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Three-generation neutrino oscillations in curved spacetime

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Abstract

Three-generation MSW effect in curved spacetime is studied and a brief discussion on the gravitational correction to the neutrino self-energy is given. The modified mixing parameters and corresponding conversion probabilities of neutrinos after traveling through celestial objects of constant densities are obtained. The method to distinguish between the normal hierarchy and inverted hierarchy is discussed in this framework. Due to the gravitational redshift of energy, in some extreme situations, the resonance energy of neutrinos might be shifted noticeably and the gravitational effect on the self-energy of neutrino becomes significant at the vicinities of spacetime singularities.

1. Introduction

The theory about neutrino has so far been well-established and oscillations among three generations are experimentally observed as the mixing angles are measured with rather high accuracy based on the Earth experiments with long and short baselines, even though the CP violating phase is not well determined yet. In terms of the measured values the solar and atmospheric neutrino phenomena are perfectly explained. Now people turn their attention to the cosmic neutrino which, as is well known, serves as a unique messenger to provide us with valuable information of properties of super-large celestial objects, explosions of supernovae and even the structure of the earlier

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universe which was not optically transparent. This topic attracts much attention from theorists and experimentalists of particle physics. On the long way from the production source to the earth detectors, the neutrino beam would pass many star clusters and galaxies (including the source of production) whose large mass would curve the spacetime of their vicinities. As it traverses the massive region, it undergoes the Mikheyev–Smirnov–Wolfenstein (MSW) effect [1] which determines its evolution. It is well understood how the electron neutrino produced in the Sun converses to other flavors due to the MSW effect. By contrast, in the massive celestial objects, the MSW mechanism would be affected by the gravitational field and have a different behavior. In this work, we are going to investigate the modification caused by the curved spacetime.

Various works have been dedicated to neutrino oscillations in curved spacetime; it was first pioneered by Stodolsky [2] and then discussed in detail by many authors [3–16]. There are several methods developed and applied: plane wave method [5,8,10], WKB approximation [2,11,13,16] and geometric treatment [6]. For various metrics: such as Schwarzschild metric [3–7,9,10], Kerr metric [8,11], Kerr–Newman metric [16], Hartle–Thorne metric [15], Lense–Thirring metric [14], and for different celestial objects: active galactic nuclei [4], neutron stars [3] and supernovae [3,12], discussions are made. The possible spin oscillation in gravitational field is investigated later [17–21].

Most of the former literatures are focused on the two-generation oscillations and to the best of our knowledge, three-generation oscillations and the MSW effect and the corresponding neutrino self-energy correction in curved spacetime have not been studied yet. This topic is directly linked to one of the most fundamental yet undetermined features of neutrinos; *Mass Hierarchy Problem*. One of the two major experimental approaches [22] to solve this problem is observing the MSW effect which would clearly indicate how the conversion probability is related to the sign of Δm^2_{31}. Furthermore, determining this puzzle may play a significant role for understanding the nature of neutrinos (Dirac or Marjonara). In this paper we present a preliminary discussion on the topic.

In Sec. 2 we give a short review of the geometric treatment of the phase differences in neutrino oscillations. In Sec. 3 we discuss the possible corrections of the neutrino self-energy in curved spacetime briefly. In Sec. 4 and Sec. 5 we make explicit calculations on three-generation neutrino oscillations in vacuum and matter in a static, spherically symmetric spacetime. In Sec. 6 the numerical estimations of the strength of such modifications are carried out and our conclusion is summarized in Sec. 7. It is noted that here we choose the sign of metric to be $diag(-,+,+,+)$ and use the natural unit system with $\hbar = c = 1$.

2. Geometric effects on the phase differences

Two critical points concerning neutrino oscillations in curved spacetime are the definitions of the energy and phase difference occurring in the formulation of neutrino oscillations. Here we apply a framework for curved spacetime where the geometric effect manifests [6], the energy is defined clearly and the phase difference is covariantly expressed. For the convenience of readers, only results of this treatment are listed here. The concerned formalism and notations are reviewed in Appendix A.

The original expression of the one-particle-neutrino wave function in flat spacetime is

$$|\Psi_\alpha(x,t)\rangle = \sum_j U_{\alpha j} e^{iP^\mu x_\mu} \left|\nu_j\right\rangle, \tag{1}$$

where $|\Psi_\alpha(x,t)\rangle$ denotes the wave function in the flavor representation and is expanded into $|\Psi_\alpha(x,t)\rangle = \sum_j U_{\alpha j} \left|\nu_j\right\rangle$ with $\left|\nu_j\right\rangle$ being the wave function in the mass representation. The roman letters $\alpha = \nu, \mu, \tau$ and the Latin letters $j = 1,2,3$ denote flavor and mass states, respectively. $U_{\alpha j}$ is the element of the neutrino mixing matrix (PMNS matrix). P^{μ} is the four-momentum of neutrino in the mass basis and $e^{iP^{\mu}x_{\mu}}$ is the vacuum evolution phase which reduces to e^{iEt} in the non-relativistic Schrödinger framework.

Generalizing Eq. (1) for the environment filled with electrons in a gravitational field, the formalism would be modified to be an equation in the curved spacetime as

$$|\Psi_{\alpha}(\tau)\rangle = \sum_{j} U_{\alpha j} e^{i\int(\frac{M^2}{2}+p^{\mu}A_{\mu})d\tau} \left|\nu_j\right\rangle, \tag{2}$$

where the position and time coordinates are replaced by an affine parameter τ, M^2 is an operator acting on neutrino wave functions and its eigenvalues are the square of neutrino masses. p^{μ} is the tangent vector to the path along which neutrinos travel. This tangent vector is chosen so that $p^0 = P^0$ and the rest three spatial components are parallel to the three spatial components of P^{μ}. A_{μ} is the effective potential caused by the electron environment. We can define the evolution phases of the corresponding wave functions as

$$\Omega \equiv e^{i\int(\frac{M^2}{2}+p^{\mu}A_{\mu})d\tau}. \tag{3}$$

Differentiating Eq. (2) a Schrödinger-like evolution equation is yielded

$$-i\frac{\partial\,|\Psi_{\alpha}(\tau)\rangle}{\partial\tau} = (\frac{M^2}{2}+p^{\mu}A_{\mu})\,|\Psi_{\alpha}(\tau)\rangle, \tag{4}$$

and it can be seen that treating $\frac{M^2}{2}+p^{\mu}A_{\mu}$ as an effective Hamiltonian would be of convenience.

Then in the mass representation, using Eq. (2) one can write down the probability of conversion for a neutrino from α flavor to β flavor after traveling a spacetime interval $\Delta s(\tau-\tau_0)$ as

$$\mathcal{Q}_{\beta\alpha} \equiv \left|\langle\Psi_{\beta}\;(\tau)\;|\Psi_{\alpha}(\tau_0)\rangle\right|^2 = \sum_{i,j} U^{\dagger}_{\alpha i} U_{\beta i} U_{\alpha j} U^{\dagger}_{\beta j} e^{i\Omega_{ji}}, \tag{5}$$

where the phase difference Ω_{ij} of two mass eigenstates is defined as

$$\Omega_{ji} \equiv \int (\frac{m_j^2-m_i^2}{2}+\Delta p^{\mu}A_{\mu})d\tau \equiv \int (\frac{\Delta_{ji}}{2}+\Delta p^{\mu}A_{\mu})d\tau, \tag{6}$$

where $\Delta_{ji} \equiv m_j^2 - m_i^2$ denotes the mass difference and m_j is the eigenvalue of the corresponding mass eigenstate. $\Delta p^{\mu}A_{\mu}$ is the difference of the electron-environmental contribution due to the asymmetric matter effect, where W boson exchange only affects electron neutrinos. Noticing these quantities are expressed in the mass basis and so that $M^2 = diag(m_1^2, m_2^2, m_3^2)$, we are able to write down Eq. (6). This phase difference and mixing matrix will compose the sufficient condition for calculating the conversion probability.

3. Neutrino self-energy in curved spacetime

We now turn to the specific expression of the effective potential A_{μ} which is induced by the corrections to the neutrino self-energy in the electron environment. The leading order contribution to the self-energy arises from the charged current and neutral current interactions. The later one has exactly the same effect on all the three neutrino generations, thus shows no significance to the MSW effect and can be neglected.

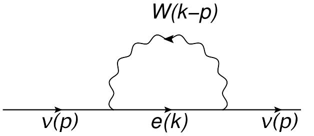

Fig. 1. Feynman diagram of charged current interaction of electron neutrinos.

As is shown in Fig. 1, the Feynman diagram of the contribution from the charged current interaction (W boson exchange) is presented. To compute this diagram in curved spacetime, two modifications should be accounted.

1. *Finite-temperature field theory in curved spacetime* [23–26]. In the gravitational background, the definitions of both vacuum and thermal equilibria become ambiguous; a vacuum state in equilibrium seen by one observer may appear as a state with particles and deviated from equilibrium [27]. Besides, the possible phase transitions are also of interest. To solve these difficulties a finite-temperature field theory in curved spacetime is needed.
2. *Quantum field theory in curved spacetime and particle creation* [28,29]. As mentioned above, the definition of vacuum is not consistent with diffeomorphisms. Since particles are created via the coupling between gravitational field and quantum field, the Fock space on which the field operators act is changed, thus one may have to distinguish between the *in*-state and *out*-state Fock space. To avoid that ambiguity, one has to construct two sets of field operators and vacuum states corresponding to the *in* and the *out* states, respectively. The unique method for defining such states is still absent at present. Furthermore, due to these *in* and *out* states there exist two types of the S-matrix elements in the amplitudes. Thus the Feynman rules for accounting the self-energy of neutrinos in the electroweak theory should be modified.

These two effects are most important for discussions in some cases such as the expanding universe, where the dynamic spacetime deviates the quantum states from equilibrium, or the ultra-strong gravitational field, where the particle creation becomes possible. Yet if we restrict ourselves to neutrino oscillations in celestial objects like sun or supernovae, the calculation can be substantially simplified. Besides, the region where the MSW resonance appears can be very narrow [30], so the variation of the metric inside this region can be neglected. Thus it can be assumed that the environmental matter is always in equilibrium and no particle creation takes place in this case. Only propagators should be modified while vertices and Feynman rules remain the same.

Under these approximations and keeping the propagators in the momentum representation, we can write down the common Feynman rules for Fig. 1 as

$$-i\Sigma(\mathbf{p})=-i\int d^4k(-\frac{i}{\sqrt{2}}g_W)\gamma^\mu\mathcal{P}_L iS_e(\mathbf{k})(-\frac{i}{\sqrt{2}}g_W)\gamma^\mu\mathcal{P}_L\frac{i}{(\mathbf{k}-\mathbf{p})^2-m_W^2}, \tag{7}$$

where g_W is the coupling constant of SU(2), $\mathcal{P}_L$ is the left-projection operator, γ^μ is the gamma matrix in curved spacetime and is defined in Eq. (71), iS_e is the propagator of electrons in thermal bath with gravitational background. The W boson propagator has been written in 't Hooft–Feynman gauge. With the gravitational modifications, the propagator of spin-1/2 fermions written in the momentum representation reads [31]

$$iS(\mathbf{k})=(1+f_1(x^\mu)(-\frac{\partial}{\partial m^2})+f_2(x^\mu)(-\frac{\partial}{\partial m^2})^2)iS_0(\mathbf{k}), \tag{8}$$

where f_1 and f_2 are functions of x^μ, $iS_0(k)$ is the corresponding electron propagator in thermal bath with spacetime being flat. Considering modifications up to the order of $\frac{\partial}{\partial m^2}$ only and ignoring the higher oder terms in $f_1(x^\mu)$ which are dependent on the electron momentum, the electron propagator in curved spacetime is

$$iS(\mathbf{k}){=}(1{+}\frac{1}{12}\mathcal{R}(-\frac{\partial}{\partial m^2}))iS_0(\mathbf{k}), \tag{9}$$

where $\mathcal{R}$ is the Ricci scalar curvature of the spacetime.

Writing down the flat-spacetime propagator of electron field in terms of the real-time formalism with finite temperature [32]:

$$iS_0(\mathbf{k}){=}(\gamma^\mu k_\mu{+}m_e)(\frac{i}{k^2{-}{m_e}^2}{-}2\pi\delta(k^2{-}{m_e}^2)f_F(k_\mu u^\mu)), \tag{10}$$

where the mass of fermion m has been replaced by the mass of electron m_e and $f_F(x)$ represents the Fermi distribution function

$$f_F(x){=}\frac{\theta(x)}{e^{\beta(x-\mu)}{+}1}{+}\frac{\theta(-x)}{e^{-\beta(x-\mu)}{+}1}, \tag{11}$$

where β is the inverse of temperature, μ is the chemical potential of the electron and $\theta(x)$ is the Heaviside step function.

Noticing that in Eq. (10), the first term is the common propagator in vacuum except modified gamma matrices being introduced and it may lead to infinity when the loop diagrams involving the propagator are integrated. The renormalization schemes in curved spacetime have been discussed in some details by several authors [31,33]. Under the condition that temperature is just above the threshold of thermal effects on electron, the propagator of W boson is unchanged and the dispersion relation of electron is non-relativistic $\omega_k{=}\frac{k^2}{2m_e}$ where ω_k is the energy of electron.

Then Eq. (7) can be evaluated up to the order of $1/m_W^2$ as

$$\Sigma(\mathbf{p}){=}(1{-}\frac{\mathcal{R}}{16m_e^2})\sqrt{2}G_F N_e e_a^\mu u^a, \tag{12}$$

where G_F is the Fermi coupling constant, N_e is the locally measured electron number density and u^a is the four-velocity of the environmental electron current. Comparing with the common result obtained in flat spacetime, the self-energy is modified by the tetrad e_a^μ and an extra term $1{-}\frac{R}{16m_e^2}$ arising from gravitation background. This term can be separated as an extra gravitational phase.

4. Neutrino propagation in vacuum with curved spacetime

In this section we consider the neutrino propagation in a general static and spherically symmetric spacetime, the metric of which is [35]

$$ds^2{=}{-}e^{2\Phi}dt^2{+}e^{2\Lambda}dr^2{+}r^2d\theta^2{+}r^2\sin^2\theta d\phi^2, \tag{13}$$

where $\Phi(r)$ and $\Lambda(r)$ are arbitrary functions of the radial coordinate r. In such a spherically symmetric spacetime the contribution of spin connection vanishes [6,34]. Thus we can safely concentrate on flavor oscillations of neutrino and ignore possible spin flips. Another characteristic of this spacetime is the existence of a Killing vector ∂_t which corresponds to a conserved quantity $P_0{=}g_{0\upsilon}P^\upsilon{\equiv}{-}E_\infty$. As will be shown later, this quantity is just the energy of neutrino measured by an observer at $r{=}\infty$.

Without losing generality, here we consider only the radial propagation of neutrino in vacuum with curved spacetime, the integral element for the propagating phase is written as a function of proper length Eq. (80):

$$d\tau = dl(-g_{00}(\frac{dx^0}{d\tau})^2)^{-1/2} = dl\frac{1}{E_\infty e^{-\Phi}}, \tag{14}$$

where $E_\infty e^{-\Phi}$ is understood as the local energy which is measured by an observer in the local inertial frame. $e^{-\Phi}$ vanishes as $r \to \infty$, leaving E_∞ only, which means it is the energy measured at $r = \infty$. The proper length Eq. (78) for radial propagation case reads

$$dl = \sqrt{g_{11}(dx^1)^2}. \tag{15}$$

Then in vacuum without the electron environment, the phase difference Eq. (6) is calculated as

$$\Omega_{ji} = \int \frac{\Delta_{ji}}{2} d\tau = \int \frac{\Delta_{ji}}{E_\infty e^{-\Phi}} e^{\Lambda} dr. \tag{16}$$

This phase difference is exactly the same as which in the two-generation case except the labels i, j range from 1 to 3 now.

5. Three-generation MSW effect in curved spacetime

Using the metric Eq. (13), assuming the electron environment is at rest with respect to the oscillation process and choosing the tetrad to be $e_a^\mu = diag(e^{-\Phi}, e^{-\Lambda}, \frac{1}{r}, \frac{1}{r\sin\theta})$, the self-energy Eq. (12) of the neutrino in environment with curved spacetime can be calculated. Then instead of Eq. (12) one can turn to the effective potential

$$A_{\mu f} = \begin{pmatrix} \sqrt{2}e^{-\Phi}G_F N_e(1 - \frac{\mathcal{R}}{16m_e^2}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

where the subscript f refers to the flavor basis. It is noted that this effective potential is expanded in the flavor basis and diagonalized.

To calculate the phase differences, one could directly substitute Eq. (17) into Eq. (6), yet in general it is more convenient if we consider the effective potential as a modification to the masses of neutrinos and calculate the resultant mixing parameters in medium with the effective masses. Then neutrinos would behave exactly like they were in vacuum except possessing different masses and different mixings.

First writing the evolution relation Eq. (77) for α-flavor neutrino in the flavor basis:

$$|\Psi_\alpha(r)\rangle = e^{-i\int \frac{1}{2E_\infty}(e^{\Phi}M_f^2 + V_f)e^{\Lambda}dr}\ |\Psi_\alpha(r_0)\rangle, \tag{18}$$

where

$$M_f^2 = U M_m^2 U^\dagger \tag{19}$$

is the mass matrix in the flavor representation, which is transformed from the mass-basis mass matrix by the PMNS matrix. The mass matrix only appears in the phase part of the neutrino evolution, thus only the relative differences of the eigenvalues matter. Therefore subtracting m_1^2 from the mass-basis mass matrix and using the mass differences Δ_{21}, Δ_{31} as new eigenvalues instead of m_1^2, m_2^2, m_3^2, we have

$$M_m^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix}. \tag{20}$$

The PMNS matrix is parameterized with mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$ and a CP violating phase δ

$$\begin{aligned} U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & e^{-i\delta} s_{13} \\ 0 & 1 & 0 \\ -e^{-i\delta} s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta} s_{13} \\ -c_{23}s_{12} - e^{i\delta} c_{12}s_{13}s_{23} & c_{12}c_{23} - e^{i\delta} s_{12}s_{13}s_{23} & c_{13}s_{23} \\ -e^{i\delta} c_{12}c_{23}s_{13} + s_{12}s_{23} & -e^{i\delta} c_{23}s_{12}s_{13} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix}, \end{aligned} \tag{21}$$

where $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$. We set $\delta = 0$ through this work.

And the electron-environmental contribution matrix V_f is written as

$$V_f = \begin{pmatrix} v(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

where $v(r) = 2\sqrt{2} e^{-\Phi} G_F E_\infty N_e (1 - \frac{\mathcal{R}}{16 m_e^2})$.

Defining the effective mass matrix as

$$\tilde{M}_f^2 = (e^{\Phi} M_f^2 + V_f) e^{\Lambda}, \tag{23}$$

we obtain the three eigenvalues of the effective mass matrix

$$x_A = 2\sqrt{\frac{l}{3}} \cos(\frac{1}{3} \arccos(\frac{3s}{2l}\sqrt{\frac{l}{3}})) - \frac{b}{3a}, \tag{24}$$

$$x_B = 2\sqrt{\frac{l}{3}} \cos(\frac{1}{3} \arccos(\frac{3s}{2l}\sqrt{\frac{l}{3}}) - \frac{2\pi}{3}) - \frac{b}{3a}, \tag{25}$$

$$x_C = 2\sqrt{\frac{l}{3}} \cos(\frac{1}{3} \arccos(\frac{3s}{2l}\sqrt{\frac{l}{3}}) - \frac{4\pi}{3}) - \frac{b}{3a}, \tag{26}$$

where the parameters a, b, c, d, s, l are defined as

$$a = 8, \tag{27}$$

$$b = -8v - 8e^{\Psi} \Delta_{21} - 8e^{\Phi} \Delta_{31}, \tag{28}$$

$$\begin{aligned} c = {} & e^{\Phi} (6v\Delta_{21} + 4v\Delta_{31} + 8e^{\Phi} \Delta_{21}\Delta_{31} + 2v\Delta_{21} \cos 2\theta_{12} + \\ & v\Delta_{21} \cos 2\,(\theta_{12} - \theta_{13}) - 2v\Delta_{21} \cos 2\theta_{13} + 4v\Delta_{31} \cos 2\theta_{13} + v\Delta_{21} \cos 2\,(\theta_{12} + \theta_{13})), \end{aligned} \tag{29}$$

$$d = -8e^{2\Phi} v\Delta_{21}\Delta_{31} \cos\theta_{12}{}^2 \cos\theta_{13}{}^2, \tag{30}$$

$$s = \frac{2b^3 - 9abc + 27a^2 d}{27a^3}, \tag{31}$$

$$l = -\frac{3ac - b^2}{3a^2}. \tag{32}$$

It should be clarified that x_A, x_B, x_C are the three eigenvalues of matrix (23) with the order $x_A \geq x_B \geq x_C$. Thus they may not be in a sequence of m_1, m_2, m_3, nor do they have the same values of the real neutrino masses due to the simplification of matrix (20). Only the values of their differences are identical to the corresponding mass differences. Because of this ambiguity, we need to discuss these solutions for normal hierarchy (NH) and inverted hierarchy (IH) respectively.

For the normal hierarchy, there are

$$\tilde{m}_1^2 = x_C + C, \tilde{m}_2^2 = x_B + C, \tilde{m}_3^2 = x_A + C, \tag{33}$$

where $C = m_1^2$ is caused by the subtraction of mass m_1^2 from the mass matrix.

And so for the inverted hierarchy, we have

$$\tilde{m}_3^2 = x_C + C, \tilde{m}_1^2 = x_B + C, \tilde{m}_2^2 = x_A + C. \tag{34}$$

Solving the resonance condition for v yields

$$v|_{res} = e^{\Phi} \, v_{flat}\big|_{res}, \tag{35}$$

where $v_{flat}\big|_{res}$ denotes the resonance condition in the Minkowski spacetime with the neutrino energy being $E_{flat}\big|_{res}$. Substituting the expression of v, it is obtained that

$$E_{\infty}|_{res} = e^{2\Phi} (1 - \frac{R}{16 m_e^2})^{-1} \, E_{flat}\big|_{res}. \tag{36}$$

The modification to the resonance energy consists of two parts, the overall redshift with a factor $e^{2\Phi}$ and an extra term coming from the gravitational effect on the neutrino self-energy. The overall redshift factor $e^{2\Phi}$ obtained here is the square of the redshift factor given in [6].

5.1. Neutrino mixing parameters in celestial objects

It would be convenient to write the medium-modified mixing matrix in the same parametrization scheme as Eq. (21), then the oscillation in matter will be of the same form as the oscillation in vacuum.

We can define the effective phase and phase difference in medium with results obtained above to be

$$\tilde{\Omega}_j \equiv \int \frac{\tilde{m}_j^2}{E_{\infty} e^{-\Phi}} e^{\Lambda} dr, \tag{37}$$

$$\tilde{\Omega}_{ji} \equiv \int \frac{\tilde{\Delta}_{ji}}{E_{\infty} e^{-\Phi}} e^{\Lambda} dr. \tag{38}$$

To calculate the mixing matrix in medium one needs the time-dependent perturbation theory and the evolution equation Eq. (4)

$$-i \frac{\partial}{\partial \tau} \tilde{U} e^{i\Omega_j} \, |\nu_j\rangle = \tilde{M}_m^2 \tilde{U} e^{i\Omega_j} \, |\nu_j\rangle, \tag{39}$$

where $\tilde{\Delta}_{ji} \equiv \tilde{m}_j^2 - \tilde{m}_i^2$ is the effective mass difference in medium. The elements of the effective mixing matrix obey differential equations

$$-i \frac{\partial}{\partial \tau} \sum_j \tilde{U}_{\alpha j} = \sum_j V_{mij} \tilde{U}_{\alpha j} e^{i \tilde{\Omega}_{ji}}. \tag{40}$$

In general it is difficult to solve for these mixing parameters analytically and in practice one would do it numerically. But if the environmental matter has uniform density and the first order derivative of metric can be neglected, Eq. (39) can be rewritten as

$$-i\frac{\partial}{\partial\tau}\sum_j \tilde{U}_{\alpha j}e^{i\tilde{\Omega}_j}\,|\nu_j\rangle = \tilde{U}^\dagger \tilde{M}_m^2 \tilde{U}\sum_j e^{i\tilde{\Omega}_j}\,|\nu_j\rangle. \tag{41}$$

Then the modified PMNS matrix reduces to be the unitary diagonalization matrix for (23) through the similarity relation

$$\tilde{U}^\dagger \tilde{M}_f^2 \tilde{U} = \tilde{M}_m^2. \tag{42}$$

Solving for the diagonalization matrix through the Gram–Schmidt process then applying re-parameterization, the modified mixing angles can be calculated in either NH or IH. For NH, there are

$$^N\tilde{t}_{12}^2 = \frac{\left(\rho_A^2\xi_B - \xi_A\rho_A\rho_B + \sigma_A(\sigma_A\xi_B - \xi_A\sigma_B)\right)^2}{\left(\xi_A^2+\rho_A^2+\sigma_A^2\right)(\sigma_A\rho_B-\rho_A\sigma_B)^2}, \tag{43}$$

$$^N\tilde{t}_{13}^2 = \frac{\xi_A^2}{\xi_A^2+\rho_A^2+\sigma_A^2}, \tag{44}$$

$$^N\tilde{t}_{23}^2 = \frac{\rho_A^2}{\sigma_A^2}, \tag{45}$$

where $\tilde{t}_{ij}^2 \equiv \tan^2\tilde{\theta}_{ij}$ and N denotes normal hierarchy. Similarly, for IH it yields

$$^I\tilde{t}_{12}^2 = \frac{\xi_A^2\left(\sigma_A^2\left(\xi_B^2+\rho_B^2\right) - 2\xi_A\sigma_A\xi_B\sigma_B - 2\rho_A\rho_B(\xi_A\xi_B+\sigma_A\sigma_B) + \rho_A^2\left(\xi_B^2+\sigma_B^2\right) + \xi_A^2\left(\rho_B^2+\sigma_B^2\right)\right)}{\left(\rho_A^2\xi_B - \xi_A\rho_A\rho_B + \sigma_A(\sigma_A\xi_B - \xi_A\sigma_B)\right)^2}, \tag{46}$$

$$^I\tilde{t}_{13}^2 = \frac{(\sigma_A\rho_B-\rho_A\sigma_B)^2}{\rho_A^2\xi_A^2 - 2\xi_A\rho_A\xi_B\rho_B + \xi_A^2\rho_B^2 + (\sigma_A\xi_B-\xi_A\sigma_B)^2}, \tag{47}$$

$$^I\tilde{t}_{23}^2 = \left(\frac{\sigma_A\xi_B-\xi_A\sigma_B}{-\rho_A\xi_B+\xi_A\rho_B}\right)^2. \tag{48}$$

The parameters appearing in the above equations are defined as

$$\xi_I \equiv -\zeta^2 + (\varepsilon - x_I)(\eta - x_I); \tag{49}$$

$$\rho_I \equiv \gamma\zeta + \beta(x_I - \eta); \tag{50}$$

$$\sigma_I \equiv \beta\zeta + \gamma(x_I - \varepsilon), \tag{51}$$

where $I = A, B, C$, and

$$\alpha \equiv e^\Lambda(v + e^\Psi\Delta_{21}c_{13}^2 s_{12}^2 + e^\Psi\Delta_{31}s_{13}^2); \tag{52}$$

$$\beta \equiv e^{\Lambda+\Psi}(\Delta_{21}(s_{12}c_{12}c_{13}c_{23} - s_{12}^2 s_{13}s_{23}c_{13}) + \Delta_{31}s_{13}s_{23}c_{13}); \tag{53}$$

$$\gamma \equiv e^{\Lambda+\Psi}(-\Delta_{21}(s_{12}s_{23}c_{12}c_{13} + s_{12}^2 s_{13}c_{13}c_{23}) + \Delta_{31}s_{13}c_{23}c_{13}); \tag{54}$$

$$\varepsilon \equiv e^{\Lambda+\Psi}(\Delta_{21}(c_{12}c_{23} - s_{12}s_{13}s_{23})^2 + \Delta_{31}s_{23}^2 c_{13}^2); \tag{55}$$

$$\zeta \equiv e^{\Lambda+\Psi}(-\Delta_{21}(s_{23}c_{12} + s_{12}s_{13}c_{23})(c_{12}c_{23} - s_{12}s_{13}s_{23}) + \Delta_{31}s_{23}c_{13}^2 c_{23}); \tag{56}$$

$$\eta \equiv e^{\Lambda+\Psi}(\Delta_{21}(s_{12}s_{13}c_{23} + s_{23}c_{12})^2 + \Delta_{31}c_{13}^2 c_{23}^2). \tag{57}$$

6. Numerical estimations

The actual structure of real celestial objects can be rather complicated and this section we present an explicit estimate for a toy model where the matter density is uniform. All quantities are written in the SI units for numerical computations. The observer is located at $r = \infty$.

Now let us estimate the matter effect when neutrinos travel inside a massive celestial object. We need an interior Schwarzschild solution to describe the background spacetime. These solutions of the Einstein field equations are usually impossible to be explicitly written in a simple form but are presented numerically. For simplicity, here we apply the easiest interior Schwarzschild solution in which the celestial object is approximated as an isotropic, uncharged non-rotating perfect fluid. The metric can be evaluated using the Oppenheimer–Volkoff equation and the result is [35]

$$ds^2 = -(\frac{3}{2}(1-2\frac{GM}{Rc^2})^{1/2} - \frac{1}{2}(1-2\frac{GMr^2}{R^3c^2})^{1/2})^2 c^2 dt^2 + \frac{1}{1-\frac{2GMr^2}{R^3c^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{58}$$

where M is the mass of the celestial object through which the neutrino traverses and R is the radius of the object, r is the radial coordinate and $r<R$. To have a noticeable effect on the oscillation, we consider a celestial object whose radius is 10^2 times larger than that of the sun and is 10^6 times heavier than the sun. In this configuration the celestial object would have the same density as the sun. $M = 10^6 M_\odot \approx 2\times10^{36}$ kg, $R = 10^2 R_\odot \approx 7\times10^{10}$ km and $\rho = \rho_\odot \approx 10$ g/cm^3 are the mass, radius and density of the celestial object, respectively.

For oscillations that take place deep inside the object, i.e. at the region where $r << R$, we obtain that

$$e^{2\Lambda} \sim 1, \tag{59}$$

$$e^{2\Phi} \sim 0.94. \tag{60}$$

Recalling that $\rho = (N_n+N_p)m_n$ and $Y_e = \frac{N_e}{N_n+N_p}$, where m_n is the mass of a nucleon, N_n and N_p are the number densities of neutron and proton, respectively. Y_e is the electron fraction and is approximated well as 0.5. We can write down the environmental contribution function υ as

$$\upsilon(r) = 1.518\times10^{-13} e^{-\Phi} E_\infty \rho Y_e (1 - \frac{\mathcal{R}\hbar^2}{16 m_e^2 c^2}), \tag{61}$$

where the unit of υ is eV and the density is written in g/cm^3, the Ricci scalar has the unit of m^{-2}. For the metric (58) the Ricci scalar curvature is calculated as Fig. 2 which has a nearly exponential dependence on r. It tends to infinity as $r\to0$ and drops drastically as $r\to R$. Except for the region of $r << 1$ which is the vicinity of the singularity, in most cases $\mathcal{R}$ is rather small and can be safely neglected, thus we will ignore this effect in the rest of this section.

Then it is of interest to investigate the influence imposed on the effective masses of neutrinos by the curved spacetime. The dependence of these masses on υ is shown in Fig. 3. It can be seen that the redshift which is denoted by the deviation of the vertical dashed lines from the solid lines in the figure. At the minimum of $\tilde{\Delta}_{32}$ the redshift is only obvious in NH. Whereas, for IH at the minima of $\tilde{\Delta}_{32}$ and $\tilde{\Delta}_{21}$ the environmental effect υ is almost the same.

Finally we would like to see the overall effect of the survival probability which may be measurable. Assuming that electron neutrinos are produced in the core of this celestial object or

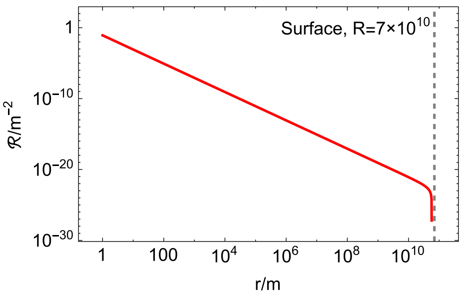

Fig. 2. Ricci scalar curvature vs. r.

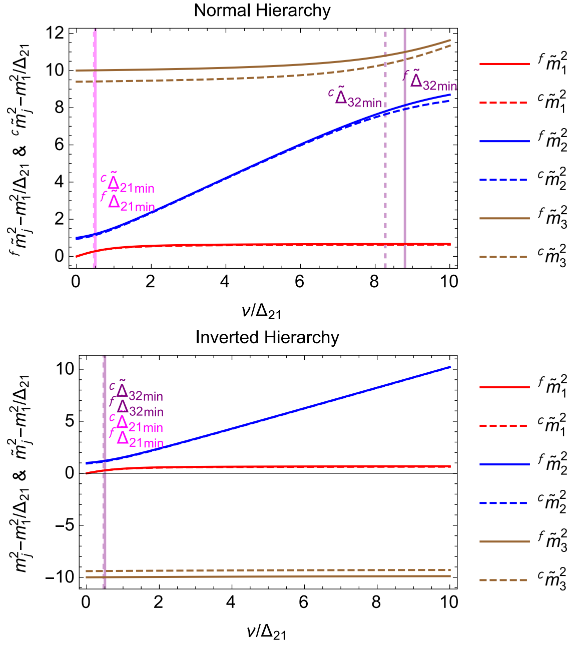

Fig. 3. Mass eigenvalues in flat and curved spacetime, respectively. Above: in NH. Below: in IH. Red, blue and brown solid (dashed) curves denote the mass eigenvalues ${}^f\tilde{m}_j^2$ (${}^c\tilde{m}_j^2$) for $j = 1, 2, 3$ in flat (curved) spacetime, respectively. f and c denote flat spacetime and curved spacetime, respectively. Solid (dashed) pink vertical lines denote the positions of the minimum of ${}^f\tilde{\Delta}_{21}$ (${}^c\tilde{\Delta}_{21}$). Solid (dashed) purple vertical lines are defined similarly except they are for ${}^f\tilde{\Delta}_{32}$ (${}^c\tilde{\Delta}_{32}$) instead. Mixing angles are chosen to be $\theta_{12} = 33.48°$ and $\theta_{13} = 8.52°$ [36]. Metric is chosen so that $e^{2\Phi} = 0.94$ and $e^{2\Lambda} = 1$. For illustration purpose, we have set $\Delta_{21} = 1$ and $\Delta_{31} = \pm 10$ for NH (IH). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

traversing it, and then departing from the region of the object to propagate towards the earth. If theses neutrinos could be measured right after they traverse through the object the overall survival probability is

$$\mathcal{Q}_{ee} = \sin^4\tilde{\theta}_{13} + \cos^4\tilde{\theta}_{13}(\cos^4\tilde{\theta}_{12} + \sin^4\tilde{\theta}_{12}). \tag{62}$$

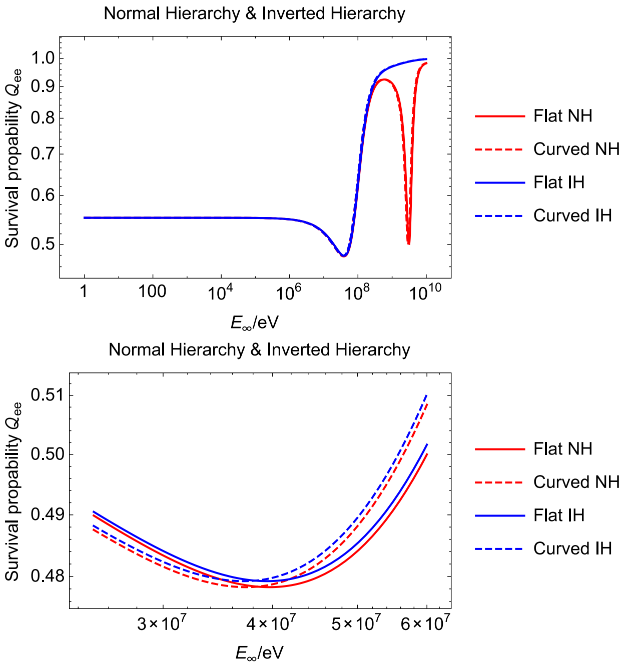

Fig. 4. Survival probability after traveling through the celestial object. Above: plot range from 1 to 10^{10} eV. Below: details around the lower energy (10 MeV) oscillation region. Red and blue solid (dashed) curves denote the survival probability $\mathcal{Q}_{ee}$ in flat (curved) spacetime, respectively. Mixing parameters are chosen as $\theta_{12} = 33.48°$, $\theta_{13} = 8.52°$, $\theta_{23} = 42.2°$, Δ_{21}=7.5×10^{-5} eV; Δ_{31}=2.5×10^{-3} eV for NH and Δ_{31}= -2.5×10^{-3} eV for IH [36]. The metric is chosen as $e^{2\Phi} = 0.94$ and $e^{2\Lambda} = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Numerical result of Eq. (62) is plotted in Fig. 4. There the gravitational background does not exert any noticeable effect on the overall oscillation behaviors, but from the figure one can notice a tiny redshift of the resonance energies.

For higher energies, there can be a second resonance dip, while the first one occurs at about a few of tens MeV which is the characteristic energy of solar and supernova neutrinos. The second dip appears at about GeV scale and can only take place in NH. Except for this resonance, the mass hierarchy plays a limited role in the mixing and the probability result, thus requires experiments of very high precision to detect.

After the neutrino beam leaves the massive celestial object, it is supposed to freely propagate towards the earth and will be detected by the earth detectors, then the later stage oscillations are completely the regular oscillations in vacuum. Now let us consider the resultant flavor survival probability determined by the oscillations in the matter and the sequent vacuum. The survival probability is then given by

$$\mathcal{Q}_{ee}=c_{12}^2\tilde{c}_{12}^2c_{13}^2\tilde{c}_{13}^2+c_{13}^2\tilde{c}_{13}^2s_{12}^2\tilde{s}_{12}^2+s_{13}^2\tilde{s}_{13}^2, \tag{63}$$

the result of which is depicted in Fig. 5. The spacetime still mainly influences the resonance energy with a gravitational redshift.

The resonant result indeed imposes an initial condition to the later vacuum oscillation. We notice that under this initial condition, for the lower energy neutrinos (about 10 MeV or below)

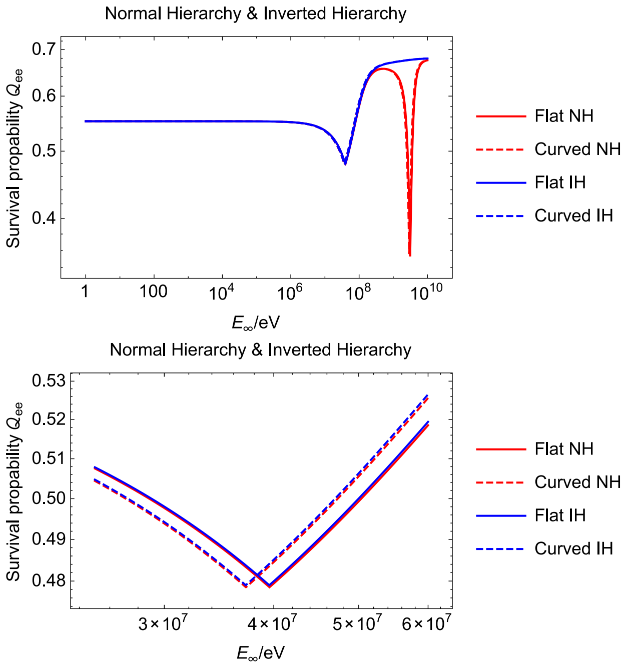

Fig. 5. Survival probability after traveling through the object and the distance between the object and the earth. Above: plot range from 1 to 10^{10} eV. Below: details around the lower energy (10 MeV) oscillation region. Red and blue solid (dashed) curves denote the survival probability $\mathcal{Q}_{ee}$ in flat (curved) spacetime, respectively. Mixing parameters are chosen as $\theta_{12} = 33.48°$, $\theta_{13} = 8.52°$, $\theta_{23} = 42.2°$, Δ_{21}=7.5×10^{-5} eV, Δ_{31}=2.5×10^{-3} for NH and Δ_{31}=-2.5×10^{-3} eV for IH [36]. Metric is chosen as $e^{2\Phi} = 0.94$ and $e^{2\Lambda} = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the vacuum propagation barely changes their behaviors, therefore the propagation of the beam from the production core to the earth is only affected by existence of the celestial matter. But by contrary, for the higher energy neutrinos (at GeV scale), the probability is seriously modified. The survival probability at this dip goes down from 0.5 to 0.3 and the upper limit of the survival probability could be suppressed from 1 to 0.7.

Keeping the density similar to that of the sun, the dependence of mass and radius on $e^{2\Phi}$ is shown in Table 1. Actually, in our case, if taking $e^{2\Phi} = 0.94$, then the energy can be shifted by 6% at most. With small mass of the celestial object the redshift would be very small. As the mass increases, the overall redshift increases quickly till at $M = 10^8 M_\odot$, $R = 10^{8/3} R_\odot$ there is $R \approx R_s$, where R_s is the Schwarzschild radius and $R_s = \frac{2GM}{c^2}$. In this extreme condition the interior Schwarzschild solution we are using fails and the object collapses, yielding an unreasonable result.

This redshift will be much more manifest in supernovae where the density of matter can be as much as $\sim 10^{13}$ times larger than the solar density. The reason why we stick to choosing the solar density is that the MSW effect in the sun has been well experimentally and theoretically confirmed, so that we have something solid to help in making sense. For the influence of density on the energy of the first oscillation dip, see Table 2. It should be clarified that this work is only to set an upper bound on the gravitational modifications to the MSW effect. The real gravitational effect may be somehow different.

Table 1
Redshift factor $e^{2\Phi}$ vs. the mass and radius of the object.

M	R	$e^{2\Phi}$
$M_{\odot}$	$R_{\odot}$	0.99999
$10M_{\odot}$	$10^{1/3}R_{\odot}$	0.99997
$10^3 M_{\odot}$	$10R_{\odot}$	0.99936
$10^6 M_{\odot}$	$10^2 R_{\odot}$	0.93678
$10^7 M_{\odot}$	$10^{7/3} R_{\odot}$	0.71304
$10^8 M_{\odot}$	$10^{8/3} R_{\odot}$	0.00329

Table 2
Energy of the first resonance dip vs. density of the celestial object. The mass of the object is selected to be $10M_{\odot}$.

$\rho/\rho_{\odot}$	10	20	30	40	50
$E_{\infty}\|_{res}$/eV	3.9×10^6	2.0×10^6	1.2×10^6	9.8×10^5	7.2×10^5

6.1. Varying density

Now let us turn to the more realistic situation in which the density profile of the celestial object is arbitrarily varying. In this scheme the neutrino is produced inside the object, or just travels through the object, then leaves the object and propagates towards the Earth and finally is received by detectors on the ground. The varying density of the celestial object renders the mixing parameters dependent on the position and time, i.e., $\tilde{\theta}_{ij}\to\tilde{\theta}_{ij}(\rho(r,t))$. This composes a new degree of freedom and induces a new effect: the change of admixtures, meaning that the mass eigenstates $|\nu_j\rangle$ are not eigenstates of propagation anymore and dependent on position and time. Furthermore, for a neutrino of certain energy, inside the celestial object there will exist a (possibly narrow) resonance layer where $\rho(r,t)=\rho_R$, with ρ_R being the resonance density.

Excluding dynamics of the medium, the density is only dependent on the position. Under these circumstances the survival probability becomes

$$\mathcal{Q}_{ee}=c_{13}^2\tilde{c}_{13}^2(r_0)(\frac{1}{2}+(\frac{1}{2}-P_C)\cos 2\theta_{12}\cos 2\tilde{\theta}_{12}(r_0))+s_{13}^2\tilde{s}_{13}^2(r_0), \quad (64)$$

where P_C stands for the ν_1/ν_2 level crossing probability and r_0 is the radial coordinate of the initial position of neutrino.

To calculate this survival probability, one still needs to find the mixing parameters at the initial point and the metric of background spacetime. Yet if taking the density to be non-constant, Eqs. (41) and (58) fail and one will have to solve differential equations Eq. (4) (evolution equation) or Eq. (40) (mixing equation) and the Oppenheimer–Volkoff equation in order to obtain the components of each flavor or mixing parameters and the metric for the spacetime. In general it would be difficult solving them analytically even for linear or exponential density profiles. For a specific problem, it is usually only solvable via numerical methods and the results are model-dependent.

What is more, because of the self-energy correction by gravitational filed discussed earlier in Sec. 3, the effective potential of neutrino acquires a dependence on the spacetime. With this new degree of freedom there will be new effects. The oscillation condition of MSW effect will be switched as

$$\rho(r)=\rho_R \rightarrow \rho(r)=\rho_R(r), \tag{65}$$

which means the resonance density ρ_R is not constant anymore and there might be zero, one, or even multiple resonance layers in the objects, depending on specific structure of density profiles $\rho(r)$ and the spacetime. Surely these effects may be fairly small if not at the vicinities of spacetime singularities.

7. Discussion

In this work we review the geometric treatment for neutrino oscillations in curved spacetime and then extend it to the three-generation cases. A discussion of the gravitational effect on neutrino self-energy is given and the corresponding correction to the neutrino effective potential is derived. Applying this method and the results, survival probabilities of oscillations in vacuum and the MSW effect in a general, static and spherically symmetric spacetime for the three-generation neutrino scenario are then calculated. For the matter effect in curved spacetime, mixing parameters of neutrinos are evaluated for celestial objects with constant densities, and both normal and inverted mass hierarchy cases are respectively discussed. In the end of this work, numerical results for the background of a static, isotropic interior Schwarzschild spacetime are shown in figures.

The modification due to the gravitational background to the neutrino self-energy Eq. (12) implies an overall energy shift that is induced by the tetrad and an extra gravitational phase, however the dependence of the phase on energy is not separable from the common matter effect. The reason is that this phase has the same dependence on G_F, N_e and E_∞ as the original MSW effect. Thus this extra phase caused by the gravitational background and the original MSW phase would mix together and become indistinguishable at the energy spectrum, resulting in an overall redshift, even though formally this geometric-induced extra phase can be separated from the original propagation phase. Furthermore, this new effect is $-\frac{\mathcal{R}\hbar^2}{16m_e^2c^2}$, which is proportional to the Ricci scalar curvature $\mathcal{R}$ of the spacetime and therefore, it will vanish in the Schwarzschild spacetime, since in the Schwarzschild metric $\mathcal{R}=0$ and it describes a vacuum spacetime. Also, for this effect to be manifest we need that $\mathcal{R}\sim\frac{m_e^2c^2}{\hbar^2}\sim10^{24}$ m^{-2}, which is dramatically large and only possible at the vicinities of spacetime singularities.

The three-generation vacuum oscillations in curved spacetime have exactly the same form as their two-generation situation and all the conclusions can be generalized directly. On the other hand, the formulation for the MSW effect with three generations in curved spacetime becomes rather lengthy and require a separate discussion for normal hierarchy and inverted hierarchy. The major influence of gravitation is the energy redshift caused by a factor $(1-\frac{\mathcal{R}\hbar^2}{16m_e^2c^2})^{-1}e^{2\Phi}$. The numerical results are then given in two scenarios; matter effect only and the overall survival probability including the matter effect and the later stage vacuum oscillations.

The extra resonance dip of higher energy only exists in the normal mass hierarchy. Therefore detecting it can help solve the mass hierarchy problem. Besides this significant difference, the mass hierarchy also plays a role in determining the survival probability curve of neutrinos and the dip at lower energy, even though detecting such effect is rather difficult, if not impossible in the future.

For astronomical objects with varying densities, a method of calculation is offered. In curved spacetime there might exist zero, one or even multiple resonance layers, though these effects require high-precision experiments in order to detect, but actually they may become significant only at the vicinities of spacetime singularities. At the present stage it would be rather difficult to acquire any detailed information about structures and density profiles of those celestial objects. But with these neutrino experiments we can expect to learn the structures of these celestial objects.

In 2016 the gravitational wave emitted from a binary black hole merger [37] was eventually discovered by the LIGO collaboration. One can immediately conjecture that besides the gravitational wave emission, the electromagnetic radiation and neutrino emission would also take place just like the supernova 1987A. Events at such scale, the gravitational field must play a role to induce local quantum effects and produce a great amount of neutrinos. The photons might be scattered away or absorbed by the celestial objects on the long journey to our earth which is much longer than the distance from Supernova 1987A to the earth, but the neutrinos should come along with the gravitational wave. So far, because of lacking detailed information, we cannot determine the neutrino energy spectra yet. Even though the IceCube and ANTARES collaborations reported that they did not detect any neutrino excess accompanying the observation of gravitational wave [38], it is really natural to believe that there is no reason to prohibit neutrino burst along with such events, so we should have received some extra neutrinos. Therefore, one may guess that due to unknown reasons, the neutrinos produced in the black hole merger disappear or somehow evade our detection. The mechanism is worth careful studies and the effects of the gravitational field which deforms the flat spacetime to curved are also needed to investigate. Our recent work is only a step towards the aim and we indeed find the curve spacetime can influence the neutrino propagation. We would like to emphasize that we have no intention to draw a comparison between the energy spectra or production mechanisms of the supernova neutrinos and that of the neutrinos from binary black hole merger. The two sources have completely the different strengths of gravitational fields, thus the possibly observable effects would be different. We are discussing the scenarios and wish to draw researchers' attention to the gravitational effect on neutrino oscillation. On other aspects, because many approximations are adopted in the study, the obtained results are not complete, more work is badly needed for getting a better understanding of mysteries of the cosmic neutrinos.

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Appendix A

In this appendix we review the geometric treatment of neutrino phase differences [6].

Any wave function of neutrino flavor eigenstates can be expanded in the space of mass eigenstates

$$|\Psi_\alpha(x,t)\rangle = \sum_j U_{\alpha j} T^\dagger \left|\nu_j\right\rangle, \tag{66}$$

where $\alpha{=}e,\mu,\tau$ represents the three neutrino flavor eigenstates and $j{=}1,2,3$ represents the three mass eigenstates. U_{ij} is the element of the PMNS matrix and $T^\dagger$ is the spacetime evolution operator. Neutrino mass states are free solutions to the corresponding Dirac equations, thus they can be expressed as plane waves with the Minkowski metric being chosen as $diag(-1,1,1,1)$

$$T^\dagger \left|\nu_j\right\rangle = e^{iP^\mu x_\mu} \left|\nu_j\right\rangle, \tag{67}$$

where $P^\mu = (E, P_j^1, 0, 0)$ is the four-momentum of the neutrino whose trajectory is assumed to be null geodesics. The mass-shell condition is $P_j^1 \approx E - \frac{M_j^2}{2E}$, where M_j is the mass operator that yields the mass of neutrino when acts on a neutrino state.

In curved spacetime where the neutrino travels along a null geodesic line instead of a straight line, Eq. (67) can be re-written as

$$T^\dagger \left|\nu_j\right\rangle = e^{i\int P^\mu p_\mu d\tau} \left|\nu_j\right\rangle, \tag{68}$$

where the tangent vector to the null geodesic line on which the neutrino travels is defined to be $p^\mu = \frac{dx^\mu}{d\tau}$ with τ an affine parameter.

After modifications of wave functions, in curved spacetime the translation of Dirac equation is also needed. On a four dimensional, torsion-free spacetime depicted by a pseudo-Riemann manifold $\mathcal{M}$, the Dirac equation in electron environment with metric sign $(-+++)$ reads

$$\gamma^\mu (i\mathcal{D}_\mu - A_\mu)\Psi + M\Psi = 0, \tag{69}$$

where A_μ denotes the effective potential caused by the electron environment, the covariant derivative $\mathcal{D}_\mu$ and Dirac matrices γ^μ on the manifold are defined as

$$\mathcal{D}_\mu = \partial_\mu + \Gamma_\mu, \tag{70}$$

$$\gamma^\mu = e_a^\mu \gamma^a. \tag{71}$$

The spinor connection Γ_μ, spin connection $\omega_{ab\mu}$ and tetrad e_a^μ are defined to be

$$\Gamma_\mu = \frac{1}{8}\omega_{ab\mu}[\gamma^a, \gamma^b], \tag{72}$$

$$\omega_{ab\mu} = e_a^\nu \partial_\mu e_{b\nu} - e_a^\nu \Gamma^\sigma_{\mu\nu} e_{b\sigma}, \tag{73}$$

$$g_{\mu\nu} e_a^\mu e_b^\nu = \eta_{ab}. \tag{74}$$

The mass shell condition (Hamilton–Jacobi equation) for neutrinos can be obtained from Eq. (69) as

$$(P^\mu + A^\mu)(P_\mu + A_\mu) = -M^2. \tag{75}$$

Since the trajectories are null geodesics where the proper time vanishes, we cannot simply use the proper time as the affine parameter to build the relation between the four-momentum and the tangent vector. But we can still demand that $P^0 = p^0$ and the rest three spatial components parallel to each other. Then from Eq. (75) we obtain

$$P^\mu p_\mu = -\left(\frac{M^2}{2} + p^\mu A_\mu\right). \tag{76}$$

Combining Eq. (66), Eq. (68) and Eq. (76) we finally obtain the propagation of a flavor state in spacetime

$$|\Psi_\alpha(\tau)\rangle = \sum_j U_{\alpha j} e^{i\int (\frac{M^2}{2} + p^\mu A_\mu) d\tau} \left|\nu_j\right\rangle. \tag{77}$$

To calculate Eq. (77), we replace the integral element with a proper length dl defined as

$$dl^2 \equiv g_{ij} dx^i dx^j. \tag{78}$$

Taking square root and dividing the affine parameter into it yields

$$dl = (g_{ij}dx^i dx^j)^{1/2} = (g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau})^{1/2}d\tau. \tag{79}$$

The integral element can be obtained as

$$d\tau = dl(-g_{00}(\frac{dx^0}{d\tau})^2)^{-1/2}. \tag{80}$$

From Eq. (79) to Eq. (80) we have applied the geodesic equation $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \Rightarrow g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = -g_{00}(\frac{dx^0}{d\tau})^2$.

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