{\mathbb{P}}$ is always bijective. However, $\eta_{\mathbb{P}}$ may or may not be collinear (cf. [7, Propositions 2.4 and 4.5]). In view of the fundamental theorem of projective geometry [1, p. 88], this fact can be expressed equivalently by stating that $\eta_{\mathbb{P}}$ may or may not be liftable to a linear bijection $\Phi : E \to E$. The present article puts a unifying perspective on this puzzling behaviour of $\eta_{\mathbb{P}}$ by showing that $\eta_{\mathbb{P}}$ in fact always admits a lifting $\Phi : E \to E$ of degree d with $1 \leqslant d \leqslant m-2$ (Theorem 2.4), where Φ is a linear bijection if and only if $d = 1$. We begin our investigation by introducing rigorous terminology.[2]

A *quasi-lifting of* $\eta_{\mathbb{P}}$ *outside* A, or briefly a *quasi-lifting* of $\eta_{\mathbb{P}}$, is a map $\Phi : E \to E^*$, $\Phi(v) = (\varphi_1(v), \ldots, \varphi_m(v))$ having the following properties:

(a) $\{\varphi_1, \ldots, \varphi_m\} \subset R_d$ for some $d \in \mathbb{N}\backslash\{0\}$;

(b) A is a proper algebraic subset of E such that $\Phi^{-1}(0) \subset A$ and $\eta_{\mathbb{P}}[v] = [\Phi(v)]$ for all $v \in E\backslash A$;

(c) $\gcd(\varphi_1, \ldots, \varphi_m) = (1)$.

A *lifting* of $\eta_{\mathbb{P}}$ is a quasi-lifting of $\eta_{\mathbb{P}}$ outside $\{0\}$.

Let $\Phi = (\varphi_1, \ldots, \varphi_m)$ be a quasi-lifting of $\eta_{\mathbb{P}}$. The *support* of Φ is defined by $I(\Phi) = \{i \in \underline{m} \mid \varphi_i \neq 0\}$. It is non-empty, according to (b). The *degree* of Φ is defined by $\deg \Phi = \deg \varphi_i$ for all $i \in I(\Phi)$. It is well-defined and satisfies $\deg \Phi \geqslant 1$, according to (a). For every $\lambda \in \mathbb{R}\backslash\{0\}$, the map $\lambda\Phi = (\lambda\varphi_1, \ldots, \lambda\varphi_m)$ is also a quasi-lifting of $\eta_{\mathbb{P}}$.

Lemma 2.1. *If Φ and Ψ are quasi-liftings of $\eta_{\mathbb{P}}$, then $\Psi = \lambda\Phi$ for a unique $\lambda \in \mathbb{R}\backslash\{0\}$.*

Proof. Let $\Phi = (\varphi_1, \ldots, \varphi_m)$ be a quasi-lifting of $\eta_{\mathbb{P}}$ outside A and $\Psi = (\psi_1, \ldots, \psi_m)$ be a quasi-lifting of $\eta_{\mathbb{P}}$ outside B. Set $C = A \cup B$. Note that $E\backslash C$ is non-empty and open. Due to (b) we have for all $v \in E\backslash C$ that

$$[(\psi_1(v), \ldots, \psi_m(v))] = [\Psi(v)] = \eta_{\mathbb{P}}[v] = [\Phi(v)] = [(\varphi_1(v), \ldots, \varphi_m(v))],$$

and hence

$$(\psi_1(v), \ldots, \psi_m(v)) = \mu(v)(\varphi_1(v), \ldots, \varphi_m(v)) \tag{1}$$

for a unique $\mu(v) \in \mathbb{R}\backslash\{0\}$. This implies $I(\Phi) = I(\Psi)$. Set $I = I(\Phi)$ and let $\mu : E\backslash C \to \mathbb{R}\backslash\{0\}$ be the function defined by (1). Then we can write (1) equivalently as system

[1] In view of the (1,2,4,8)-theorem of Hopf [13], Bott and Milnor [2] and Kervaire [15] (cf. [12]), combined with Osborn's theorem [19] (cf. [6]), this hypothesis implies that $m \in \{3, 7\}$. Nevertheless we shall develop our theory of liftings independent of the (1,2,4,8)-theorem, up to Corollary 3.3.

[2] In the present section we consider, for technical reasons, E^* instead of E as the codomain of a quasi-lifting Φ of $\eta_{\mathbb{P}}$. Accordingly we consider $\mathbb{P}(E^*)$ instead of $\mathbb{P}(E)$ as the codomain of $\eta_{\mathbb{P}}$, i.e. we interpret $\eta_{\mathbb{P}}[v] = (\eta(v \wedge E))^{\perp}$ as $(\eta(v \wedge E))^{\perp} = \{\xi \in E^* \mid \eta(v \wedge E) \subset \ker \xi\} \in \mathbb{P}(E^*)$. We identify $E^* = (\mathbb{R}^{m\times 1})^*$ with $\mathbb{R}^{1\times m}$ and R_1, respectively.

of equations

$$\psi_i = \mu\varphi_i \quad \forall i \in I \tag{2}$$

of real-valued functions defined on $E\backslash C$. Multiplying for arbitrary $j \in I$ all equations in (2) by φ_j and eliminating the unknown function μ by means of $\psi_j = \mu\varphi_j$, we obtain the new system of equations

$$\psi_i\varphi_j = \varphi_i\psi_j \quad \forall (i, j) \in I^2 \tag{3}$$

of real-valued polynomial functions defined on $E\backslash C$. Because $E\backslash C$ is non-empty and open, (3) remains valid when viewed as a system of equations in the polynomial ring $\mathbb{R}[X_1, \ldots, X_m]$. Thus we can conclude with (c) that

$$(\psi_i) = \gcd(\psi_i\varphi_j \mid j \in I) = \gcd(\varphi_i\psi_j \mid j \in I) = (\varphi_i)$$

holds for all $i \in I$, and hence

$$\psi_i = \mu_i\varphi_i \quad \forall i \in I \tag{4}$$

for uniquely determined numbers $\mu_i \in \mathbb{R}\backslash\{0\}$. From (3) and (4) we deduce the system of equations

$$\mu_i i = \frac{\psi_i}{\varphi_i} = \frac{\psi_j}{\varphi_j} = \mu_j \quad \forall (i, j) \in I^2$$

in the rational function field $\mathbb{R}(X_1, \ldots, X_m)$. Accordingly $\lambda = \mu_i$ for all $i \in I$ is a well-defined non-zero real number which, substituted into (4), yields $\Psi = (\psi_1, \ldots, \psi_m) = \lambda(\varphi_1, \ldots, \varphi_m) = \lambda\Phi$. Moreover, λ is uniquely determined by Φ and Ψ, because $\Phi \neq 0$. $\square$

Let H be a hyperplane in E and let $\mathbf{h} = (h_2, \ldots, h_m)$ be a basis in H. The data η and $\mathbf{h}$ determine a map $\Phi^{\mathbf{h}} : E \to E^*$, defined for all $v \in E$ by

$$\Phi^{\mathbf{h}}(v) = \det(? \mid \eta(v \wedge h_2) \mid \ldots \mid \eta(v \wedge h_m)).$$

Lemma 2.2. *The map* $\Phi^{\mathbf{h}} : E \to E^*$, $\Phi^{\mathbf{h}}(v) = (\varphi_1^{\mathbf{h}}(v), \ldots, \varphi_m^{\mathbf{h}}(v))$ *has the following properties.*

(i) $\{\varphi_1^{\mathbf{h}}, \ldots, \varphi_m^{\mathbf{h}}\} \subset R_{m-1}$;
(ii) $(\Phi^{\mathbf{h}})^{-1}(0) = H$;
(iii) $\eta_{\mathbb{P}}[v] = [\Phi^{\mathbf{h}}(v)]$ *for all* $v \in E\backslash H$;
(iv) *Each of the real polynomials* $\varphi_1^{\mathbf{h}}, \ldots, \varphi_m^{\mathbf{h}}$ *vanishes on the complex hyperplane* $H \oplus iH$ *in* $\mathbb{C}^m$.

Proof. (i) Each vector $v \in E$ determines a real matrix

$$M^{\mathbf{h}}(v) = (\eta(v \wedge h_2) \mid \ldots \mid \eta(v \wedge h_m)) \in \mathbb{R}^{m\times(m-1)}$$

and, deleting from $M^{\mathbf{h}}(v)$ the i-th row, a real quadratic matrix $M^{\mathbf{h}}_{\hat{i}}(v) \in \mathbb{R}^{(m-1)\times(m-1)}$. Considering $v \in E$ as indeterminate vector, we obtain matrices $M^{\mathbf{h}} \in R_1^{m\times(m-1)}$ and $M^{\mathbf{h}}_{\hat{i}} \in R_1^{(m-1)\times(m-1)}$ with entries in the dual space $E^* = R_1$ of E. Thus we conclude that

$$\varphi_i^{\mathbf{h}} = \det(e_i \mid \eta(? \wedge h_2) \mid \ldots \mid \eta(? \wedge h_m)) = (-1)^{i+1} \det M^{\mathbf{h}}_{\hat{i}} \in R_{m-1}$$

for all $i \in \underline{m}$.

(ii) For any $v \in E$, the identity $\Phi^{\mathbf{h}}(v) = 0$ holds if and only if the columns of $M^{\mathbf{h}}(v)$ are linearly dependent. Because η is dissident, the latter holds if and only if $v \in H$.

(iii) If $v \in E\backslash H$, then $\ker \Phi^{\mathbf{h}}(v) = [\eta(v \wedge h_2), \ldots, \eta(v \wedge h_m)] = \eta(v \wedge H) = \eta(v \wedge E)$. Consequently $\Phi^{\mathbf{h}}(v) \in (\eta(v \wedge E))^{\perp}\backslash\{0\}$, and hence $[\Phi^{\mathbf{h}}(v)] = (\eta(v \wedge E))^{\perp} = \eta_{\mathbb{P}}[v]$.

(iv) Complexify the given $\mathbb{R}$-linear dissident map $\eta : E \wedge E \to E$ to a $\mathbb{C}$-linear map $\eta_{\mathbb{C}} : \mathbb{C}^m \wedge \mathbb{C}^m \to \mathbb{C}^m$, and define the complex polynomials $\varphi^{\mathbf{h}}_{\mathbb{C},1}, \ldots, \varphi^{\mathbf{h}}_{\mathbb{C},m}$ in $\mathbb{C}[X_1, \ldots, X_m]$ by $\varphi^{\mathbf{h}}_{\mathbb{C},i}(z) = \det(e_i \mid \eta_{\mathbb{C}}(z \wedge h_2) \mid \ldots \mid \eta_{\mathbb{C}}(z \wedge h_m))$ for all $i \in \underline{m}$ and $z \in \mathbb{C}^m$. Observing that $\varphi^{\mathbf{h}}_{\mathbb{C},i}(v) = \varphi^{\mathbf{h}}_i(v)$ for all $i \in \underline{m}$ and $v \in \mathbb{R}^m$ we deduce that $\varphi^{\mathbf{h}}_{\mathbb{C},i} = \varphi^{\mathbf{h}}_i$ for all $i \in \underline{m}$. If $z \in H \oplus iH$, then the columns $\eta_{\mathbb{C}}(z \wedge h_2), \ldots, \eta_{\mathbb{C}}(z \wedge h_m)$ are $\mathbb{C}$-linearly dependent and hence $\varphi^{\mathbf{h}}_i(z) = \varphi^{\mathbf{h}}_{\mathbb{C},i}(z) = 0$ for all $i \in \underline{m}$. □

Given η and $\mathbf{h}$ as before, with corresponding map $\Phi^{\mathbf{h}} = (\varphi^{\mathbf{h}}_1, \ldots, \varphi^{\mathbf{h}}_m)$, we choose now $\varphi^{\mathbf{h}} \in R\backslash\{0\}$ such that $(\varphi^{\mathbf{h}}) = \gcd(\varphi^{\mathbf{h}}_1, \ldots, \varphi^{\mathbf{h}}_m)$ and set $\varphi_i = \varphi^{\mathbf{h}}_i/\varphi^{\mathbf{h}}$ for all $i \in \underline{m}$.

Proposition 2.3. *The map* $\Phi : E \to E^*$, $\Phi(v) = (\varphi_1(v), \ldots, \varphi_m(v))$ *is a quasi-lifting of* $\eta_{\mathbb{P}}$ *outside* H, *satisfying* $1 \leqslant \deg \Phi \leqslant m - 2$.

Proof. First we verify that Φ satisfies the defining conditions (a)–(c) for a quasi-lifting of $\eta_{\mathbb{P}}$ outside H.

(a) Because every factor of a non-zero homogeneous polynomial is homogeneous, Lemma 2.2(i) implies that $\{\varphi_1, \ldots, \varphi_m\} \subset R_d$ for $d = m - 1 - \deg \varphi^{\mathbf{h}}$. According to Lemma 2.2(ii) and (iii), the identity

$$\eta_{\mathbb{P}}[v] = [\Phi^{\mathbf{h}}(v)] = [\varphi^{\mathbf{h}}(v)(\varphi_1(v), \ldots, \varphi_m(v))] = [(\varphi_1(v), \ldots, \varphi_m(v))] \tag{5}$$

holds for all $v \in E\backslash H$. Because $\eta_{\mathbb{P}}$ is bijective, its restriction to $\mathbb{P}(E)\backslash\mathbb{P}(H)$ is injective which, together with (5), implies that $d \geqslant 1$.

(b) The hyperplane H is a proper algebraic subset of E. The inclusion $\Phi^{-1}(0) \subset (\Phi^{\mathbf{h}})^{-1}(0) = H$ holds by definition of Φ and Lemma 2.2(ii). The identity $\eta_{\mathbb{P}}[v] = [(\varphi_1(v), \ldots, \varphi_m(v))] = [\Phi(v)]$ holds for all $v \in E\backslash H$ by (5) and definition of Φ.

(c) $\gcd(\varphi_1, \ldots, \varphi_m) = (1)$ holds by definition of the polynomials $\varphi_1, \ldots, \varphi_m$.

It remains to show that $d \leqslant m - 2$, or equivalently that $\deg \varphi^{\mathbf{h}} \geqslant 1$. Choose $\xi \in R_1\backslash\{0\}$ such that $\xi(H) = \{0\}$. Then the zero set of ξ in $\mathbb{C}^m$ is the complex hyperplane $\mathscr{Z}(\xi) = H \oplus iH$ and the vanishing ideal of $H \oplus iH$ in R is $\mathscr{I}(H \oplus iH) = \mathscr{I}\mathscr{Z}(\xi) = \mathrm{Rad}(\xi) = (\xi)$, by Hilbert's Nullstellensatz. We conclude with Lemma 2.2(iv) that $\{\varphi^{\mathbf{h}}_1, \ldots, \varphi^{\mathbf{h}}_m\} \subset \mathscr{I}(H \oplus iH) = (\xi)$ and hence $(\varphi^{\mathbf{h}}) = \gcd(\varphi^{\mathbf{h}}_1, \ldots, \varphi^{\mathbf{h}}_m) \subset (\xi)$, proving that $\deg \varphi^{\mathbf{h}} \geqslant \deg \xi = 1$. □

Two functions $F : E \to E^*$ and $G : E \to E^*$ are called *associated* if and only if $G = \lambda F$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. The set of all associated pairs (F, G) forms an equivalence relation, called *association*, on the vector space $(E^*)^E$ of all functions from E to E^*. The equivalence class of $F : E \to E^*$ under association is called the *associate class* of F. If $F \neq 0$ then its associate class is the punctured line $[F]\backslash\{0\}$.

Theorem 2.4. *Let η be a dissident map on the m-dimensional Euclidean column space E, where $3 \leqslant m < \infty$. Then the following assertions hold true.*

(i) *There exists a lifting of $\eta_{\mathbb{P}}$.*
(ii) *The set of all liftings of $\eta_{\mathbb{P}}$ forms precisely one associate class in $(E^*)^E\backslash\{0\}$.*
(iii) *The degree of a lifting Φ of $\eta_{\mathbb{P}}$ is uniquely determined by η. It satisfies $1 \leqslant \deg \Phi \leqslant m - 2$. Moreover, $\deg \Phi = 1$ if and only if $\eta_{\mathbb{P}}$ is collinear.*
(iv) *Every lifting of $\eta_{\mathbb{P}}$ has support $\underline{m}$.*
(v) *Every quasi-lifting of $\eta_{\mathbb{P}}$ is a lifting of $\eta_{\mathbb{P}}$.*

Proof. (i) For each $i \in \underline{m}$ choose a basis $\mathbf{h}_i$ in the coordinate hyperplane $H_i = e_i^{\perp}$ in E. The data η and $\mathbf{h}_i$ determine, according to Proposition 2.3, a quasi-lifting $\Phi_i : E \to E^*$ of $\eta_{\mathbb{P}}$ outside H_i. Set $\Phi = \Phi_1$. Due to Lemma 2.1 there are $\lambda_1, \ldots, \lambda_m$ in $\mathbb{R}\backslash\{0\}$ such that $\Phi = \lambda_i \Phi_i$ for all $i \in \underline{m}$. Thus Φ is a quasi-lifting of $\eta_{\mathbb{P}}$ outside H_i for all $i \in \underline{m}$. Hence Φ is a quasi-lifting of $\eta_{\mathbb{P}}$ outside $\bigcap_{i \in \underline{m}} H_i = \{0\}$, i.e. a lifting of $\eta_{\mathbb{P}}$.

(ii) follows directly from (i), the definition of a lifting of $\eta_{\mathbb{P}}$ and Lemma 2.1.

(iii) The degree of a lifting Φ of $\eta_{\mathbb{P}}$ does not depend on the choice of Φ, because any two liftings of $\eta_{\mathbb{P}}$ are associated. The lifting Φ of $\eta_{\mathbb{P}}$ constructed in the proof of (i) satisfies $1 \leqslant \deg \Phi \leqslant m - 2$, according to Proposition 2.3. If $\deg \Phi = 1$ then $\Phi : E \to E^*$ is a linear bijection and hence $\eta_{\mathbb{P}} = \mathbb{P}(\Phi)$ is collinear. Conversely, if $\eta_{\mathbb{P}}$ is collinear then the fundamental theorem of projective geometry asserts the existence of a lifting Φ of $\eta_{\mathbb{P}}$ such that $\deg \Phi = 1$.

(iv) For every lifting Φ of $\eta_{\mathbb{P}}$, the assumption $i \in \underline{m}\backslash I(\Phi)$ implies $\operatorname{im} \eta_{\mathbb{P}} \subset \mathbb{P}(H_i)$, contradicting the surjectivity of $\eta_{\mathbb{P}}$.

(v) According to (i) and Lemma 2.1, every quasi-lifting Ψ of $\eta_{\mathbb{P}}$ is associated with a lifting Φ of $\eta_{\mathbb{P}}$. Hence Ψ is a lifting of $\eta_{\mathbb{P}}$. $\square$

Identifying E^* with E via the linear bijection $E^* \to E$, $\xi = \langle x, ?\rangle \mapsto x$ and identifying $\mathbb{P}(E^*)$ with $\mathbb{P}(E)$ accordingly via the collinear bijection $\mathbb{P}(E^*) \to \mathbb{P}(E)$, $[\xi] = [\langle x, ?\rangle] \mapsto [x]$ we recover our original view of $\eta_{\mathbb{P}} : \mathbb{P}(E) \to \mathbb{P}(E)$ as a selfbijection of $\mathbb{P}(E)$ and of its lifting $\Phi : E \to E$ as a selfmap of E. Henceforth we shall maintain this original viewpoint.

3. Liftings of isomorphic dissident maps

Let $\eta : E \wedge E \to E$ and $\eta' : E \wedge E \to E$ be dissident maps on the m-dimensional Euclidean column space E, where $3 \leqslant m < \infty$. Let $\Phi : E \to E$ be a lifting of $\eta_{\mathbb{P}} : \mathbb{P}(E) \to \mathbb{P}(E)$ and let $\Phi' : E \to E$ be a lifting of $\eta'_{\mathbb{P}} : \mathbb{P}(E) \to \mathbb{P}(E)$.

Proposition 3.1. *If $\sigma : \eta \to \eta'$ is an isomorphism, then*
(i) $\sigma\Phi\sigma^{-1} = \lambda\Phi'$ *for a unique* $\lambda \in \mathbb{R}\backslash\{0\}$, *and*
(ii) $\deg \Phi = \deg \Phi'$.

Proof. (i) By Lemma 2.1 it suffices to show that the map $\sigma\Phi\sigma^{-1} : E \to E$ satisfies the defining conditions (a)–(c) for a quasi-lifting of $\eta'_{\mathbb{P}}$. To do so, let $d = \deg \Phi$ and $\varphi_1, \ldots, \varphi_m$ in $R_d\backslash\{0\}$ such that $\Phi(v) = (\varphi_1(v) \ldots \varphi_m(v))^{\mathrm{T}}$ for all $v \in E$. Moreover, let $S \in \mathrm{O}_m(\mathbb{R})$ be the orthogonal matrix representing σ in $\mathbf{e}$.

(a) We introduce the functions $\varrho_i : E \to \mathbb{R}$ and $\psi_i : E \to \mathbb{R}$, where $i \in \underline{m}$, on setting $\varrho_i = \varphi_i\sigma^{-1}$ and $\psi_i = \sum_{j=1}^{m} S_{ij}\varrho_j$. Then $\{\varphi_1, \ldots, \varphi_m\} \subset R_d$ together with the linearity of σ^{-1} implies that $\{\varrho_1, \ldots, \varrho_m\} \subset R_d$ and hence $\{\psi_1, \ldots, \psi_m\} \subset R_d$. On the other hand, the functions $\psi_1, \ldots, \psi_m$ are just the component functions of $\sigma\Phi\sigma^{-1}$, because

$$\sigma\Phi\sigma^{-1}(v) = S \begin{pmatrix} \varphi_1\sigma^{-1}(v) \\ \vdots \\ \varphi_m\sigma^{-1}(v) \end{pmatrix} = S \begin{pmatrix} \varrho_1(v) \\ \vdots \\ \varrho_m(v) \end{pmatrix} = \begin{pmatrix} \psi_1(v) \\ \vdots \\ \psi_m(v) \end{pmatrix}$$

holds for all $v \in E$.

(b) Because σ is linear bijective and Φ is a lifting of $\eta_{\mathbb{P}}$ we have that $(\sigma\Phi\sigma^{-1})^{-1}(0) = \{0\}$. If $v \in E\backslash\{0\}$, then $[\Phi(\sigma^{-1}v)] = \eta_{\mathbb{P}}[\sigma^{-1}v] = (\eta(\sigma^{-1}v \wedge E))^{\perp}$ implies that $\langle \Phi(\sigma^{-1}v), \eta(\sigma^{-1}v \wedge E)\rangle = \{0\}$. Hence $\langle \sigma\Phi\sigma^{-1}(v), \eta'(v \wedge E)\rangle = \langle \sigma\Phi\sigma^{-1}(v), \sigma\eta(\sigma^{-1}v \wedge E)\rangle = \{0\}$ which in turn implies that $[\sigma\Phi\sigma^{-1}(v)] = (\eta'(v \wedge E))^{\perp} = \eta'_{\mathbb{P}}[v]$.

(c) Choose $\psi \in R\backslash\{0\}$ such that $(\psi) = \gcd(\psi_1, \ldots, \psi_m)$. From $\Phi = (\varphi_1 \ldots \varphi_m)^{\mathrm{T}}$ and $\sigma\Phi\sigma^{-1} = (\psi_1 \ldots \psi_m)^{\mathrm{T}}$ we deduce via $\Phi = \sigma^{-1}(\sigma\Phi\sigma^{-1})\sigma$ that $\varphi_i = \sum_{j=1}^{m} S_{ji}\psi_j\sigma$ for all $i \in \underline{m}$. Now $\psi|\psi_j$ for all $j \in \underline{m}$ implies that $\psi\sigma|\psi_j\sigma$ for all $j \in \underline{m}$ and hence $\psi\sigma|\varphi_i$ for all $i \in \underline{m}$. Thus we have both $(\varphi_1, \ldots, \varphi_m) \subset (\psi\sigma)$ and $\gcd(\varphi_1, \ldots, \varphi_m) = (1)$. This implies $(\psi\sigma) = (1)$ and hence $(\psi) = (1)$.

(ii) Part (a) of the proof of (i) shows that $\deg \Phi = \deg \sigma\Phi\sigma^{-1}$, and (i) implies that $\deg \sigma\Phi\sigma^{-1} = \deg \Phi'$. $\square$

To enable a concise summary of what we have achieved so far, we introduce further notation. With the m-dimensional Euclidean column space $E = \mathbb{E}^m$, where $3 \leqslant m < \infty$, we associate the set

$$\mathscr{E}(E) = \{(V, \eta) \in \mathscr{E} \mid V = E\}$$

of all dissident maps on E, the set

$$\mathscr{L}(E) = \{\Phi : E \to E \mid \Phi \text{ lifts } \eta_{\mathbb{P}} \text{ for some } \eta \in \mathscr{E}(E)\}$$

of all liftings on E, and the set

$$\mathscr{A}\mathscr{L}(E) = \{[\Phi]\backslash\{0\} \mid \Phi \in \mathscr{L}(E)\}$$

of all associate classes of liftings on E. The orthogonal group $\mathrm{O}_m = \mathrm{O}_m(\mathbb{R})$ acts from the left on each of these sets by

$$\begin{array}{ll} \mathrm{O}_m \times \mathscr{E}(E) \to \mathscr{E}(E), & \sigma \cdot \eta = \sigma\eta(\sigma^{-1} \wedge \sigma^{-1}); \\ \mathrm{O}_m \times \mathscr{L}(E) \to \mathscr{L}(E), & \sigma \cdot \Phi = \sigma\Phi\sigma^{-1}; \\ \mathrm{O}_m \times \mathscr{A}\mathscr{L}(E) \to \mathscr{A}\mathscr{L}(E), & \sigma \cdot ([\Phi]\backslash\{0\}) = [\sigma\Phi\sigma^{-1}]\backslash\{0\}. \end{array}$$

Corollary 3.2. *Let $E = \mathbb{E}^m$, where $3 \leqslant m < \infty$. Then the following statements hold true.*

(i) *The lift mapping $\ell : \mathscr{E}(E) \to \mathscr{A}\mathscr{L}(E)$, $\ell(\eta) = [\Phi]\backslash\{0\}$ where Φ lifts $\eta_{\mathbb{P}}$, is well-defined, surjective and O_m-equivariant.*
(ii) *The degree mapping* $\deg : \mathscr{A}\mathscr{L}(E) \to \underline{m-2}$, $\deg([\Phi]\backslash\{0\}) = \deg \Phi$, *is well-defined and constant on each O_m-orbit of $\mathscr{A}\mathscr{L}(E)$.*
(iii) *The lift mapping $\ell : \mathscr{E}(E) \to \mathscr{A}\mathscr{L}(E)$ induces a surjective mapping of orbit sets $\overline{\ell} : \mathscr{E}(E)/\mathrm{O}_m \to \mathscr{A}\mathscr{L}(E)/\mathrm{O}_m$, $\overline{\ell}(\mathrm{O}_m \cdot \eta) = \mathrm{O}_m \cdot \ell(\eta)$.*
(iv) *The degree mapping* $\deg : \mathscr{A}\mathscr{L}(E) \to \underline{m-2}$ *induces a mapping* $\overline{\deg} : \mathscr{A}\mathscr{L}(E)/\mathrm{O}_m \to \underline{m-2}$, $\overline{\deg}(\mathrm{O}_m \cdot ([\Phi]\backslash\{0\})) = \deg \Phi$.

Proof. (i) and (ii) follow immediately from Theorem 2.4 and Proposition 3.1, while (iii) and (iv) are trivial consequences of (i) and (ii). □

Corollary 3.2 establishes the commutative diagram

$$\begin{array}{ccccc} & & \mathscr{L}(E) & & \\ & \ell & \downarrow & \deg & \\ \mathscr{E}(E) & \longrightarrow & \mathscr{A}\mathscr{L}(E) & \longrightarrow & \underline{m-2} \\ \downarrow & & \downarrow & & \| \\ \mathscr{E}(E)/\mathrm{O}_m & \longrightarrow & \mathscr{A}\mathscr{L}(E)/\mathrm{O}_m & \longrightarrow & \underline{m-2} \\ & \overline{\ell} & & \overline{\deg} & \end{array}$$

where $\ell, \overline{\ell}$ and the canonical vertical maps are surjective. The fibres of $\overline{\deg} \circ \overline{\ell}$ form a partition for the isoclasses of dissident maps on E, and the fibres of $\overline{\ell}$ form a partition for the fibres of $\overline{\deg} \circ \overline{\ell}$. Thus the analysis of the above diagram provides a strategy towards a possible classification of all dissident maps, up to isomorphism. The remaining part of the present article will exemplify this view.

We begin by gathering obvious information on the sets $\mathscr{E}(E)$ and $\mathscr{L}(E)$. For every $d \in \underline{m}$ we denote by $\mathscr{F}_{md}$ the set of all maps $\Phi : E \to E$, $\Phi(v) = (\varphi_1(v) \dots \varphi_m(v))^{\mathrm{T}}$ such that $\{\varphi_1, \dots, \varphi_m\} \subset R_d\backslash\{0\}$, $\Phi^{-1}(0) = \{0\}$ and $\gcd(\varphi_1, \dots, \varphi_m) = (1)$. Note that $\mathscr{F}_{m1} = \mathrm{GL}(E)$.

Corollary 3.3. *Let $E = \mathbb{E}^m$, where $3 \leqslant m < \infty$. Then the sets $\mathscr{E}(E)$ and $\mathscr{L}(E)$ admit the following description.*

(i) *If $m \notin \{3, 7\}$, then $\mathscr{E}(E) = \emptyset$ and $\mathscr{L}(E) = \emptyset$.*
(ii) *If $m = 3$, then $\mathscr{E}(E) \neq \emptyset$ and $\{\mathbb{I}_E\} \subset \mathscr{L}(E) \subset \mathrm{GL}(E)$.*
(iii) *If $m = 7$, then $\mathscr{E}(E) \neq \emptyset$ and $\{\mathbb{I}_E\} \subset \mathscr{L}(E) \subset \mathrm{GL}(E) \cup (\bigcup_{d=2}^{5} \mathscr{F}_{7d})$.*

Proof. (i) If $\eta \in \mathscr{E}(E)$, then the vector space $A_\eta = \mathbb{R} \times \mathbb{R}^m$, endowed with the $\mathbb{R}$-bilinear multiplication $(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle, \alpha w + \beta v + \eta(v \wedge w))$, is a real quadratic division algebra, by Osborn's theorem (cf. [19,6]). The (1,2,4,8)-theorem (cf. [12]) implies that $m + 1 \in \{1, 2, 4, 8\}$, and hence that $m \in \{3, 7\}$.

(ii) Let $m = 3$. The standard vector product[3] $\pi_3 : E \wedge E \to E$ is given by $\pi_3(e_1 \wedge e_2) = e_3$, $\pi_3(e_2 \wedge e_3) = e_1$ and $\pi_3(e_3 \wedge e_1) = e_2$. Being a vector product, π_3 is a dissident map on E such that $\pi_{3\mathbb{P}} = \mathbb{I}_{\mathbb{P}(E)}$. If $\Phi \in \mathscr{L}(E)$, then $\deg \Phi = 1$ by Theorem 2.4(iii), and hence $\Phi \in \mathscr{F}_{31} = \mathrm{GL}(E)$.

(iii) Let $m = 7$. Writing vectors in E in the form $\left(\begin{smallmatrix} v \\ \alpha \\ w \end{smallmatrix}\right)$, where $(v, \alpha, w) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$, the standard vector product $\pi_7 : E \wedge E \to E$ is given by

$$\pi_7\left(\begin{pmatrix} v \\ \alpha \\ w \end{pmatrix} \wedge \begin{pmatrix} v' \\ \alpha' \\ w' \end{pmatrix}\right) = \begin{pmatrix} \alpha w' - \alpha' w + \pi_3(v \wedge v') - \pi_3(w \wedge w') \\ -\langle v, w'\rangle + \langle w, v'\rangle \\ -\alpha v' + \alpha' v - \pi_3(v \wedge w') - \pi_3(w \wedge v') \end{pmatrix}$$

(cf. [7] or [17]). Being a vector product, π_7 is a dissident map on E such that $\pi_{7\mathbb{P}} = \mathbb{I}_{\mathbb{P}(E)}$. If $\Phi \in \mathscr{L}(E)$, then $1 \leqslant \deg \Phi \leqslant 5$ by Theorem 2.4(iii), and hence $\Phi \in \bigcup_{d=1}^{5} \mathscr{F}_{7d} = \mathrm{GL}(E) \cup (\bigcup_{d=2}^{5} \mathscr{F}_{7d})$. $\square$

In the next section we shall refine the preliminary information contained in Corollary 3.3 by giving, in case $m \in \{3, 7\}$, both an exact description of the set $\mathscr{L}(E) \cap \mathrm{GL}(E)$ of all linear liftings on E and a complete description of the set $\mathscr{E}^1(E) = \{\eta \in \mathscr{E}(E) \mid \deg(\ell(\eta)) = 1\}$ of all dissident maps on E having degree one.

4. Dissident maps of degree one

Recall that an arbitrary algebra A is said to be *flexible* if and only if $(ab)a = a(ba)$ holds for all $(a, b) \in A^2$. Besides, we call a dissident map $\eta : V \wedge V \to V$ *flexible* if and only if $\langle \eta(u \wedge v), w\rangle = \langle u, \eta(v \wedge w)\rangle$ for all $(u, v, w) \in V^3$. This terminology is justified by the equivalence (i) $\Leftrightarrow$ (iv) in Proposition 4.1 below. Flexible dissident maps generalize vector products, by definition. But on the level of the induced selfbijection $\eta_{\mathbb{P}}$ they are no longer distinguishable from vector products. This fact even characterizes the flexible dissident maps, according to the equivalence (i) $\Leftrightarrow$ (iii) in the following proposition.

Proposition 4.1. *For each dissident map $\eta : V \wedge V \to V$, the following assertions are equivalent*:

(i) *η is flexible.*
(ii) $\langle v, \eta(v \wedge w)\rangle = 0$ *for all* $(v, w) \in V^2$.
(iii) $\eta_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}(V)}$.
(iv) *The real quadratic division algebra $A_\eta = \mathbb{R} \times V$, with multiplication $(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle, \alpha w + \beta v + \eta(v \wedge w))$, is flexible.*

[3] A *vector product* on a Euclidean space V is a linear map $\pi : V \wedge V \to V$ such that $\langle \pi(u \wedge v), w\rangle = \langle u, \pi(v \wedge w)\rangle$ for all $(u, v, w) \in V^3$, and $|\pi(u \wedge v)| = 1$ for all orthonormal pairs $(u, v) \in V^2$.

Proof. (i) $\Rightarrow$ (ii). If $(v, w) \in V^2$, then $\langle v, \eta(v \wedge w)\rangle = \langle \eta(v \wedge v), w\rangle = 0$.
(ii) $\Rightarrow$ (i). If $(u, v, w) \in V^3$, then $0 = \langle w + u, \eta((w + u) \wedge v)\rangle = \langle w, \eta(u \wedge v)\rangle + \langle u, \eta(w \wedge v)\rangle = \langle \eta(u \wedge v), w\rangle - \langle u, \eta(v \wedge w)\rangle$.
(ii) $\Leftrightarrow$ (iii) holds by definition of $\eta_{\mathbb{P}}$.
(ii) $\Leftrightarrow$ (iv). A routine calculation shows that

$$((\alpha, v)(\beta, w))(\alpha, v) = (\alpha, v)((\beta, w)(\alpha, v))$$

is equivalent to $\langle v, \eta(v \wedge w)\rangle = 0$, for all $((\alpha, v), (\beta, w)) \in A_\eta \times A_\eta$. $\square$

We denote by $\mathscr{E}^f$ the full subcategory of $\mathscr{E}$ formed by all flexible dissident maps. Theorems of Darpö, Cuenca Mira et al. assert that a classification of $\mathscr{E}_3^f$ is obtained from the standard vector product π_3 by homothety [4], and a complete description of $\mathscr{E}_7^f$ is obtained from the standard vector product[4] π_7 by vectorial isotopy [3].

Heading for a precise version of these statements (Theorem 4.2 below), we denote by $\gamma^* \in \mathrm{GL}(\mathbb{E}^7)$ the adjoint of any $\gamma \in \mathrm{GL}(\mathbb{E}^7)$, defined by $\langle \gamma(v), w\rangle = \langle v, \gamma^*(w)\rangle$ for all $(v, w) \in (\mathbb{E}^7)^2$. The right group action

$$Hom_{\mathbb{R}}(\mathbb{E}^7 \wedge \mathbb{E}^7, \mathbb{E}^7) \times \mathrm{GL}(\mathbb{E}^7) \to Hom_{\mathbb{R}}(\mathbb{E}^7 \wedge \mathbb{E}^7, \mathbb{E}^7), \quad \mu \cdot \gamma = \gamma^* \mu(\gamma \wedge \gamma),$$

called "vectorial isotopy" in [3], is easily seen to induce a map

$$\mathrm{GL}(\mathbb{E}^7) \to \mathscr{E}_7^f, \quad \gamma \mapsto \pi_7 \cdot \gamma.$$

By $\mathbb{G}_2$ we denote both the automorphism group $\mathrm{Aut}(\pi_7) = \{\sigma \in \mathrm{O}(\mathbb{E}^7) \mid \pi_7 \cdot \sigma = \pi_7\}$ and its matrix version $\{S \in \mathrm{O}_7(\mathbb{R}) \mid \pi_7 \cdot \underline{S} = \pi_7\}$. This notation is justified by the fact that $\mathrm{Aut}(\pi_7)$ is a compact, connected, simple real Lie group of dimension 14 and therefore an exceptional compact Lie group of type $\mathbb{G}_2$ (cf. [20, Theorem 11.33]).

Theorem 4.2. (i) *The family* $\{\lambda\pi_3 \mid \lambda \in \mathbb{R}_{>0}\}$ *is a cross-section for* $\mathrm{Ob}(\mathscr{E}_3^f)/\tilde{\to}$.
(ii) *The family* $\{\pi_7 \cdot \delta \mid \delta \in \mathrm{Pds}(\mathbb{E}^7)\}$ *exhausts* $\mathrm{Ob}(\mathscr{E}_7^f)$.
(iii) *For all* $\delta, \overline{\delta} \in \mathrm{Pds}(\mathbb{E}^7)$, *the set of all morphisms* $\sigma : \pi_7 \cdot \delta \to \pi_7 \cdot \overline{\delta}$ *in* $\mathscr{E}$ *admits the description* $\mathrm{Mor}(\pi_7 \cdot \delta, \pi_7 \cdot \overline{\delta}) = \{\sigma \in \mathbb{G}_2 \mid \sigma\delta\sigma^{-1} = \overline{\delta}\}$.

Proof. For proofs of (i) and (ii) we refer to [4, proof of Proposition 6.1] and [3, proof of Theorem 5.7]. Our proof of (iii) is a refinement of the proof of the corresponding (but slightly weaker) statement contained in [3, Theorem 5.7]. We include it here for the reader's convenience.

Let $\delta, \overline{\delta} \in \mathrm{Pds}(\mathbb{E}^7)$ be given. If $\sigma \in \mathrm{Mor}(\pi_7 \cdot \delta, \pi_7 \cdot \overline{\delta})$, then $\sigma \in \mathrm{O}(\mathbb{E}^7)$ such that $\pi_7 \cdot \delta = (\pi_7 \cdot \overline{\delta}) \cdot \sigma = \pi_7 \cdot (\overline{\delta}\sigma)$. Thus $\tau = \overline{\delta}\sigma\delta^{-1}$ is in $\mathrm{GL}(\mathbb{E}^7)$ and satisfies $\pi_7 \cdot \tau = \pi_7$. We infer from [3, Proposition 3.8] that $\tau \in \mathbb{G}_2$. Now $\mathbb{I} = \sigma\sigma^* = \overline{\delta}^{-1}\tau\delta\delta\tau^{-1}\overline{\delta}^{-1}$ implies that $\overline{\delta}^2 = \tau\delta^2\tau^{-1} = (\tau\delta\tau^{-1})^2$, and hence that $\overline{\delta} = \tau\delta\tau^{-1}$. So $\sigma = \overline{\delta}^{-1}\tau\delta = (\tau\delta^{-1}\tau^{-1})\tau\delta = \tau$, and therefore $\sigma \in \mathbb{G}_2$ such that $\sigma\delta\sigma^{-1} = \overline{\delta}$.

[4] See proof of Corollary 3.3 for definition of π_3 and π_7.

Conversely, if $\sigma \in \mathbb{G}_2$ satisfies $\sigma\delta\sigma^{-1} = \overline{\delta}$, then $(\pi_7 \cdot \overline{\delta}) \cdot \sigma = \pi_7 \cdot (\overline{\delta}\sigma) = \pi_7 \cdot (\sigma\delta) = (\pi_7 \cdot \sigma) \cdot \delta = \pi_7 \cdot \delta$ shows that $\sigma \in \mathrm{Mor}(\pi_7 \cdot \delta, \pi_7 \cdot \overline{\delta})$. $\square$

From now on let $m \in \{3, 7\}$. We define the *degree* of a dissident map $\eta \in \mathscr{E}(\mathbb{E}^m)$ to be the degree of a lifting Φ of $\eta_{\mathbb{P}}$. More generally, we define the *degree* of a dissident map $\upsilon \in \mathscr{E}_m$ to be the degree of any $\eta \in \mathscr{E}(\mathbb{E}^m)$ such that $\upsilon \tilde{\to} \eta$. This notion is well-defined by Proposition 3.1(ii), and it gives rise to the degree map $\deg : \mathscr{E}_m \to \underline{m-2}$ by Theorem 2.4(iii).

Setting $\mathscr{E}^f(\mathbb{E}^m) = \mathscr{E}(\mathbb{E}^m) \cap \mathscr{E}^f$ and $\mathscr{E}^1(\mathbb{E}^m) = \{\eta \in \mathscr{E}(\mathbb{E}^m) \mid \deg \eta = 1\}$, we obtain the filtration of full subcategories $\mathscr{E}^f(\mathbb{E}^m) \subset \mathscr{E}^1(\mathbb{E}^m) \subset \mathscr{E}(\mathbb{E}^m)$. More generally, setting $\mathscr{E}^1_m = \{\eta \in \mathscr{E}_m \mid \deg \eta = 1\}$, we obtain the filtration of full subcategories $\mathscr{E}^f_m \subset \mathscr{E}^1_m \subset \mathscr{E}_m$.

In the remaining part of the present section we shall demonstrate how our theory of liftings can be applied in order to extend the above classification of $\mathscr{E}^f_3$ to a classification of $\mathscr{E}_3$, and the above complete description of $\mathscr{E}^f_7$ to a complete description of $\mathscr{E}^1_7$.

Proposition 4.3. *Let $E = \mathbb{E}^m$, where $m \in \{3, 7\}$.*

(i) *If $\varphi \in \mathscr{E}^f(E)$ and $\varepsilon \in \mathrm{Def}(E)$, then $\varepsilon\varphi \in \mathscr{E}^1(E)$ and ε^{-*} lifts $(\varepsilon\varphi)_{\mathbb{P}}$.*
(ii) *If $\eta \in \mathscr{E}^1(E)$ and Φ lifts $\eta_{\mathbb{P}}$, then $\Phi \in \mathrm{Def}(E)$ and $\Phi^*\eta \in \mathscr{E}^f(E)$.*

Proof. (i) Given $(\varepsilon, \varphi) \in \mathrm{Def}(E) \times \mathscr{E}^f(E)$, let $(v, w) \in E^2$ be a non-proportional pair and assume that $(v, w, \varepsilon\varphi(v \wedge w))$ is linearly dependent. Then $\varepsilon\varphi(v \wedge w) \in [v, w]$. On the other hand $\varphi(v \wedge w) \in [v, w]^\perp$, since φ is flexible. Hence $\langle \varphi(v \wedge w), \varepsilon\varphi(v \wedge w)\rangle = 0$. Definiteness of ε implies that $\varphi(v \wedge w) = 0$, contradicting the dissidence of φ. So $(v, w, \varepsilon\varphi(v \wedge w))$ must be linearly independent. Accordingly $\varepsilon\varphi \in \mathscr{E}(E)$.

The identities $\langle \varepsilon^{-*}(v), \varepsilon\varphi(v \wedge w)\rangle = \langle v, \varphi(v \wedge w)\rangle = 0$, valid for all $(v, w) \in E^2$, show that $[\varepsilon^{-*}(v)] = (\varepsilon\varphi)_{\mathbb{P}}[v]$ holds for all $v \in E\backslash\{0\}$. Hence ε^{-*} lifts $(\varepsilon\varphi)_{\mathbb{P}}$, and thus $\varepsilon\varphi \in \mathscr{E}^1(E)$.

(ii) If $\eta \in \mathscr{E}^1(E)$ and Φ lifts $\eta_{\mathbb{P}}$, then $\Phi \in \mathrm{GL}(E)$ and $\langle v, \Phi^*\eta(v \wedge w)\rangle = \langle \Phi(v), \eta(v \wedge w)\rangle = 0$ holds for all $(v, w) \in E^2$. Hence $\Phi^*\eta(v \wedge w) \in [v, w]^\perp$. If $(v, w) \in E^2$ is non-proportional, then $\Phi^*\eta(v \wedge w) \neq 0$ because $\Phi^* \in \mathrm{GL}(E)$ and η is dissident, and therefore $\Phi^*\eta(v \wedge w) \notin [v, w]$. Accordingly $\Phi^*\eta \in \mathscr{E}(E)$ and so, by Proposition 4.1, even $\Phi^*\eta \in \mathscr{E}^f(E)$.

To prove definiteness of Φ, let $v \in E$ be such that $\langle v, \Phi(v)\rangle = 0$. If $v \neq 0$, then $v \in [\Phi(v)]^\perp = \eta(v \wedge E)$, contradicting the dissidence of η. Hence $v = 0$. Accordingly Φ is anisotropic. By Sylvester's inertia theorem this implies that Φ is definite. $\square$

Corollary 4.4. *Let $E = \mathbb{E}^m$, where $m \in \{3, 7\}$.*

(i) $\mathscr{L}(E) \cap \mathrm{GL}(E) = \mathrm{Def}(E)$.
(ii) *The map $\mathrm{Def}(E) \times \mathscr{E}^f(E) \to \mathscr{E}^1(E)$, $(\varepsilon, \varphi) \mapsto \varepsilon\varphi$ is well-defined and surjective.*
(iii) *For all $(\varepsilon, \varphi), (\overline{\varepsilon}, \overline{\varphi}) \in \mathrm{Def}(E) \times \mathscr{E}^f(E)$, the identity $\varepsilon\varphi = \overline{\varepsilon}\,\overline{\varphi}$ holds if and only if $(\lambda\varepsilon, \frac{1}{\lambda}\varphi) = (\overline{\varepsilon}, \overline{\varphi})$ for some $\lambda \in \mathbb{R}\backslash\{0\}$.*

(iv) *For all* $(\varepsilon, \varphi), (\bar{\varepsilon}, \overline{\varphi}) \in \mathrm{Def}(E) \times \mathscr{E}^f(E)$*, the set of all morphisms* $\sigma : \varepsilon\varphi \to \overline{\varepsilon\varphi}$ *in* $\mathscr{E}$ *admits the description*

$$\mathrm{Mor}(\varepsilon\varphi, \bar{\varepsilon}\,\overline{\varphi}) = \{\sigma \in \mathrm{O}(E) | (\lambda\sigma\varepsilon\sigma^{-1}, \lambda^{-1}\sigma\varphi(\sigma \wedge \sigma)^{-1}) = (\bar{\varepsilon}, \overline{\varphi}) \\ \textit{for some } \lambda \in \mathbb{R}\backslash\{0\}\}.$$

Proof. (i) and (ii) are immediate consequences of Proposition 4.3.

(iii) If $\varepsilon\varphi = \overline{\varepsilon\varphi}$, then Proposition 4.3(i) and Theorem 2.4(ii) imply that $\varepsilon^{-*} = \lambda\bar{\varepsilon}^{-*}$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. Consequently $\lambda\varepsilon = \bar{\varepsilon}$ and $\frac{1}{\lambda}\varphi = \overline{\varphi}$. The converse implication is trivially true.

(iv) Observing that $(\sigma\varepsilon\sigma^{-1}, \sigma\varphi(\sigma \wedge \sigma)^{-1}) \in \mathrm{Def}(E) \times \mathscr{E}^f(E)$ whenever $(\varepsilon, \varphi) \in \mathrm{Def}(E) \times \mathscr{E}^f(E)$ and $\sigma \in \mathrm{O}(E)$, the statement is a straightforward consequence of (iii). $\square$

Focusing now on $m = 3$, we proceed to demonstrate in several steps how from the above results a classification of $\mathscr{E}_3$ can be derived in a streamlined manner.

Proposition 4.5. (i) *The family* $\{\varepsilon\pi_3 \mid \varepsilon \in \mathrm{Pd}(\mathbb{E}^3)\}$ *exhausts* $\mathrm{Ob}(\mathscr{E}_3)$.

(ii) *For all* $\varepsilon, \bar{\varepsilon} \in \mathrm{Pd}(\mathbb{E}^3)$*, the set of all morphisms* $\sigma : \varepsilon\pi_3 \to \bar{\varepsilon}\pi_3$ *in* $\mathscr{E}$ *admits the description* $\mathrm{Mor}(\varepsilon\pi_3, \bar{\varepsilon}\pi_3) = \{\sigma \in \mathrm{SO}(\mathbb{E}^3) \mid \sigma\varepsilon\sigma^{-1} = \bar{\varepsilon}\}$.

Proof. (i) If $\varepsilon \in \mathrm{Pd}(\mathbb{E}^3)$ then $\varepsilon\pi_3 \in \mathscr{E}_3$, by Proposition 4.3(i). Conversely, if any $\upsilon = (V, \upsilon) \in \mathscr{E}_3$ is given, choose an orthonormal basis in V to obtain $\eta \in \mathscr{E}(\mathbb{E}^3) = \mathscr{E}^1(\mathbb{E}^3)$ such that $\upsilon \tilde{\to} \eta$. Choose a lifting Φ of $\eta_{\mathbb{P}}$. Proposition 4.3(ii) ensures that $\Phi \in \mathrm{Def}(\mathbb{E}^3)$. Moreover, Proposition 4.3(ii) and Theorem 4.2(i) imply that $\Phi^*\eta \tilde{\to} \lambda\pi_3$ for some $\lambda \in \mathbb{R}_{>0}$. Hence there exists a $\sigma \in \mathrm{O}(\mathbb{E}^3)$ such that $(\sigma\Phi^*\sigma^{-1})(\sigma\eta(\sigma \wedge \sigma)^{-1}) = \sigma(\Phi^*\eta)(\sigma \wedge \sigma)^{-1} = \lambda\pi_3$. This implies that $\eta \tilde{\to} \sigma\eta(\sigma \wedge \sigma)^{-1} = \sigma\Phi^{-*}\sigma^{-1}\lambda\pi_3 = \varepsilon\pi_3$, where $\varepsilon = \lambda\sigma\Phi^{-*}\sigma^{-1}$ is definite because Φ is. Finally, $-\mathbb{I}_{\mathbb{E}^3} : \varepsilon\pi_3 \tilde{\to} (-\varepsilon)\pi_3$ is an isomorphism, and one of $\varepsilon, -\varepsilon$ is positive definite.

(ii) If $\sigma \in \mathrm{Mor}(\varepsilon\pi_3, \bar{\varepsilon}\pi_3)$, then Corollary 4.4(iv) asserts that

$$(\lambda\sigma\varepsilon\sigma^{-1}, \lambda^{-1}\sigma\pi_3(\sigma \wedge \sigma)^{-1}) = (\bar{\varepsilon}, \pi_3)$$

for some $\lambda \in \mathbb{R}\backslash\{0\}$. Hence

$$|\lambda| = |\lambda|\,|\pi_3(e_1 \wedge e_2)| = |\sigma\pi_3(\sigma \wedge \sigma)^{-1}(e_1 \wedge e_2)| = 1.$$

Since both $\bar{\varepsilon}$ and $\frac{1}{\lambda}\bar{\varepsilon} = \sigma\varepsilon\sigma^{-1}$ are positive definite, it follows that $\lambda = 1$. Moreover, $\sigma\pi_3(\sigma \wedge \sigma)^{-1} = \pi_3$ implies that $\sigma \in \mathrm{SO}(\mathbb{E}^3)$.

Conversely, if $\sigma \in \mathrm{SO}(\mathbb{E}^3)$ satisfies $\sigma\varepsilon\sigma^{-1} = \bar{\varepsilon}$, then $\sigma(\varepsilon\pi_3)(\sigma \wedge \sigma)^{-1} = (\sigma\varepsilon\sigma^{-1})(\sigma\pi_3(\sigma \wedge \sigma)^{-1}) = \bar{\varepsilon}\pi_3$, because π_3 is $\mathrm{SO}(\mathbb{E}^3)$-equivariant.[5] Hence $\sigma \in \mathrm{Mor}(\varepsilon\pi_3, \bar{\varepsilon}\pi_3)$. $\square$

Recall that a groupoid is a small category all of whose morphisms are isomorphisms. The object set $\mathrm{Pd}(\mathbb{E}^3)$ becomes a groupoid by viewing as morphisms $\sigma : \varepsilon \to \varepsilon'$ the

[5] The $\mathrm{SO}(\mathbb{E}^3)$-equivariance of π_3 follows easily from the fact that $\langle \pi_3(u \wedge v), w\rangle = \det(u|v|w)$ holds for all $(u, v, w) \in (\mathbb{E}^3)^3$.

special orthogonal endomorphisms $\sigma \in \mathrm{SO}(\mathbb{E}^3)$ which satisfy $\sigma\varepsilon\sigma^{-1} = \varepsilon'$. Then the functor $\mathscr{G} : \mathrm{Pd}(\mathbb{E}^3) \to \mathscr{E}_3$, given on objects by $\mathscr{G}(\varepsilon) = \varepsilon\pi_3$ and on morphisms by $\mathscr{G}(\sigma) = \sigma$, is well-defined, dense and full by Proposition 4.5, and faithful by definition.

Corollary 4.6. *The functor $\mathscr{G} : \mathrm{Pd}(\mathbb{E}^3) \to \mathscr{E}_3$ is an equivalence of categories.*

Setting $\mathscr{T} = \{d \in \mathbb{R}^3 \mid 0 < d_1 \leqslant d_2 \leqslant d_3\}$ we denote, for any $d \in \mathscr{T}$, by D_d the diagonal matrix in $\mathbb{R}^{3\times 3}$ with diagonal sequence d. We endow the object set $\mathbb{R}^3 \times \mathscr{T}$ with the structure of a groupoid by declaring as morphisms $S : (y, d) \to (y', d')$ the special orthogonal matrices $S \in \mathrm{SO}_3(\mathbb{R})$ satisfying $(Sy, SD_dS^{\mathrm{T}}) = (y', D_{d'})$.

Every $y \in \mathbb{R}^3$ determines an antisymmetric linear endomorphism $\mu_y = \pi_3(y \wedge ?)$ of $\mathbb{E}^3$. Every $d \in \mathscr{T}$ determines a symmetric positive definite linear endomorphism $\delta_d = \underline{D_d}$ of $\mathbb{E}^3$. Hence every pair $(y, d) \in \mathbb{R}^3 \times \mathscr{T}$ determines a positive definite linear endomorphism $\varepsilon_{yd} = \mu_y + \delta_d$ of $\mathbb{E}^3$. The functor $\mathscr{F} : \mathbb{R}^3 \times \mathscr{T} \to \mathrm{Pd}(\mathbb{E}^3)$, given on objects by $\mathscr{F}(y, d) = \varepsilon_{yd}$ and on morphisms by $\mathscr{F}(S) = \underline{S}$ is well-defined, again due the $\mathrm{SO}(\mathbb{E}^3)$-equivariance of π_3.

Corollary 4.7. *The functor $\mathscr{F} : \mathbb{R}^3 \times \mathscr{T} \to \mathrm{Pd}(\mathbb{E}^3)$ is an equivalence of groupoids.*

Proof. The functor $\mathscr{F}$ is dense by Jacobi's spectral theorem,[6] faithful by definition and full by $\mathrm{SO}(\mathbb{E}^3)$-equivariance of π_3 and uniqueness of the decomposition of a positive definite matrix into its antisymmetric and positive definite symmetric part. □

As an immediate consequence of Corollaries 4.6 and 4.7 we obtain the following description of the category $\mathscr{E}_3$ in terms of the groupoid $\mathbb{R}^3 \times \mathscr{T}$.

Theorem 4.8. *The composed functor $\mathscr{G}\mathscr{F} : \mathbb{R}^3 \times \mathscr{T} \to \mathscr{E}_3$ is an equivalence of categories.*

Due to the equivalences of categories

$$\mathbb{R}^3 \times \mathscr{T} \xrightarrow{\mathscr{F}} \mathrm{Pd}(\mathbb{E}^3) \xrightarrow{\mathscr{G}} \mathscr{E}_3$$

the problems of classifying $\mathscr{E}_3$, $\mathrm{Pd}(\mathbb{E}^3)$ and $\mathbb{R}^3 \times \mathscr{T}$ up to isomorphism are all equivalent. The latter one can be solved without effort by means of geometrical conception!

Indeed, let us interpret the groupoid $\mathbb{R}^3 \times \mathscr{T}$ geometrically by identifying its objects (y, d) with those configurations in $\mathbb{R}^3$ which are composed of a point y and an ellipsoid $E_d = \{z \in \mathbb{R}^3 \mid z^{\mathrm{T}} D_d z = 1\}$ in normal position. A morphism $(y, d) \to (y', d')$ in $\mathbb{R}^3 \times \mathscr{T}$ exists only if $d = d'$. Accordingly, identifying $\mathrm{SO}_3(\mathbb{R})$ with $\mathrm{SO}(\mathbb{E}^3)$, the morphisms $(y, d) \to (y', d')$ in $\mathbb{R}^3 \times \mathscr{T}$ are identified with those rotation symmetries of the ellipsoid $E_d = E_{d'}$ which send y to y'. Thus the problem of classifying $\mathbb{R}^3 \times \mathscr{T}$ up to isomorphism splits into the $\mathscr{T}$-family of normal form problems given by the natural group actions $G_d \times \mathbb{R}^3 \to \mathbb{R}^3$, $(S, y) \mapsto Sy$ where $G_d = \{S \in \mathrm{SO}_3(\mathbb{R}) \mid SD_dS^{\mathrm{T}} = D_d\}$ is the rotation symmetry group of the ellipsoid E_d,

[6] We choose to translate the German standard name "Hauptachsentransformation" for this theorem as "spectral theorem". We attribute it to Jacobi due to his article [14] in which he presented a constructive method for the numerical solution of the characteristic equation of a real symmetric matrix. See also [21, Vorwort and Section 13].

for any $d \in \mathscr{T}$. The solution of these normal form problems amounts to an easy exercise in elementary geometry. Leaving the verification of details to the interested reader, we content ourselves with presenting the results.

By $\mathscr{G}r$ we denote the category of groups. The image of the map $G_? : \mathscr{T} \to \mathscr{G}r,\ d \mapsto G_d$ consists of four subgroups of $\mathrm{SO}_3(\mathbb{R})$ only, namelynamely

$$G_1 = \mathrm{SO}_3(\mathbb{R}),$$

$$G_2 = \left\{ \left(\begin{array}{c|c} S & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0\ 0 & \det S \end{array} \right) \middle|\ S \in \mathrm{O}_2(\mathbb{R}) \right\},$$

$$G_3 = \left\{ \left(\begin{array}{c|c} \det S & 0\ 0 \\ \hline \begin{matrix} 0 \\ 0 \end{matrix} & S \end{array} \right) \middle|\ S \in \mathrm{O}_2(\mathbb{R}) \right\},$$

$$G_4 = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \right\}.$$

Their fibres $\mathscr{T}_i = \{d \in \mathscr{T} \mid G_d = G_i\}$ are

$$\mathscr{T}_1 = \{d \in \mathscr{T} \mid d_1 = d_2 = d_3\},$$
$$\mathscr{T}_2 = \{d \in \mathscr{T} \mid d_1 = d_2 < d_3\},$$
$$\mathscr{T}_3 = \{d \in \mathscr{T} \mid d_1 < d_2 = d_3\},$$
$$\mathscr{T}_4 = \{d \in \mathscr{T} \mid d_1 < d_2 < d_3\}.$$

Cross-sections $\mathscr{C}_i$ for the orbit sets $\mathbb{R}^3/G_i$ are given by

$$\mathscr{C}_1 = \mathbb{R}_{\geqslant 0} \times \{0\} \times \{0\},$$
$$\mathscr{C}_2 = \{0\} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0},$$
$$\mathscr{C}_3 = \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \times \{0\},$$
$$\mathscr{C}_4 = (\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}) \cup \{y \in (\mathbb{R}_{\geqslant 0})^3 \mid y_1 y_2 = 0\}.$$

This accomplishes the classification of $\mathbb{R}^3 \times \mathscr{T}$, along with the classifications of $\mathrm{Pd}(\mathbb{E}^3)$ and $\mathscr{E}_3$. Let us summarize the result.

Theorem 4.9. (i) *The set of configurations* $\mathscr{C} = \bigcup_{i=1}^4 (\mathscr{C}_i \times \mathscr{T}_i)$ *classifies* $\mathbb{R}^3 \times \mathscr{T}$.
(ii) *The set of positive definite endomorphisms* $\mathscr{F}(\mathscr{C}) = \{\varepsilon_{yd} \mid (y, d) \in \mathscr{C}\}$ *classifies* $\mathrm{Pd}(\mathbb{E}^3)$.
(iii) *The set of dissident maps* $\mathscr{G}\mathscr{F}(\mathscr{C}) = \{\varepsilon_{yd}\pi_3 \mid (y, d) \in \mathscr{C}\}$ *classifies* $\mathscr{E}_3$.

Let us now switch to $m = 7$. Here, a pattern of reasoning can be carried through which is reminiscent of the above one for $m = 3$, although not quite analogous. It yields a description of the category $\mathscr{E}_7^1$ in terms of the matrix triple category $\mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{pds}} \times \mathbb{R}^{7\times 7}_{\mathrm{spds}}$ to be defined below. We start with the following fundamental lemma.

Lemma 4.10. *For all* $(\varepsilon, \delta), (\overline{\varepsilon}, \overline{\delta}) \in \mathrm{Pd}(\mathbb{E}^7) \times \mathrm{Pds}(\mathbb{E}^7)$, *the set of all morphisms* $\sigma : \varepsilon(\pi_7 \cdot \delta) \to \overline{\varepsilon}(\pi_7 \cdot \overline{\delta})$ *in* $\mathscr{E}$ *admits the description*

$$\begin{aligned}\mathrm{Mor}(\varepsilon(\pi_7\cdot\delta),\overline{\varepsilon}(\pi_7\cdot\overline{\delta})) &= \{\sigma\in\mathbb{G}_2 | (\lambda\sigma\varepsilon\sigma^{-1},\lambda^{-1/3}\sigma\delta\sigma^{-1})\\ &= (\overline{\varepsilon},\overline{\delta})\ \textit{for some}\ \lambda\in\mathbb{R}_{>0}\}.\end{aligned}$$

Proof. If $\sigma\in\mathrm{Mor}(\varepsilon(\pi_7\cdot\delta),\overline{\varepsilon}(\pi_7\cdot\overline{\delta}))$, then Theorem 4.2(ii) and Corollary 4.4(iv) imply that $\sigma\in\mathrm{O}(\mathbb{E}^7)$ such that

$$(\lambda\sigma\varepsilon\sigma^{-1},\lambda^{-1}\sigma(\pi_7\cdot\delta)(\sigma\wedge\sigma)^{-1}) = (\overline{\varepsilon},\pi_7\cdot\overline{\delta})$$

for some $\lambda\in\mathbb{R}\backslash\{0\}$. Since both $\overline{\varepsilon}$ and $\lambda^{-1}\overline{\varepsilon}=\sigma\varepsilon\sigma^{-1}$ are positive definite, it follows that $\lambda>0$. Equality of the second components in the above identity can be expressed in the form $(\pi_7\cdot\delta)\cdot\sigma^{-1}=\lambda(\pi_7\cdot\overline{\delta})=\pi_7\cdot(\lambda^{1/3}\overline{\delta})$ which means that $\sigma\in\mathrm{Mor}(\pi_7\cdot\delta,\pi_7\cdot(\lambda^{\frac{1}{3}}\overline{\delta}))$. We conclude with Theorem 4.2(iii) that $\sigma\in\mathbb{G}_2$ and $\sigma\delta\sigma^{-1}=\lambda^{\frac{1}{3}}\ \overline{\delta}$.

Conversely, if $\sigma\in\mathbb{G}_2$ satisfies $(\lambda\sigma\varepsilon\sigma^{-1},\lambda^{-\frac{1}{3}}\sigma\delta\sigma^{-1})=(\overline{\varepsilon},\overline{\delta})$ for some $\lambda>0$, then we conclude with Theorem 4.2(iii) that

$$\sigma\in\mathrm{Mor}(\pi_7\cdot(\lambda^{-\frac{1}{3}}\delta),\pi_7\cdot\overline{\delta}) = \mathrm{Mor}(\lambda^{-1}(\pi_7\cdot\delta),\pi_7\cdot\overline{\delta}).$$

Hence $\lambda^{-1}\ \sigma(\pi_7\cdot\delta)(\sigma\wedge\sigma)^{-1}=\pi_7\cdot\overline{\delta}$. Now Corollary 4.4(iv) implies that $\sigma\in\mathrm{Mor}(\varepsilon(\pi_7\cdot\delta),\overline{\varepsilon}(\pi_7\cdot\overline{\delta}))$. $\square$

In order to eliminate the troublesome factor λ appearing in Lemma 4.10, we restrict $\mathrm{Pds}(\mathbb{E}^7)$ to the subset $\mathrm{Spds}(\mathbb{E}^7)=\mathrm{SL}(\mathbb{E}^7)\cap\mathrm{Pds}(\mathbb{E}^7)$. Viewing $\mathrm{Pd}(\mathbb{E}^7)\times\mathrm{Spds}(\mathbb{E}^7)$ as the object set of a groupoid whose morphisms $\sigma:(\varepsilon,\delta)\to(\overline{\varepsilon},\overline{\delta})$ are the orthogonal endomorphisms $\sigma\in\mathbb{G}_2$ which satisfy $(\sigma\varepsilon\sigma^{-1},\sigma\delta\sigma^{-1})=(\overline{\varepsilon},\overline{\delta})$, we easily obtain the following result.

Proposition 4.11. (i) *The family* $\{\varepsilon(\pi_7\cdot\delta)\mid(\varepsilon,\delta)\in\mathrm{Pd}(\mathbb{E}^7)\times\mathrm{Spds}(\mathbb{E}^7)\}$ *exhausts* $\mathrm{Ob}(\mathscr{E}_7^1)$.

(ii) *For all* $(\varepsilon,\delta),(\overline{\varepsilon},\overline{\delta})\in\mathrm{Pd}(\mathbb{E}^7)\times\mathrm{Spds}(\mathbb{E}^7)$, *the set of all morphisms* $\sigma:\varepsilon(\pi_7\cdot\delta)\to\overline{\varepsilon}(\pi_7\cdot\overline{\delta})$ *in* $\mathscr{E}$ *admits the description*

$$\mathrm{Mor}(\varepsilon(\pi_7\cdot\delta),\overline{\varepsilon}(\pi_7\cdot\overline{\delta})) = \{\sigma\in\mathbb{G}_2\mid(\sigma\varepsilon\sigma^{-1},\sigma\delta\sigma^{-1})=(\overline{\varepsilon},\overline{\delta})\}.$$

(iii) *The functor* $\mathscr{G}:\mathrm{Pd}(\mathbb{E}^7)\times\mathrm{Spds}(\mathbb{E}^7)\to\mathscr{E}_7^1$, *given on objects by* $\mathscr{G}(\varepsilon,\delta)=\varepsilon(\pi_7\cdot\delta)$ *and on morphisms by* $\mathscr{G}(\sigma)=\sigma$, *is an equivalence of categories.*

Proof. (i) If $(\varepsilon,\delta)\in\mathrm{Pd}(\mathbb{E}^7)\times\mathrm{Spds}(\mathbb{E}^7)$, then $\varepsilon(\pi_7\cdot\delta)\in\mathscr{E}_7^1$ by Theorem 4.2(ii) and Corollary 4.3(i). Conversely, let $\upsilon\in\mathscr{E}_7^1$ be given. Arguing as in the proof of Proposition 4.5(i), however with application of Theorem 4.2(ii) instead of Theorem 4.2(i), one finds a pair $(\overline{\varepsilon},\overline{\delta})\in\mathrm{Pd}(\mathbb{E}^7)\times\mathrm{Pds}(\mathbb{E}^7)$ such that $\overline{\varepsilon}(\pi_7\cdot\overline{\delta})\tilde{\to}\upsilon$. Set $\lambda=(\det\overline{\delta})^{-\frac{3}{7}}$. Then $(\varepsilon,\delta)=(\lambda\overline{\varepsilon},\lambda^{-\frac{1}{3}}\overline{\delta})\in\mathrm{Pd}(\mathbb{E}^7)\times\mathrm{Spds}(\mathbb{E}^7)$such that $\varepsilon(\pi_7\cdot\delta)=\overline{\varepsilon}(\pi_7\cdot\overline{\delta})\tilde{\to}\upsilon$.

(ii) follows directly from Lemma 4.10, together with $\det\delta=1=\det\overline{\delta}$.

(iii) The functor $\mathscr{G}$ is well-defined on objects and dense by (i), well-defined on morphisms and full by (ii), and faithful by definition. $\square$

Passing to the level of matrices, we restrict likewise $\mathbb{R}^{7\times7}_{\mathrm{pds}}$ to the subset $\mathbb{R}^{7\times7}_{\mathrm{spds}}=\mathrm{SL}_7(\mathbb{R})\cap\mathbb{R}^{7\times7}_{\mathrm{pds}}$. We view $\mathbb{R}^{7\times7}_{\mathrm{ant}}\times\mathbb{R}^{7\times7}_{\mathrm{pds}}\times\mathbb{R}^{7\times7}_{\mathrm{spds}}$ as the object set of a groupoid whose morphisms S :

$(B, C, D) \to (\overline{B}, \overline{C}, \overline{D})$ are the orthogonal matrices $S \in \mathbb{G}_2$ which satisfy $(SBS\mathrm{T}, SCS\mathrm{T}, SDS\mathrm{T}) = (\overline{B}, \overline{C}, \overline{D})$.

Proposition 4.12. *The functor $\mathscr{F} : \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}} \to \mathrm{Pd}(\mathbb{E}^7) \times \mathrm{Spds}(\mathbb{E}^7)$, given on objects by $\mathscr{F}(B, C, D) = (\underline{B} + \underline{C}, \underline{D})$ and on morphisms by $\mathscr{F}(S) = \underline{S}$, is an equivalence of categories.*

Proof. The functor $\mathscr{F}$ is dense and full by the unique decomposition of a positive definite endomorphism into its antisymmetric and positive definite symmetric part, and faithful by construction. □

Now the announced matricial description of the category $\mathscr{E}^1_7$ is an immediate consequence of Propositions 4.11(iii) and 4.12.

Theorem 4.13. *The composed functor $\mathscr{G}\mathscr{F} : \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}} \to \mathscr{E}^1_7$ is an equivalence of categories.*

It seems worth while to restate in explicit terms some of the information encoded in Theorem 4.13.

Corollary 4.14. (i) *For each matrix triple $(B, C, D) \in \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}}$, the linear map $\mathscr{G}\mathscr{F}(B, C, D) : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7,\ v\wedge w \mapsto (B+C)D\pi_7(Dv \wedge Dw)$, is a 7-dimensional dissident map of degree one.*

(ii) *For each 7-dimensional dissident map η of degree one there exists a matrix triple $(B, C, D) \in \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}}$ such that $\mathscr{G}\mathscr{F}(B, C, D) \tilde{\to} \eta$.*

(iii) *For all matrix triples $(B, C, D), (\overline{B}, \overline{C}, \overline{D}) \in \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}}$, the dissident maps $\mathscr{G}\mathscr{F}(B, C, D)$ and $\mathscr{G}\mathscr{F}(\overline{B}, \overline{C}, \overline{D})$ are isomorphic if and only if there exists an orthogonal matrix $S \in \mathbb{G}_2$ such that*

$$(SBS^{\mathrm{T}}, SCS^{\mathrm{T}}, SDS^{\mathrm{T}}) = (\overline{B}, \overline{C}, \overline{D}).$$

Moreover, the classification problems for the three categories involved in the sequence of equivalences

$$\mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}} \xrightarrow{\mathscr{F}} \mathrm{Pd}(\mathbb{E}^7) \times \mathrm{Spds}(\mathbb{E}^7) \xrightarrow{\mathscr{G}} \mathscr{E}^1_7$$

are all equivalent.

There seems to be no easy way to find an explicit cross-section for $\mathrm{Ob}(\mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}})/\tilde{\to}$. Yet there do exist cumbersome strategies which eventually may succeed. Interesting first steps in this direction are to be found in [4, Section 7].

5. Real quadratic division algebras of degree one

Let us briefly recall some basic notions from real algebra. A real *algebra* is a real vector space A, endowed with an $\mathbb{R}$-bilinear multiplication $A \times A \to A, (x, y) \mapsto xy$. By a real *division algebra* we mean a real algebra satisfying $0 < \dim A < \infty$ and having no zero divisors (i.e. $xy = 0$ only if $x = 0$ or $y = 0$). By a real *quadratic algebra* we mean a real algebra A such that $0 < \dim A < \infty$, there exists an identity element $1 \in A$ and $1, x, x^2$ are linearly dependent for each $x \in A$. Morphisms between quadratic algebras are algebra morphisms respecting the identity elements. A *dissident triple* (V, ξ, η) consists of a Euclidean space V, a linear form $\xi : V \wedge V \to \mathbb{R}$ and a dissident map $\eta : V \wedge V \to V$. A morphism $\sigma : (V, \xi, \eta) \to (\overline{V}, \overline{\xi}, \overline{\eta})$ between dissident triples is an orthogonal map $\sigma : V \to \overline{V}$ satisfying both $\xi = \overline{\xi}(\sigma \wedge \sigma)$ and $\sigma\eta = \overline{\eta}(\sigma \wedge \sigma)$.

Now we switch our investigation from the category $\mathscr{E}$ of all dissident maps to the related category $\mathscr{Q}$ of all real quadratic division algebras. The crucial link between these categories is provided by the category $\mathscr{D}$ of all dissident triples. Indeed, each dissident triple (V, ξ, η) determines a real quadratic division algebra $\mathscr{H}(V, \xi, \eta) = \mathbb{R} \times V$, with multiplication given by

$$\begin{pmatrix} \alpha \\ v \end{pmatrix} \begin{pmatrix} \beta \\ w \end{pmatrix} = \begin{pmatrix} \alpha\beta - \langle v, w \rangle + \xi(v \wedge w) \\ \alpha w + \beta v + \eta(v \wedge w) \end{pmatrix}.$$

The assignments $(V, \xi, \eta) \mapsto \mathscr{H}(V, \xi, \eta)$ and $\sigma \mapsto \mathbb{I}_{\mathbb{R}} \times \sigma$ establish a functor $\mathscr{H} : \mathscr{D} \to \mathscr{Q}$ which, by Osborn's theorem [19, p. 204], is an equivalence of categories. It induces equivalences of full subcategories $\mathscr{H} : \mathscr{D}_{n-1} \to \mathscr{Q}_n$, for all $n \in \mathbb{N} \backslash \{0\}$.

We aspire to classify $\mathscr{Q}$. The (1,2,4,8)-theorem implies that the problem of classifying $\mathscr{Q}$ is equivalent to the problem of classifying $\mathscr{Q}_n$ for all $n \in \{1, 2, 4, 8\}$. In view of the equivalences of categories $\mathscr{H} : \mathscr{D}_{n-1} \to \mathscr{Q}_n$ the latter problem is equivalent to the problem of classifying $\mathscr{D}_m$ for all $m \in \{0, 1, 3, 7\}$. For trivial reasons, $\{(\mathbb{E}^0, o, o)\}$ classifies $\mathscr{D}_0$ and $\{(\mathbb{E}^1, o, o)\}$ classifies $\mathscr{D}_1$. Since $\mathscr{H}(\mathbb{E}^0, o, o) \tilde{\to} \mathbb{R}$ and $\mathscr{H}(\mathbb{E}^1, o, o) \tilde{\to} \mathbb{C}$, we conclude that $\{\mathbb{R}\}$ classifies $\mathscr{Q}_1$ and $\{\mathbb{C}\}$ classifies $\mathscr{Q}_2$. Thus the problem of classifying $\mathscr{Q}$ is reduced to the problem of classifying $\mathscr{D}_3$ and $\mathscr{D}_7$.

We define the *degree* of a dissident triple $(V, \xi, \eta) \in \mathscr{D}_3 \cup \mathscr{D}_7$ by $\deg(V, \xi, \eta) = \deg \eta$. Theorem 2.4(iii) implies that $\deg(V, \xi, \eta) = 1$ for all $(V, \xi, \eta) \in \mathscr{D}_3$ and $1 \leqslant \deg(V, \xi, \eta) \leqslant 5$ for all $(V, \xi, \eta) \in \mathscr{D}_7$. For all $d \in \underline{5}$ we denote by $\mathscr{D}_7^d$ the full subcategory of $\mathscr{D}_7$ formed by $\{(V, \xi, \eta) \mid \deg \eta = d\}$. Similarly we define the *degree* of a real quadratic division algebra $A \in \mathscr{Q}_4 \cup \mathscr{Q}_8$ by $\deg A = \deg(V, \xi, \eta)$ for any $(V, \xi, \eta) \in \mathscr{D}_3 \cup \mathscr{D}_7$ such that $\mathscr{H}(V, \xi, \eta) \tilde{\to} A$. Again we have that $\deg A = 1$ for all $A \in \mathscr{Q}_4$ and $1 \leqslant \deg A \leqslant 5$ for all $A \in \mathscr{Q}_8$. For all $d \in \underline{5}$ we denote by $\mathscr{Q}_8^d$ the full subcategory of $\mathscr{Q}_8$ formed by $\{A \mid \deg A = d\}$. The equivalence of categories $\mathscr{H} : \mathscr{D}_7 \to \mathscr{Q}_8$ induces equivalences of full subcategories $\mathscr{H} : \mathscr{D}_7^d \to \mathscr{Q}_8^d$, for all $d \in \underline{5}$.

In the present section we extend the classification of $\mathscr{E}_3$ (Theorem 4.9(iii)) to a classification of $\mathscr{Q}_4$ (Theorem 5.2(iv)), and the complete description of $\mathscr{E}_7^1$ (Theorem 4.13) to a complete description of $\mathscr{Q}_8^1$ (Theorem 5.3). This is achieved by adapting the pattern of reasoning which in Section 4 was designed for $\mathscr{E}_3$ and $\mathscr{E}_7^1$ to the enlarged context of $\mathscr{D}_3$ and $\mathscr{D}_7^1$, respectively. In particular we exhibit new functors $\mathscr{G}$ and $\mathscr{F}$ extending the equivalences

$\mathscr{G}$ and $\mathscr{F}$ from Corollaries 4.6 and 4.7 and Propositions 4.11(iii) and 4.12 respectively. We skip however to verify that these new functors again are dense, full and faithful, because all required arguments partly are quotations from Section 4, partly routine.

Let us begin with the equivalence of categories $\mathscr{H} : \mathscr{D}_3 \to \mathscr{Q}_4$. We view $\mathrm{Ant}(\mathbb{E}^3)\times\mathrm{Pd}(\mathbb{E}^3)$ as the object set of a groupoid whose morphisms $\sigma : (\mu, \varepsilon) \to (\overline{\mu}, \overline{\varepsilon})$ are the special orthogonal endomorphisms $\sigma \in \mathrm{SO}(\mathbb{E}^3)$ which satisfy $(\sigma\mu\sigma^{-1}, \sigma\varepsilon\sigma^{-1}) = (\overline{\mu}, \overline{\varepsilon})$. Then the functor

$$\mathscr{G} : \mathrm{Ant}(\mathbb{E}^3) \times \mathrm{Pd}(\mathbb{E}^3) \to \mathscr{D}_3,$$

given on objects by $\mathscr{G}(\mu, \varepsilon) = (\mathbb{E}^3, \xi_\mu, \varepsilon\pi_3)$ where $\xi_\mu(v \wedge w) = \langle v, \mu(w)\rangle$, and on morphisms by $\mathscr{G}(\sigma) = \sigma$, is an equivalence of categories.[7] Moreover we view $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$ as the object set of a groupoid whose morphisms $S : (x, y, d) \to (\overline{x}, \overline{y}, \overline{d})$ are the special orthogonal matrices $S \in SO_3(\mathbb{R})$ which satisfy $(Sx, Sy, SD_dS^{\mathrm{T}}) = (\overline{x}, \overline{y}, D_{\overline{d}})$. Then the functor

$$\mathscr{F} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T} \to \mathrm{Ant}(\mathbb{E}^3) \times \mathrm{Pd}(\mathbb{E}^3),$$

given on objects by[8] $\mathscr{F}(x, y, d) = (\mu_x, \varepsilon_{yd})$ and on morphisms by $\mathscr{F}(S) = \underline{S}$, is an equivalence of groupoids. Altogether we have reached the following description of the category $\mathscr{Q}_4$ in terms of the groupoid $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$.

Theorem 5.1. *The composed functor* $\mathscr{H}\mathscr{G}\mathscr{F} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T} \to \mathscr{Q}_4$ *is an equivalence of categories*.

The sequence of equivalences

$$\mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T} \xrightarrow{\mathscr{F}} \mathrm{Ant}(\mathbb{E}^3) \times \mathrm{Pd}(\mathbb{E}^3) \xrightarrow{\mathscr{G}} \mathscr{D}_3 \xrightarrow{\mathscr{H}} \mathscr{Q}_4$$

together with the cross-section $\mathscr{C}$ for $\mathrm{Ob}(\mathbb{R}^3\times\mathbb{R}^3\times\mathscr{T})/\tilde{\to}$, obtained by elementary geometric considerations analogous to those explained in Section 4 and displayed in [8, Proposition 4.3(i)], yields classifications of all four involved categories at once.

Theorem 5.2. (i) *The set of configurations* $\mathscr{C}$ *classifies* $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$.

(ii) *The set of pairs of endomorphisms* $\mathscr{F}(\mathscr{C}) = \{(\mu_x, \varepsilon_{yd}) \mid (x, y, d) \in \mathscr{C}\}$ *classifies* $\mathrm{Ant}(\mathbb{E}^3) \times \mathrm{Pd}(\mathbb{E}^3)$.

(iii) *The set of dissident triples* $\mathscr{G}\mathscr{F}(\mathscr{C}) = \{(\mathbb{E}^3, \xi_{\mu_x}, \varepsilon_{yd}\pi_3) \mid (x, y, d) \in \mathscr{C}\}$ *classifies* $\mathscr{D}_3$.

(iv) *The set of real quadratic division algebras* $\mathscr{H}\mathscr{G}\mathscr{F}(\mathscr{C}) =$

$$\left\{ \mathbb{R} \times \mathbb{R}^3, \begin{pmatrix} \alpha \\ v \end{pmatrix} \begin{pmatrix} \beta \\ w \end{pmatrix} = \begin{pmatrix} \alpha\beta - v^{\mathrm{T}}w + \det(v \mid x \mid w) \\ \alpha w + \beta v + \varepsilon_{yd}\pi_3(v \wedge w) \end{pmatrix} \middle| (x, y, d) \in \mathscr{C} \right\}$$

classifies $\mathscr{Q}_4$.

[7] To prove that $\mathscr{G}$ is full, apply Proposition 4.5(ii).

[8] See paragraph preceding Corollary 4.7 for definition of μ_x and ε_{yd}.

Let us resume with the equivalence $\mathscr{H} : \mathscr{D}_7^1 \to \mathscr{Q}_8^1$. We view

$$\mathrm{Ant}(\mathbb{E}^7) \times \mathrm{Pd}(\mathbb{E}^7) \times \mathrm{Spds}(\mathbb{E}^7)$$

as the object set of a groupoid whose morphisms

$$\sigma : (\mu, \varepsilon, \delta) \to (\overline{\mu}, \overline{\varepsilon}, \overline{\delta})$$

are the orthogonal endomorphisms $\sigma \in \mathbb{G}_2$ which satisfy

$$(\sigma\mu\sigma^{-1}, \sigma\varepsilon\sigma^{-1}, \sigma\delta\sigma^{-1}) = (\overline{\mu}, \overline{\varepsilon}, \overline{\delta}).$$

Then the functor

$$\mathscr{G} : \mathrm{Ant}(\mathbb{E}^7) \times \mathrm{Pd}(\mathbb{E}^7) \times \mathrm{Spds}(\mathbb{E}^7) \to \mathscr{D}_7^1,$$

given on objects by $\mathscr{G}(\mu, \varepsilon, \delta) = (\mathbb{E}^7, \xi_\mu, \varepsilon(\pi_7 \cdot \delta))$, where $\xi_\mu(v \wedge w) = \langle v, \mu(w)\rangle$, and on morphisms by $\mathscr{G}(\sigma) = \sigma$, is an equivalence of categories. Moreover we view

$$\mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}}$$

as the object set of a groupoid whose morphisms

$$S : (A, B, C, D) \to (\overline{A}, \overline{B}, \overline{C}, \overline{D})$$

are the orthogonal matrices $S \in \mathbb{G}_2$ which satisfy

$$(SAS^{\mathrm{T}}, SBS^{\mathrm{T}}, SCS^{\mathrm{T}}, SDS^{\mathrm{T}}) = (\overline{A}, \overline{B}, \overline{C}, \overline{D}).$$

Then the functor

$$\mathscr{F} : \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}} \to \mathrm{Ant}(\mathbb{E}^7) \times \mathrm{Pd}(\mathbb{E}^7) \times \mathrm{Spds}(\mathbb{E}^7),$$

given on objects by $\mathscr{F}(A, B, C, D) = (\underline{A}, \underline{B} + \underline{C}, \underline{D})$ and on morphisms by $\mathscr{F}(S) = \underline{S}$, is an equivalence of groupoids. Composing these equivalences $\mathscr{H}, \mathscr{G}$ and $\mathscr{F}$ we obtain the following matricial description of the category $\mathscr{Q}_8^1$.

Theorem 5.3. *The composed functor*

$$\mathscr{H}\mathscr{G}\mathscr{F} : \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}} \to \mathscr{Q}_8^1$$

is an equivalence of categories.

Let us restate explicitly three items which are implicit in Theorem 5.3.

Corollary 5.4. (i) *For each matrix quadruple* $(A, B, C, D) \in \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}}$, *the algebra* $\mathscr{H}\mathscr{G}\mathscr{F}(A, B, C, D) = \mathbb{R} \times \mathbb{R}^7$, *with multiplication*

$$\begin{pmatrix} \alpha \\ v \end{pmatrix} \begin{pmatrix} \beta \\ w \end{pmatrix} = \begin{pmatrix} \alpha\beta - v^{\mathrm{T}}w + v^{\mathrm{T}}Aw \\ \alpha w + \beta v + (B + C)D\pi_7(Dv \wedge Dw) \end{pmatrix},$$

is an eight-dimensional real quadratic division algebra of degree one.

(ii) *For each eight-dimensional real quadratic division algebra Q of degree one there exists a matrix quadruple* $(A, B, C, D) \in \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}}$ *such that* $\mathscr{H}\mathscr{G}\mathscr{F}(A, B, C, D) \tilde{\to} Q$.

(iii) *For all matrix quadruples* $(A, B, C, D), (\overline{A}, \overline{B}, \overline{C}, \overline{D}) \in \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}}$, *the algebras* $\mathscr{H}\mathscr{G}\mathscr{F}(A, B, C, D)$ *and* $\mathscr{H}\mathscr{G}\mathscr{F}(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ *are isomorphic if and only if there exists an orthogonal matrix* $S \in \mathbb{G}_2$ *such that*

$$(SAS^{\mathrm{T}}, SBS^{\mathrm{T}}, SCS^{\mathrm{T}}, SDS^{\mathrm{T}}) = (\overline{A}, \overline{B}, \overline{C}, \overline{D}).$$

Moreover, the classification problems for the four categories involved in the sequence of equivalences $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are all equivalent. Regarding the prospects of finding an explicit cross-section for $\mathrm{Ob}(\mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{ant}} \times \mathbb{R}^{7\times7}_{\mathrm{pds}} \times \mathbb{R}^{7\times7}_{\mathrm{spds}})/\tilde{\to}$, the comment we added to Corollary 4.14 applies even stronger to the present situation.

6. Real quadratic division algebras of higher degree

Recall that the *double* of a real quadratic algebra A is defined by $\mathscr{V}(A) = A \times A$ with multiplication $(w, x)(y, z) = (wy - \overline{z}x, x\overline{y} + zw)$, where $\overline{y}, \overline{z}$ denote the conjugates of y, z.[9] This doubling construction is easily seen to provide an endofunctor $\mathscr{V}$ on the category of all real quadratic algebras, acting on morphisms by $\mathscr{V}(\varphi) = \varphi \times \varphi$. The endofunctor $\mathscr{V}$ in turn induces, due to [5, p. 946], a functor $\mathscr{V} : \mathscr{Q}_4 \to \mathscr{Q}_8$. Recall that the category $\mathscr{Q}_4$ is fully understood, in view of the equivalence $\mathscr{H}\mathscr{G}\mathscr{F} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T} \to \mathscr{Q}_4$ (cf. Theorems 5.1 and 5.2). Thus the context

$$\begin{array}{ccc} & & \mathscr{Q}_8 \\ & \mathscr{H}\mathscr{G}\mathscr{F} & \uparrow \quad \mathscr{V} \\ \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T} & \longrightarrow & \mathscr{Q}_4 \end{array}$$

provides a source for the construction of 8-dimensional real quadratic division algebras which appear to be of interest. We simplify notation on setting $A(\kappa) = \mathscr{H}\mathscr{G}\mathscr{F}(\kappa)$ and $B(\kappa) = \mathscr{V}(A(\kappa))$ for all $\kappa \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$, and $1_3 = (1\ 1\ 1)^{\mathrm{T}}$.

Theorem 6.1. *For every* $\kappa \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$, *the following assertions are equivalent*:

(i) $\kappa = (0, 0, \lambda 1_3)$ *for some* $\lambda > 0$.
(ii) $A(\kappa)$ *is flexible*.
(iii) $B(\kappa)$ *is flexible*.
(iv) $\deg B(\kappa) = 1$.

Proof. For a proof of (i) ⇔ (ii) see [4, Proposition 6.1]. For a proof of (i) ⇔ (iii) see [7, Proposition 4.5]. Given $\kappa \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$, let $(\mathbb{E}^7, \xi(\kappa), \eta(\kappa))$ be a dissident triple such that $\mathscr{H}(\mathbb{E}^7, \xi(\kappa), \eta(\kappa)) \tilde{\to} B(\kappa)$. Then $\deg B(\kappa) = \deg \eta(\kappa) = \deg \Phi(\kappa)$, where $\Phi(\kappa)$ is a

[9] Frobenius's Lemma [9,16] asserts that each vector y in a real quadratic algebra A decomposes uniquely according to $y = \alpha 1 + v$, where $\alpha \in \mathbb{R}$ and $v \in A$ is purely imaginary, i.e. $v^2 \in \mathbb{R}1$ but $v \notin \mathbb{R}1 \setminus \{0\}$. The *conjugate* of y is defined as $\overline{y} = \alpha 1 - v$.

lifting of $\eta(\kappa)_{\mathbb{P}}$. By Theorem 2.4(iii), $\deg \Phi(\kappa) = 1$ if and only if $\eta(\kappa)_{\mathbb{P}}$ is collinear. By [7, Proposition 4.5], $\eta(\kappa)_{\mathbb{P}}$ is collinear if and only if $\kappa = (o, o, \lambda 1_3)$ for some $\lambda > 0$. Thus (i) $\Leftrightarrow$ (iv) holds true. □

In analogy to the geometrical interpretation of $\mathbb{R}^3 \times \mathscr{T}$ preceding Theorem 4.9 we likewise interpret the category $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$ geometrically by identifying its objects (x, y, d) with those configurations in $\mathbb{R}^3$ which are composed of a pair of points (x, y) and an ellipsoid $E_d = \{z \in \mathbb{R}^3 \mid z^{\mathrm{T}} D_d z = 1\}$ in normal position. The morphisms $(x, y, d) \to (x', y', d')$ are identified with those rotation symmetries of $E_d = E_{d'}$ which simultaneously send x to x' and y to y'. In these terms, Theorem 6.1 has the following immediate consequence.

Corollary 6.2. *If a configuration $\kappa \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$ is not a sphere with a double point in the origin, then the degree of the eight-dimensional real quadratic division algebra $B(\kappa)$ is greater than* 1.

Thus the composed functor $\mathscr{V}\mathscr{H}\mathscr{G}\mathscr{F} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T} \to \mathscr{Q}_8,\ \kappa \mapsto B(\kappa)$ yields a wealth of real quadratic division algebras of higher degree. Simultaneously this is, to our knowledge, the only to date known construction of division algebras of that type.

One may wonder which values the map $\deg B(?) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T} \to \underline{5}, \kappa \mapsto \deg B(\kappa)$ actually attains. Given $\kappa \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$, let $\eta(\kappa) \in \mathscr{E}(\mathbb{E}^7)$ be the dissident map appearing in the proof of Theorem 6.1. In order to determine the degree of a lifting $\Phi(\kappa)$ of $\eta(\kappa)_{\mathbb{P}}$ and hence $\deg B(\kappa) = \deg \Phi(\kappa)$, it suffices in view of Theorem 2.4(v) to calculate $\Phi(\kappa)$ as a quasi-lifting of $\eta(\kappa)_{\mathbb{P}}$ outside the hyperplane H with basis $\mathbf{h} = (e_2, \ldots, e_7)$. In accordance with Proposition 2.3 we have that

$$\Phi(\kappa) = \left(\frac{\varphi_1^{\mathbf{h}}(\kappa)}{\varphi^{\mathbf{h}}(\kappa)}, \ldots, \frac{\varphi_7^{\mathbf{h}}(\kappa)}{\varphi^{\mathbf{h}}(\kappa)} \right),$$

where the real homogeneous polynomials $\varphi_i^{\mathbf{h}}(\kappa)$ and $\varphi^{\mathbf{h}}(\kappa)$ in $\mathbb{R}[X_1, \ldots, X_7]$ are given by

$$\varphi_i^{\mathbf{h}}(\kappa) = \det(e_i \mid \eta(\kappa)(? \wedge e_2) \mid \ldots \mid \eta(\kappa)(? \wedge e_7))$$

for all $i \in \underline{7}$, and $(\varphi^{\mathbf{h}}(\kappa)) = \gcd(\varphi_1^{\mathbf{h}}(\kappa), \ldots, \varphi_7^{\mathbf{h}}(\kappa))$, respectively.

In these terms we calculated explicitly the lifting $\Phi(\kappa)$ of $\eta(\kappa)_{\mathbb{P}}$ for a general configuration $\kappa \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$, using maple 9.5. The curious reader may verify these calculations by using the work sheet found under the web address http://www.math.uu.se/~lars/liftings.

Reading off the degree of $\Phi(\kappa)$ one obtains the following refinement of Corollary 6.2.

Proposition 6.3. *Let* $\kappa = (x, y, d) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{T}$.

(i) *If* $y = 0$ *and* $(x, d) = (0, d_1 1_3)$, *then* $\deg B(\kappa) = 1$.
(ii) *If* $y = 0$ *and* $(x, d) \neq (0, d_1 1_3)$, *then* $\deg B(\kappa) = 3$.
(iii) *If* $y \neq 0$, *then* $\deg B(\kappa) = 5$.

This material motivates the conjecture that the degree of a real quadratic division algebra always is odd. An equivalent formulation is the following.

Conjecture 6.4. *If* $B \in \mathcal{Q}_8$, *then* $\deg B \in \{1, 3, 5\}$.

Being functorial, the construction $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{T} \to \mathcal{Q}_8$, $\kappa \mapsto B(\kappa)$ induces a mapping $\mathrm{Ob}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{T})/\tilde{\to} \;\to\; \mathrm{Ob}(\mathcal{Q}_8)/\tilde{\to}$, $[\kappa] \mapsto [B(\kappa)]$ between the sets of isoclasses of the involved categories. The functor $\kappa \mapsto B(\kappa)$ is faithful by construction, but not full. (If e.g. $\kappa = (0, 0, 1_3)$, then $\mathrm{Aut}(\kappa) = \mathrm{SO}_3(\mathbb{R})$ is a real Lie group of dimension 3, while $\mathrm{Aut}(B(\kappa)) = \mathrm{Aut}(\mathbb{O}) = \mathbb{G}_2$ is a real Lie group of dimension 14.) Nevertheless, we conjecture that the induced mapping $[\kappa] \mapsto [B(\kappa)]$ is injective. Equivalently this may be formulated as follows.

Conjecture 6.5. *If* $\kappa, \kappa' \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{T}$ *satisfy* $B(\kappa) \tilde{\to} B(\kappa')$, *then* $\kappa \tilde{\to} \kappa'$.

A partial proof of Conjecture 6.5 which, among other arguments, also makes use of Proposition 6.3, can be found in [18, Section 7].

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