Consistent derivation of the constitutive algorithm for plane stress isotropic plasticity. Part I: Theoretical formulation

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A B S T R A C T

A derivation of the projected algorithm for general isotropic three-invariant plasticity models under plane stress conditions is presented. It is obtained by consistently specializing the 3D formulation to the 2D subspace defined by the plane stress condition. Closed-form intrinsic algorithm linearization and a novel expression of the consistent tangent tensor are provided; these are also shown to directly emanate from the analogous quantities pertaining to the fully 3D case. A detailed discussion of the proposed implementation along with a representative set of numerical examples is provided in the second part of this paper [Valoroso, N., Rosati, L., 2008. Consistent derivation of the constitutive algorithm for plane stress isotropic plasticity. Part II: Computational issues. International Journal of Solids and Structures, 46, 92–124.

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1. Introduction

Following the pioneering contributions of Wilkins (1964) and Krieg and Key (1976), early numerical solutions in elastoplasticity have been mainly concerned with 3D and plane strain problems within the so-called $J_2$-flow theory. Plane stress problems have been treated only subsequently, and in many cases by developing computational schemes specifically designed for the case at hand.

Obviously, implicit formulations for plane stress plasticity problems are in principle the same as for the general 3D setting, i.e., they rely upon use of return mapping algorithms and tangent moduli obtained from consistent linearization; however, in this context the zero-stress condition has nonlinear character and a number of techniques have been proposed for dealing with it.

In particular, aiming at avoiding any re-formulation of the local constitutive problem, use can be made of a fully 3D (or plane strain) algorithm with the out-of-plane total strain components, in full or incremental form, treated as additional unknowns that are used to locally enforce the zero-stress constraint, see for instance Hallquist and Benson (1986), Dodds (1987) and Whirley et al. (1989). Basically, the iteration schemes proposed in these papers differ only in the computation of the successive estimations of the out-of-plane strains and exhibit quite different convergence properties. To the best of authors’ knowledge, within this family of methods the most effective implementation is the one proposed by Klinkel and Govindjee (2002), where the out-of-plane strain components are true Lagrange multipliers and are updated based on a Taylor series expansion, that allows a quadratic convergence rate of the local iteration method; moreover, consistent tangent moduli are provided that require only the standard static condensation of the 3D material tangent.

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In the algorithm presented by de Borst (1991) the zero-stress condition is considered in a way as an equilibrium condition in that it is accounted for at the global iteration level, while the local constitutive equations are integrated using the standard 3D stress computation scheme. This gives rise to a form of the equilibrium equations containing the condensed material tangent and an initial stress term vanishing at the converged state. A brief discussion of Klinkel-Govindjee and de Borst’s methods is postponed to Appendix D.

Unlike the mentioned procedures, in the projected algorithm initially proposed by Schreyer et al. (1979) and Jetteur (1986), and later generalized by Simo and Taylor (1986) for use with combined isotropic and kinematic hardening and in conjunction with the generalized midpoint rule (see also Fuschi et al. (1992)) use is made of a 2D formulation. The basic idea developed in these works is that of projecting the constitutive equations onto the subspace defined by the plane stress condition so that the zero-stress constraint is identically satisfied by construction. This algorithm has gained great popularity in recent years and it has been implemented in various forms (see e.g. Fuschi et al. (1994), Lourenço et al. (1997), Lee et al. (1998), Montáns (2004), among others.

The interpretation of the plane stress condition in the more general setting of plasticity under mixed stress-strain control is discussed by Klisinski et al. (1992) and the relevant implementation is addressed by Ritto-Corrêa and Camotim (2001) where, although limited to the $J_2$ case, the family of algorithms for the entire range of mixed control situations is derived. Finally, for the plane stress $J_2$ model a closed-form solution is also available; it has been provided by Simo and Govindjee (1988) for the case of linear kinematic hardening and later extended by Alfano et al. (1999) to encompass linear isotropic hardening and Perzyna-like viscoplasticity.

For the isotropic case of interest, a common feature of the solution schemes based on the projected algorithm and presented in the quoted contributions is that they directly address the case of $J_2$ plasticity, what naturally prompts, due to the inherent simplicity of von Mises criterion, the adoption of very special relationships, such as the one occurring between the in-plane components of the back-stress and the plastic strain tensors; more generally, particular hypotheses are made which hinder from developing an approach that can be easily generalized to more complex models, the only paper by Fuschi et al. (1994) partly representing an exception.

Objectives of this work are the extension of the treatment first developed by Palazzo et al. (2001) to general isotropic plasticity models under plane stress conditions and the development of the relevant implementation in a way to take full advantage of the isotropic character of the elastic constitution and of the yield function.

In this respect, in this first part of the paper is presented a general methodology for deriving the plane stress projected algorithm for three-invariant plasticity models. The generality lies in the fact that the proposed solution scheme is consistently derived from the 3D algorithm; this is obtained by suitably specializing the 3D formulation to a 2D ambient space, what allows, besides the fully intrinsic (matrix-free) implementation of the whole computational procedure, the structure of the return mapping algorithm and the formal expression of the consistent tangent tensor of the 3D case to be completely preserved.

The outline of the paper is as follows.

In Section 2 it is presented a summary of the the 3D continuum problem; the time-discretized problem and the implementation of the stress computation algorithm are given in Section 2.1. A novel expression for the consistent tangent tensor, more compact and effective with respect to other existing expressions, is then provided in Section 3. In Section 4 is discussed the mapping of tensor objects from the 3D ambient space, where they are initially defined, to a suitable subspace of the original one. The consistent mapping methodology is exploited in Section 5 to derive the projected return algorithm for the plane stress case from that of the 3D problem; the plane stress specialization of the elastoplastic compliance and the expression of the algorithmic tangent tensor are also presented. Algorithmic issues concerning the implementation of the proposed procedure along with a representative set of numerical examples are provided in the second part of this paper (Valoroso and Rosati, 2008).

2. Constitutive model. Basic continuum formulation

Let $\varepsilon$ be the strain measure at a point $X$ of a structural model, i.e., a typical quadrature point of the finite element mesh; addressing small deformations one has the additive decomposition:

$$
\varepsilon = \varepsilon + p
$$

(1)

$\varepsilon$ and $p$, respectively, denoting the elastic and plastic strain tensors.

For the purely mechanical case, the stored energy function can be given in fully decoupled form as (Lubliner, 1990)

$$
\psi(\varepsilon, \alpha) = \psi_\sigma(\varepsilon) + \psi_h(\alpha)
$$

where $\psi_\sigma$ and $\psi_h$ are isotropic functions, both assumed to be twice differentiable with positive definite Hessian. They represent in turn the elastic energy and the hardening potential, the latter characterizing the inelastic response in terms of a strain-like variable $\alpha$ that, for the ensuing developments, will be partitioned into a tensorial variable $\eta$ and a scalar variable $\zeta$, respectively, accounting for the kinematic and isotropic hardening mechanisms.

The constitutive relationships for the stress-like variables follow from the standard thermodynamic argument (Coleman and Gurtin, 1967). In particular, for linear elasticity one has the constitutive law for the Cauchy stress:

$$
\sigma = \mathbf{C} : \varepsilon
$$
\[ \sigma = \mathbb{E}(\varepsilon - \mathbf{p}) \]  
\( \mathbb{E} \) being the elastic tensor:

\[ \mathbb{E} = 2G\mathbb{I} + \lambda(\mathbb{1} \otimes \mathbb{I}) = 2G(\mathbb{1} \mathbb{E} \mathbb{1}) + \lambda(\mathbb{1} \otimes \mathbb{1}) \]

where \( G \) and \( \lambda \) are the Lamé's moduli. The symbols \( \mathbb{I} \) and \( \mathbb{1} \) in (3), respectively, denote the rank-four and rank-two identity tensors that are related each other through the so-called square tensor product (Del Piero, 1979); its definition along with the relevant composition rules are given in Appendix A.

Kinematic and isotropic hardening of the model are assumed to be governed by the relationships:

\[ \dot{\beta} = \dot{h}_{\text{kin}} \eta \]
\[ \dot{\vartheta} = \dot{h}_{\text{iso}}(\zeta) \]

where \( \dot{h}_{\text{kin}} = h_{\text{kin}} \mathbb{I} \)

\( h_{\text{kin}} \) being the kinematic hardening modulus and \( h_{\text{iso}} \) a general nonlinear isotropic hardening function. The pair of internal variables \( \beta \) and \( \vartheta \), usually referred to as the back stress and the drag stress, account for the evolution in stress space of the yield locus, defined as the level set of a scalar function that will be assumed strictly convex and smooth. In particular, in the remainder we shall make reference to a general isotropic yield function given as

\[ \phi(\sigma, \beta, \vartheta) = \phi(\sigma, \vartheta) = \varphi(I_1, J_2, J_3) - \vartheta - Y_0 \]

where \( Y_0 \) depends upon the initial yield limits of the material,

\[ I_1 = \text{tr}(\sigma); \quad J_2 = \frac{1}{2} \text{tr}S^2; \quad J_3 = \frac{1}{3} \text{tr}S^3 \]

are stress invariants and \( S \) is the deviator of the relative stress \( \sigma = \mathbf{p} + \beta \).

The definition of the model is completed by providing the evolutionary equations for the strain-like variables; for standard materials they follow from the principle of maximum plastic dissipation as

\[ \begin{align*}
\dot{\mathbf{p}} &= \nabla_{\text{kin}} \eta = \eta \\
\dot{\zeta} &= \dot{\gamma}
\end{align*} \]

where \( \nabla_{\text{kin}} \) is the gradient of the yield function with respect to \( \tau \):

\[ \nabla_{\text{kin}} = d_1 \phi(\tau, \vartheta) = \frac{\partial \phi}{\partial I_1} \mathbb{1} + \frac{\partial \phi}{\partial J_2} S + \frac{\partial \phi}{\partial J_3} \left( S^2 - \frac{2}{3} J_2 \mathbb{1} \right) \]

\[ = \left( d_1 \phi - \frac{2}{3} J_2 d_2 \phi \right) \mathbb{1} + d_2 \phi S + d_3 \phi S^2 \]

\[ = n_{i1} \mathbb{1} + n_{i2} S + n_{i3} S^2 \]

and \( \dot{\gamma} \) is the plastic consistency parameter that is characterized as a Lagrange multiplier subject to the Karush–Kuhn–Tucker (KKT) conditions (Bertsekas, 1982):

\[ \phi(\tau, \vartheta) \leq 0; \quad \dot{\gamma} \geq 0; \quad \dot{\gamma} \phi(\tau, \vartheta) = 0 \]

Assuming, without loss of generality, that prior to any loading the plastic strain \( \mathbf{p} \) and the internal variable \( \eta \) are both identically zero, one infers the equality \( \eta = \mathbf{p} \) from (8), whence the constitutive equation for the back stress tensor:

\[ \beta = \dot{h}_{\text{kin}} \mathbf{p} \]

currently addressed in the literature (Lemaitre and Chaboche, 1990). Accordingly, for the problem at hand the variables \( \eta \) and \( \beta \) can be dropped out from the formulation and the stress computation can be carried out by making reference only to the relative stress tensor:

\[ \tau = \mathbb{E} \varepsilon - \mathbb{H} \mathbf{p} \]

\( \mathbb{H} \) being the elasto-hardening operator:

\[ \mathbb{H} = \mathbb{E} + H_{\text{kin}} = (2G + h_{\text{kin}}) \mathbb{I} + \lambda(\mathbb{1} \otimes \mathbb{1}) \]

2.1. Discrete formulation and stress computation for 3D

The strain partition hypothesis (1) entails a similar additive structure for the evolution problem (8) that can be recast in the form:
where the coefficients of the expansion are given by

\[
\begin{align*}
\mathbf{d}^T \mathbf{\phi} = n_{12} d \mathbf{S} + n_{21} d \mathbf{S}^T + (1 \otimes d) \mathbf{n}_{12} + \mathbf{S} \otimes d \mathbf{n}_{12} + n_{12} d \mathbf{n}_{12} + (1 \otimes d) \mathbf{n}_{12} + (\mathbf{S} \otimes 1) + (1 \otimes \mathbf{S}) + e_1(\mathbf{1} \otimes \mathbf{1}) + e_2(\mathbf{S} \otimes \mathbf{S}) + e_3(\mathbf{1} \otimes \mathbf{S}) + e_4(\mathbf{S} \otimes \mathbf{1}) + e_5(\mathbf{S} \otimes \mathbf{S}) + e_6(\mathbf{S} \otimes \mathbf{S}) + e_7(\mathbf{S} \otimes \mathbf{S})
\end{align*}
\]
\[ e_1 = d_2 \varphi = n_{u2}; \quad e_2 = d_3 \varphi = n_{u3} \]
\[ e_3 = d_{11} \varphi - \frac{1}{3} d_2 \varphi - \frac{4}{3} f_2 d_{13} \varphi + \frac{4}{9} f_2^2 d_{33} \varphi \]
\[ e_4 = d_{12} \varphi - \frac{2}{3} d_1 \varphi - \frac{2}{3} f_2 d_{23} \varphi \]
\[ e_5 = d_{13} \varphi - \frac{2}{3} j_1 d_{33} \varphi \]
\[ e_6 = d_{22} \varphi; \quad e_7 = d_{23} \varphi; \quad e_8 = d_{33} \varphi \]
\( d_\varphi \) (\( d_2^2 \varphi \)) being the first (second) derivative of the yield function with respect to the generic (pair of) invariant(s). With specific reference to the celebrated five-parameter concrete model (Willam and Warnke, 1974), the explicit evaluation of the coefficients \( e_1, \ldots, e_8 \) is provided in the second part of the paper.

Based on the previous considerations one has the representation formula for \( G_H \):
\[ G_H = g_1 (1 \otimes 1) + g_2 (S \otimes 1 + 1 \otimes S) + g_3 (1 \otimes 1) + g_4 (S \otimes 1 + 1 \otimes S) + g_5 (S^2 \otimes 1 + 1 \otimes S^2) + g_6 (S \otimes S) + g_7 (S^2 \otimes S + S \otimes S^2) + g_8 (S^2 \otimes S^2) \]
\[ (20) \]
where it has been set:
\[ g_1 = \frac{1}{2G + h_{\text{kin}}} + \gamma e_1; \quad g_3 = -\frac{\lambda}{(2G + h_{\text{kin}})(2G + h_{\text{kin}} + 3\lambda)} + \gamma e_3 \]
\[ g_i = \gamma e_i, \quad i \neq \{1, 3\} \]

The assumed isotropic elastic behaviour and the isotropy of the yield function allow one to find out an explicit representation formula for \( G_H^{-1} \) by means of basic theorems for isotropic tensor-valued functions of tensor arguments. Omitting the details, the final result reads:
\[ G_H^{-1} = i_1 (1 \otimes 1) + i_2 (S \otimes 1 + 1 \otimes S) + i_3 (S \otimes S) + i_4 (1 \otimes 1) + i_5 (S \otimes 1 + 1 \otimes S) + i_6 (S^2 \otimes 1 + 1 \otimes S^2) + i_7 (S \otimes S) + i_8 (S^2 \otimes S + S \otimes S^2) + i_9 (S^2 \otimes S^2) \]
\[ (22) \]
where the unknown coefficients \( i_1, i_2, \ldots, i_9 \) can be determined by enforcing the condition \( G_H G_H^{-1} = 1 \), see Palazzo et al. (2001) for a detailed discussion.

3. A novel expression of the consistent tangent tensor

The consistent tangent moduli tensor, introduced in the seminal paper by Simo and Taylor (1985), expresses the elastoplastic constitutive equation in a linearized sense, i.e.
\[ ds = (d_\sigma, d_\epsilon) de = E_{\text{tan}} de \]
\[ (23) \]
and represents an essential ingredient for the effective solution of the elastoplastic boundary value problem via the full Newton–Raphson algorithm.

The expression of the consistent tangent presented by Alfano and Rosati (1998):
\[ E_{\text{tan}} = E - E_{\text{tan}} E_H^{-1} E = \frac{(E - E_{\text{tan}} E_H^{-1} E_H) (E - E_{\text{tan}} E_H^{-1} E_H) \cdot n_H \cdot n_H}{(E - E_{\text{tan}} E_H^{-1} E_H) \cdot n_H \cdot n_H + H_{\text{iso}}} \]
\[ (24) \]
where
\[ E_{\text{tan}} = (\gamma d_\varphi^2 \phi) - (\gamma d_\varphi^2 \phi) G_H^{-1} (\gamma d_\varphi^2 \phi) \]
generalizes the one originally contributed for \( J_2 \) plasticity in Simo and Taylor, 1985 since it applies to general isotropic elastoplastic models endowed with linear hardening.

Relationship (24) is however not optimal in view of actual implementation since it is obtained by considering the problem of the construction of the consistent tangent tensor completely disjoint from that of the stress computation. Indeed, relationship (24) fails to exploit the fact that all what is needed for implementing the consistent tangent can be readily obtained from the linearized form of the residual equations, thus resulting in an expression which is quite complicated and not prone to an immediate specialization to the plane stress case, what ultimately represents the main target of the present work. For this reason we provide hereafter a more direct and effective derivation of the consistent tangent.

Since the return mapping algorithm is formulated in terms of the relative stress tensor \( \epsilon \), we start by considering the linearizations of (2) and (11):
\[ d_\epsilon \sigma = E - E d_p \]
\[ d_\epsilon \epsilon = E - E d_p p \]
whose comparison yields the relationship between the consistent tangent and the rank-four tensor \( d_\epsilon \epsilon \) as
\[ d_c \sigma = E_{\tan} = E - EE^{-1}E + EE^{-1}d_c \tau \]

The previous equation can be effectively exploited for computing the consistent tangent since the derivative \( d_c \tau \) is easily obtained from the residual equations at the local converged state, i.e., from the system:

\[
\begin{bmatrix}
-E_{H}^{-1}E \delta \varepsilon^e \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
G_H & 0 & n_H \\
0 & H_{iso}^{-1} & 0 \\
(n_H)^T & 0 & -1
\end{bmatrix} \begin{bmatrix}
d_c \tau \delta \varepsilon^e \\
d_r \delta \varepsilon^e \\
d_r \delta \varepsilon^e
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix} \quad \forall \delta \varepsilon^e
\]

embodiing the full linearization of (11) and (4) along with Prager’s consistency condition (Lubliner, 1990).

Solving for \( d_c \tau \) gives then:

\[ d_c \tau = D_{\tan} E_{H}^{-1}E \]

for

\[ D_{\tan} = G_{H}^{-1} - \frac{G_{H}^{-1}n_{H} \otimes G_{H}^{-1}n_{H} + H_{iso}}{G_{H}^{-1}n_{H} \cdot n_{H} + H_{iso}} \]

On account of (25), the expression of the consistent tangent is thus obtained as

\[ E_{\tan} = E - EE_{H}^{-1}E + (EE_{H}^{-1} \otimes EE_{H}^{-1}) \ D_{\tan} \]

where the definition of the square product between rank-four tensors follows from the generalization of that between rank-two tensors, see Appendix A.

In the light of the previous considerations it is apparent that, once the tensor \( G_{H}^{-1} \) is known, the expression of the consistent tangent can be arrived at via elementary algebraic operations since one basically needs to compute the symmetric tensor \( EE_{H}^{-1} \), whose expression reads:

\[ EE_{H}^{-1} = E_{H}^{-1}E = m_1(1 \otimes 1) + m_2(1 \otimes 1) \]

with coefficients \( m_1 \) and \( m_2 \) given by

\[ m_1 = \frac{2G}{2G + h_{lin}}; \quad m_2 = \frac{\lambda h_{lin}}{(2G + h_{lin})(2G + h_{lin} + 3\lambda)} \]

Combining the previous relationships with (9) and (22), one has then:

\[ E_{\tan} = t_1(1 \otimes 1) + t_2(S \otimes 1 + 1 \otimes S) + t_3(S \otimes S) + t_4(1 \otimes 1) + t_5(S \otimes 1 + 1 \otimes S) + t_6(S^2 \otimes 1 + 1 \otimes S^2) + t_7(S \otimes S) + t_8(S^2 \otimes S + S \otimes S^2) + t_9(S^2 \otimes S^2) \]

where \( t_1, \ldots, t_9 \) are polynomials functions of \( J_2, J_3, i_1, \ldots, i_6, h_{lin} \) and of the Lamé’s moduli.

4. The plane stress problem

Let us now turn to discuss the plane stress problem, i.e., the 3D problem for which the plane stress condition:

\[ \sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \]

is enforced.

In particular, this section is devoted to obtain some preliminary results that will be later exploited to show that the projected algorithm for plane stress isotropic elastoplasticity can be derived from the corresponding 3D one and that reference can always be made to the expression of the original yield function, i.e., the one given in terms of the 3D invariants (7). In this respect, it is well known that in the plane stress problem the out-of-plane strain components are basically dependent variables; hence, the constitutive problem can be formulated in terms of the in-plane stress tensor:

\[ \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \]

assuming the in-plane total strain as driving variable. Before going into further details let us draw the reader’s attention on the terminology adopted in (28). We shall always carefully distinguish between the symbols \( \sigma \) and \( \bar{\sigma} \) to denote, respectively, 3D and 2D tensors although in the plane stress case the non-zero components of the former are the same as those of the latter. This aspect purports to the more general issue, discussed in the next subsection, of the mapping onto a 2D ambient space of tensor objects originally defined in a 3D setting.
4.1. Mapping of 3D tensors to a 2D ambient space

Given a symmetric 3D tensor \( \mathbf{A} \in \text{Lin} \) we shall use the symbols \( \mathbf{A}_p \) and \( \mathbf{A}_c \) to denote, respectively, the 3D tensors collecting the components of \( \mathbf{A} \) spanning the complementary subspaces \( \mathcal{S}_p \) and \( \mathcal{S}_c \) of Lin defined by the in-plane and out-of-plane components of \( \mathbf{A} \).

In other words we introduce the splitting:

\[
\mathbf{A} = \mathbf{A}_p + \mathbf{A}_c
\]

(29)

to indicate synthetically the decomposition of \( \mathbf{A} \) whose associated matrix form is:

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & A_{13} \\
0 & 0 & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix}
\]

Further, the notation \( \mathbf{A} \) (\( \mathbf{\hat{A}} \)) will be used to denote the 2D tensor collecting the in-plane (out-of-plane) components of \( \mathbf{A} \) and we shall write:

\[
\mathbf{A}_p := \mathbf{\hat{A}}; \quad \mathbf{A}_c := \mathbf{\hat{A}}
\]

(30)

where the symbol := has been used to emphasize that the 3D tensors on the left are isometrically mapped to the 2D tensors on the right.

The matrix form of the 2D rank-two tensors \( \mathbf{\hat{A}} \) and \( \mathbf{\hat{A}} \) read, respectively:

\[
\begin{bmatrix}
\mathbf{\hat{A}}_{11} & \mathbf{\hat{A}}_{12} \\
\mathbf{\hat{A}}_{12} & \mathbf{\hat{A}}_{22}
\end{bmatrix}; \quad \begin{bmatrix}
\mathbf{\hat{A}}_{11} & \mathbf{\hat{A}}_{13} \\
\mathbf{\hat{A}}_{13} & \mathbf{\hat{A}}_{33}
\end{bmatrix}
\]

To make the reader fully acquainted with the above terminology, let us make reference to the yield function \( \phi \) defined as in (6). The tensor \( \mathbf{d}_r \phi \), see also (9), is a rank-two 3D tensor representing the derivative of \( \phi \) with respect to the 3D tensor \( \mathbf{r} \) while:

\[
d_{r_i} \phi; \quad d_i \phi
\]

(31)

denote in turn the rank-two 2D tensors obtained as the derivatives of \( \phi \) with respect to the in-plane (\( \mathbf{r}_i \)) and out-of-plane (\( \mathbf{r}_i \)) parts of \( \mathbf{r} \).

The 2D tensors (31) are isomorphic to the rank-two tensors \( \mathbf{d}_p \phi \) and \( \mathbf{d}_c \phi \) defined on a 3D ambient space, i.e.

\[
d_p \phi = d_i \phi; \quad d_c \phi = d_i \phi
\]

that are in turn related to \( d_i \phi \) via the decomposition (29) as

\[
d_i \phi = (d_i \phi)_p + (d_i \phi)_c = d_p \phi + d_c \phi
\]

A decomposition analogous to (29) can be given for a generic rank-four symmetric tensor \( \mathbf{A} \) defined on a 3D space by setting:

\[
\mathbf{A}_r = \mathbf{A}_{pp} + \mathbf{A}_{pc} + \mathbf{A}_{cp} + \mathbf{A}_{cc}
\]

(32)

To be more specific one can appeal to the well-known technique of representing 3D fourth-order tensors as \( 6 \times 6 \) matrices. Hence, adopting the component ordering of (A.4), see Appendix A, \( \mathbf{A}_{pp}, \mathbf{A}_{pc}, \mathbf{A}_{cp} \) and \( \mathbf{A}_{cc} \) will denote in turn rank-four tensors whose representative matrix has only nine entries different from zero, namely, the components of \( \mathbf{A} \) contained in the \( 3 \times 3 \) upper-left, upper-right, bottom-left and bottom-right submatrices of the matrix form of \( \mathbf{A} \).

However, as a general rule, the vector (matrix) representation of rank-two (-four) tensors can be left in the backstage since it has to be invoked only for numerical implementation, an issue addressed in the second part of this paper. In order to emphasize this point we now proceed to a formal derivation of relationships (29) and (32).

To this end we introduce the rank-two symmetric tensor

\[
\mathbf{P} = e_1 \otimes e_1 + e_2 \otimes e_2
\]

with \( e_1, e_2 \) basis vectors for the plane stress subspace. An obvious choice provides the matrix representation of \( \mathbf{P} \) as

\[
[\mathbf{P}] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

It is immediate to verify that the rank-four complementary projectors:

\[
\mathbf{P}_c = \mathbf{I} - \mathbf{P}
\]

(33)

respective map a second-order symmetric tensor \( \mathbf{A} \) onto the subspaces \( \mathcal{S}_p \) and \( \mathcal{S}_c \) of Lin, that is:
\[ A_p = P A = P A P^T; \quad A_c = P_c A = A - P A P^T \]  
(34)

and fulfill the properties:
\[ P = P^T; \quad P_c = P_c^T; \quad P P_c = P_c P = O; \quad P^n = P \quad \forall n > 0 \]  
(35)

In the same way, for a symmetric rank-four 3D tensor \( A \), the decomposition (32) can be expressed as
\[ A = P A P^T + 2 \text{sym}(P A P^T) + P_c A P_c^T \]  
(36)

and we shall write the relevant matrix form as
\[ \begin{bmatrix} A_{pp} & A_{pc} \\ \vdots & \vdots \\ A_{cp} & A_{cc} \end{bmatrix} = \begin{bmatrix} [P A P^T] & [P A P^T] \\ \vdots & \vdots \\ [P_c A P_c^T] & [P_c A P_c^T] \end{bmatrix} \]  
(37)

Note that each tensor on the right-hand side of (36) has only nine independent components. When such tensors are represented in matrix form, the relevant non-zero components do occupy, in turn, the upper-left \([P A P^T]\), the upper-right \([P A P^T]\), the bottom-left \([P_c A P_c^T]\) and the bottom-right \([P_c A P_c^T]\) \(3 \times 3\) submatrix of the relevant \(6 \times 6\) matrix. Such submatrices are the matrix form of the rank-four 2D tensors isometric to the rank-four 3D tensors appearing in (36); it is precisely in this sense that the symbols \([P A P^T]\), \([P A P^T]\), \([P_c A P_c^T]\) and \([P_c A P_c^T]\) have to be understood.

In the following the diagonal terms of the matrix representation (37) will be synthetically addressed by adopting the following terminology:
\[ P A P^T \leadsto \hat{A}; \quad P_c A P_c^T \leadsto \hat{A} \]  
(38)

the symbol \( \leadsto \) denoting the isometric isomorphism between the rank-four 3D tensor on the left and rank-four 2D tensor on the right-hand side. Application of the previous definitions to the rank-four 3D tensor (19) representing the second derivative of \( \phi \) gives:
\[ d^2_{\sigma \sigma} \phi = d^2_{\eta \eta} \phi + d^2_{\nu \nu} \phi + d^2_{\eta \nu} \phi + d^2_{\nu \eta} \phi \]
whose matrix form reads:
\[ \begin{bmatrix} d^2_{\sigma \sigma} \phi \\ \vdots \\ d^2_{\nu \nu} \phi \end{bmatrix} = \begin{bmatrix} d^2_{\eta \eta} \phi \\ \vdots \\ d^2_{\nu \nu} \phi \end{bmatrix} \]  
(39)

where
\[ d^2_{\sigma \sigma} \phi; \quad d^2_{\eta \eta} \phi; \quad d^2_{\nu \nu} \phi \]
are 2D rank-four tensors.

### 4.2. Formal derivation of the plane elasticity tensor

Use of the previous formalism allows for an immediate derivation of the plane stress elastic relationship from the 3D one. From (2) one has the inverse relation:
\[ E^{-1} \sigma = \varepsilon - p \]  
(40)

\( E^{-1} \) being the 3D elastic compliance:
\[ E^{-1} = \frac{1}{2G} (1 \otimes I) - \frac{\lambda}{2G(2G + 3\lambda)} (I \otimes I) \]  
(41)

Premultiplication of (40) by \( P \) and \( P_c \) gives, respectively:
\[ P E^{-1} P^T \sigma = P \varepsilon - P p \]
\[ P_c E^{-1} P_c^T \sigma = P_c \varepsilon - P_c p \]  
(42)

since \( \sigma = P \sigma = P^T \sigma \) owing to the plane stress condition. Accordingly, application of the mapping (30), to (42), yields:
\[ E^{-1} \sigma = \varepsilon - p \]  
(43)

where
\[ P E^{-1} P^T = \tilde{I} \]  
(44)

that is, recalling (33) and the composition rules (A.3)

\[ PE^{-1} P^T = \frac{2G}{2G + \lambda} \mathbf{P} \otimes \mathbf{P} - \frac{\lambda}{2G(2G + \lambda)} \mathbf{P} \otimes \mathbf{P} = \frac{1}{2G} \mathbf{1} \otimes \mathbf{1} - \frac{\lambda}{2G(2G + \lambda)} \mathbf{1} \otimes \mathbf{1} = \tilde{I} \]  
(45)

having denoted by \( \mathbf{1} \) and \( \mathbf{1} \otimes \mathbf{1} \) the 2D rank-two and rank-four identity tensors.

Finally, using Sherman–Morrison–Woodbury’s formula, one gets:

\[ (\tilde{E})^{-1} = 2G\tilde{I} + \frac{2G\lambda}{2G + \lambda} (\mathbf{1} \otimes \mathbf{1}) = 2G\tilde{I} + \tilde{\lambda} (\mathbf{1} \otimes \mathbf{1}) = 2G\tilde{I} + \frac{vE}{1 - v^2} (\mathbf{1} \otimes \mathbf{1}) = \tilde{E} \]  
(46)

where

\[ \tilde{\lambda} = \frac{2G\lambda}{2G + \lambda} \]  
(47)

is the plane Lamé constant \( \lambda \) while \( E \) and \( v \) are the Young modulus and the Poisson’s ratio. Note that the last equality in (46) is inferred by the definition of \( \tilde{E} \) as the rank-four 2D tensor mapping the in-plane elastic strain \( \mathbf{e} = \mathbf{\varepsilon} - \mathbf{p} \) to the in-plane stress \( \tilde{\sigma} \).

Furthermore, Eq. (42) establishes a linear relation between the out-of-plane components of \( \varepsilon \), \( \mathbf{p} \), and the in-plane components of the stress tensor. Actually, from (41) and (A.5), see Appendix A, one recovers the well-known relationship:

\[ e_{33} - p_{33} = - \frac{\lambda}{2G(2G + \lambda)} (\sigma_{11} + \sigma_{22}) = -\frac{\lambda}{E} (\sigma_{11} + \sigma_{22}) = e_{33} \]

that allows the evaluation of the out-of-plane component of the total strain \( \varepsilon \) once the 33-component of the plastic strain \( \mathbf{p} \) has been computed. This last one can in turn be updated as \((\mathbf{p})_{33} + \gamma (\mathbf{m})_{33}\), where the last addend follows from (9) evaluated for the stress tensor at solution.

5. The projected return mapping algorithm for plane stress

Following the general ideas detailed in the previous sections we shall now provide a consistent derivation of the projected plane stress constitutive algorithm.

To this end we recall the 3D time-discretized counterpart of the flow equation (13)1:

\[ \mathbf{e} - \mathbf{e}^\varepsilon = \mathbf{p} + \mathbf{p}_0 = -\gamma \mathbf{d}_t \phi \]  
(48)

\( \mathbf{e}^\varepsilon \) being the trial elastic strain. The splitting of (48) into two nonlinear equations is straightforward since it only requires the use of (30) to get:

\[ \dot{\mathbf{e}} - \mathbf{e}^\varepsilon = -\mathbf{p} + \mathbf{p}_0 = -\gamma \mathbf{d}_t \phi \]
\[ \dot{\mathbf{e}} - \mathbf{e}^\varepsilon = -\mathbf{p} + \mathbf{p}_0 = -\gamma \mathbf{d}_t \phi \]  
(49)

This is however less so for the residual equation expressed in terms of the relative stress, i.e., (14)1, since its in-plane counterpart has to be constructed in a way to account for the plane stress constraint. The formal derivation of the in-plane and out-of-plane parts of the residual equation (14)1 can be obtained by following a path of reasoning analogous to the one adopted for the construction of the 2D elasticity tensor (46) in Section 4.2, i.e., by carrying out the static condensation of the 3D elastic relationship via the systematic use of the mappings (30) and (38) along with Sherman–Morrison–Woodbury formula, see also Appendix B. In this respect, the following considerations are in order.

Use of (29) and (34) allows one to rephrase Eq. (11) as

\[ P^T \tau + P^T \tau = E P^T \varepsilon + E P^T \varepsilon - E H P^T \mathbf{p} - E H P^T \mathbf{p} \]  
(50)

whereby, on account of (35), one obtains

\[ P E^{-1} P^T \tau + P E^{-1} P^T \tau = P \varepsilon - P E^{-1} E H P^T \mathbf{p} - P E^{-1} E H P^T \mathbf{p} \]  
(51)

via premultiplication by \( P E^{-1} \); moreover, one has the out-of-plane relationship:

\[ P^T \tau = P^T \varepsilon (\mathbf{e} - \mathbf{p}) - \varepsilon_{\text{kin}} P^T \mathbf{p} \]  
(52)

that follows from (11) using (12), premultiplying by \( P^T \) and noting that \( \varepsilon_{\text{kin}} \) and \( P^T \) commute, see also (5). The plane stress condition (27) is equivalent to

\[ P^T \varepsilon (\mathbf{e} - \mathbf{p}) = 0 \]  
(53)

so that, premultiplication of (52) by \( P E^{-1} \) yields:
\[ \mathbf{PE}^{-1} \mathbf{P}^T \mathbf{\tau} = -\mathbf{PE}^{-1} \mathbf{\mathbb{E}}_{\text{kin}} \mathbf{P}^T \mathbf{p} = \mathbf{PE}^{-1} (\mathbf{E} - \mathbf{E}_H) \mathbf{P}^T \mathbf{p} = -\mathbf{PE}^{-1} \mathbf{E}_H \mathbf{P}^T \mathbf{p} \]  

(54)

where the last equality is inferred from (35)2 and (35)3.

Accordingly, Eq. (51) reduces to

\[ \mathbf{PE}^{-1} \mathbf{P}^T \mathbf{\tau} = \mathbf{PE} - \mathbf{PE}^{-1} \mathbf{E}_H \mathbf{P}^T \mathbf{p} \]  

(55)

whose mapping to the plane stress subspace gives, on account of (54):

\[ \mathbf{\tilde{E}}^{-1} \mathbf{\tau} = \mathbf{\tilde{e}} - \mathbf{\tilde{p}}_H \]

or equivalently,

\[ \mathbf{\tau} = \mathbf{\tilde{e}} - \mathbf{\tilde{E}} \mathbf{\tilde{p}}_H \]  

(56)

being

\[ \mathbf{\tilde{p}}_H := \mathbf{PE} - \mathbf{PE}^{-1} \mathbf{E}_H \mathbf{P}^T \mathbf{p} \]  

(57)

The explicit expression of the term \( \mathbf{\tilde{p}}_H \) is obtained using (12) and (41) to compute the rank-four tensor \( \mathbf{PE}^{-1} \mathbf{E}_H \mathbf{P}^T \) as

\[ \mathbf{PE}^{-1} \mathbf{E}_H \mathbf{P}^T = \frac{2\mathbf{G} + \mathbf{h}_\text{kin}}{2\mathbf{G}} \mathbf{p}\mathbf{p} - \frac{\lambda \mathbf{h}_\text{kin}}{2\mathbf{G}(2\mathbf{G} + 3\lambda)} \mathbf{p} \otimes \mathbf{p} \]

whence

\[ \mathbf{p} \mathbf{p}_H = \mathbf{PE}^{-1} \mathbf{E}_H \mathbf{P}^T \mathbf{p} = \frac{2\mathbf{G} + \mathbf{h}_\text{kin}}{2\mathbf{G}} \mathbf{p}\mathbf{p} - \frac{\lambda \mathbf{h}_\text{kin}}{2\mathbf{G}(2\mathbf{G} + 3\lambda)} \mathbf{p} \otimes \mathbf{p} \]

Accordingly, on account of (57):

\[ \mathbf{p} \mathbf{p}_H = (\mathbf{p}_H)\mathbf{p} = \mathbf{PE} \mathbf{p}_H \]

it is also:

\[ \mathbf{p}_H = \left[ \frac{2\mathbf{G} + \mathbf{h}_\text{kin}}{2\mathbf{G}} (\mathbf{1} \otimes \mathbf{1}) - \frac{\lambda \mathbf{h}_\text{kin}}{2\mathbf{G}(2\mathbf{G} + 3\lambda)} (\mathbf{1} \otimes \mathbf{1}) \right] \mathbf{p} \]

Use of equation (46) supplies then:

\[ \mathbf{\tilde{E}} \mathbf{p}_H = \mathbf{\tilde{e}} \mathbf{p}_H \]  

(58)

where

\[ \mathbf{\tilde{E}} = \mathbf{\tilde{E}} - \mathbf{\tilde{H}}_{\text{kin}} = (2\mathbf{G} + \mathbf{h}_\text{kin}) (\mathbf{1} \otimes \mathbf{1}) + \lambda \mathbf{p} (\mathbf{1} \otimes \mathbf{1}) \]  

(59)

is the 2D elasto-hardening stiffness tensor. One has then the following constitutive relation for the in-plane relative stress:

\[ \mathbf{\tau} = \mathbf{\tilde{e}} - \mathbf{\tilde{H}}_{\text{kin}} \mathbf{p} \]  

(60)

in place of (56).

Obviously, the out-of-plane part of (11) follows from (52) and (53) using the mapping onto the out-of-plane subspace as

\[ \mathbf{\tau} = -\mathbf{\tilde{H}}_{\text{kin}} \mathbf{p} \]  

(61)

In summary, the staggered form of the residual Eq. (14) is arrived at as

\[ \begin{align*}
\mathbf{r}_e &= \mathbf{\tilde{H}}_{\text{kin}}^{-1} (\mathbf{\tau} - \mathbf{\tau}^r) + \gamma \mathbf{d}_\phi (\mathbf{\tau}, \mathbf{\tau}, \vartheta) = \mathbf{0} \\
\mathbf{r}_s &= \mathbf{\tilde{E}}^{-1} (\mathbf{\tau} - \mathbf{\tau}^s) + \gamma \mathbf{d}_\phi (\mathbf{\tau}, \mathbf{\tau}, \vartheta) = \mathbf{0} \\
\mathbf{r}_c &= \mathbf{H}_{\text{kin}}^{-1} (\vartheta - \vartheta^c) - \gamma = \mathbf{0} \\
\mathbf{r}_\phi &= \phi (\vartheta, \mathbf{\tau}, \vartheta) = \mathbf{0}
\end{align*} \]  

(62)

where the dependence of \( \phi \) upon the in-plane and out-of-plane parts of \( \mathbf{\tau} \) has been emphasized by writing \( \phi (\mathbf{\tau}, \mathbf{\tau}, \vartheta) \).

Note that the variables in (62) interact with those in (63) through the tensor \( \mathbf{\tau} \) and that (49)2 establishes a non-linear relation between the out-of-plane plastic strain and the stresses \( \mathbf{\tau} \) and \( \gamma \); accordingly, the linearization of (49)2 yields:

\[ \mathbf{d}_p = \gamma \mathbf{d}_\phi (\mathbf{\tau}, \mathbf{\tau}, \vartheta) + \gamma (\mathbf{d}_{\tau} \mathbf{\phi}) \mathbf{d}_\tau + \mathbf{d}_\tau \mathbf{\phi} \otimes \mathbf{d}_{\gamma} \]  

(64)

that, once combined with the derivative of (61) with respect to \( \mathbf{\tau} \), i.e.

\[ \mathbf{d}_\tau = -\mathbf{\tilde{H}}_{\text{kin}} \mathbf{d}_\tau \]  

(65)
provides:
\[ d_i \dot{\tau} = -\hat{A}_H^{-1} (\gamma d_i^2 \phi + d_i \phi \otimes d_i \gamma) \]  
(66)
where \( \hat{A}_H \) is the 2D rank-four tensor:
\[ \hat{A}_H = \beta H_{\text{kin}} + \gamma d_i^2 \phi \]  
(67)
The perfect analogy between the plane stress return mapping and the 3D stress computation scheme (15) is self-evident by considering the linearized expression of Eq. (63):
\[
\begin{bmatrix}
    \delta \dot{r}_e^{(k)} \\
    \delta \dot{r}_\gamma^{(k)} \\
    \delta \dot{r}_s^{(k)}
\end{bmatrix} =
\begin{bmatrix}
    \hat{G}_H^{(k)} & 0 & \hat{n}_i^{(k)} \\
    0 & (H_{\text{iso}}^{(k)})^{-1} & -1 \\
    \hat{n}_i^{(k)^T} & -1 & -\Theta^{(k)}
\end{bmatrix}
\begin{bmatrix}
    \delta \dot{r}_e^{(k+1)} \\
    \delta \dot{\gamma}^{(k+1)} \\
    \delta \dot{\tau}^{(k+1)}
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}
\]
(68)
where it has been set:
\[
\begin{align*}
    \hat{n}_i^{(k)} &= d_i \phi^{(k)} - (\gamma^{(k)} d_i^2 \phi^{(k)}) (A_H^{(k)})^{-1} d_i \phi^{(k)} \\
    \Theta^{(k)} &= (A_H^{(k)})^{-1} d_i \phi^{(k)} \cdot d_i \phi^{(k)}
\end{align*}
\]
\( \hat{G}_H \) being the rank-four 2D tensor:
\[
\hat{G}_H^{(k)} = \hat{G}_H^{-1} + \gamma^{(k)} d_i^2 \phi^{(k)} - (\gamma^{(k)} d_i^2 \phi^{(k)}) (A_H^{(k)})^{-1} (\gamma^{(k)} d_i^2 \phi^{(k)})
\]
(69)
As shown later in Appendix C, \( \hat{G}_H \) and \( A_H \) are both positive-definite; since it is also \( H_{\text{iso}}^{(k)} > 0 \), one can solve for the increment \( \delta \dot{\gamma}^{(k+1)} \) to get:
\[
\delta \dot{\gamma}^{(k+1)} = \frac{\delta \dot{r}_e^{(k)} - (\hat{G}_H^{(k)})^{-1} \hat{n}_i^{(k)} + H_{\text{iso}}^{(k)} \hat{r}_s^{(k)}}{(\hat{G}_H^{(k)})^{-1} \hat{n}_i^{(k)^T} \cdot \hat{n}_i^{(k)} + \Theta^{(k)} + H_{\text{iso}}^{(k)}}
\]
(70)
Once the increments \( \delta \dot{r}_e^{(k+1)} \) and \( \delta \dot{\gamma}^{(k+1)} \) have been computed from (68), it is necessary to evaluate the out-of-plane stress tensor \( \dot{\tau} \). This can be done via the updating formula:
\[
\delta \dot{r}_e^{(k+1)} = -(A_H^{(k)})^{-1} (\hat{r}_e^{(k)} + \gamma d_i^2 \phi^{(k)} \delta \dot{\gamma}^{(k+1)} + d_i \phi^{(k)} \delta \dot{\tau}^{(k+1)})
\]
that follows from the linearization of (62).

**Remark 5.1.** In absence of kinematic hardening the linearization by (66) becomes superfluous since in this case the out-of-plane relative stress \( \dot{\tau} \) is identically zero, see e.g. (61); this circumstance can be accounted for by setting formally \( A_H^{-1} = 0 \). However, even in this case the derivative \( d_i \phi \) is in general different from zero since, according to (9), \( (d_i \phi)_{33} \neq 0 \) even for vanishing \( \dot{\tau} \), being \( S_{33} \) and \( (S^2)_{33} \) non-zero also for \( \tau_{33} = 0 \).

**Remark 5.2.** The solution of the plane stress constitutive problem could have been obtained via the simultaneous linearization of (62) and (63). As in the 3D case we have chosen to solve (68) via (70) mainly in view of the actual implementation of the consistent tangent, whose derivation is detailed in Section 6.

### 5.1. Remarks on the specialization of \( \hat{G}_H \) to \( \hat{G}_H \)

A direct comparison between the expression (16) of the 3D tensor \( \hat{G}_H \) and that of the 2D tensor \( \hat{G}_H \) given in (69) shows that this last one contains an extra term originating from the linearization of the out-of-plane part of the relative stress, see also (66).

Postponing to the second part of the paper the explicit evaluation of this term, we shall now focus on the first two addends of \( \hat{G}_H \), that will be referred to in the following as \( \hat{G}_{H_1} \). Since these terms have the same formal expressions as those of their 3D counterparts, it is quite natural to wonder if \( \hat{G}_{H_1} \) can be obtained as the mapping of the 3D tensor \( \hat{G}_H \), that is:
\[
\hat{P} \hat{G}_H \hat{P}^T = \hat{P} (E_H^{-1} + \gamma d_i^2 \phi) \hat{P}^T = \hat{E}_H^{-1} + \gamma d_i^2 \phi = \hat{G}_{H_1}
\]
In this respect, being by definition \( \hat{P} (d_i^2 \phi) \hat{P}^T = d_i^2 \phi \), it only remains to address if the 3D elasto-hardening compliance tensor can be directly mapped to the plane stress subspace, i.e.,
\[
\hat{P} E_H^{-1} \hat{P}^T = \hat{E}_H^{-1}
\]
(71)
Unfortunately, this is not so. This fact can be easily recognized by comparing the mapping of (18) to the plane stress subspace, i.e.
\[
\hat{E}_H^{-1} = \frac{1}{2G + \lambda} \left[ \lambda (1 \otimes 1) - \frac{\lambda}{2G + \lambda + 3\lambda} (1 \otimes 1) \right]
\]
with the expression of the inverse of \((59)\) computed via Sherman–Morrison–Woodbury formula:

\[
\tilde{\varepsilon}_H^{-1} = \frac{1}{2G + h_{\text{kin}}} \left[ \mathbf{1} \otimes \mathbf{1} \right] - \frac{\lambda_p}{(2G + h_{\text{kin}} + 2\lambda_p)} \left( \mathbf{1} \otimes \mathbf{1} \right)
\]

(72)

A possible way to ensure that \(\tilde{\varepsilon}_H^{-1}\) matches with \(\tilde{\varepsilon}_H^{-1}\) or, equivalently, that the tensor \((\tilde{\varepsilon}_H^{-1})^{-1}\) coincides with \(\tilde{\varepsilon}_H\) is that of introducing a “modified” elastic tensor \(\varepsilon^*\) obtained from two fictitious Lamé moduli \(G^*\) and \(\lambda^*\)

\[
\tilde{\varepsilon}_H = \tilde{\varepsilon}^* + h_{\text{kin}} = (2G^* + h_{\text{kin}})(\mathbf{1} \otimes \mathbf{1}) + \lambda^*(\mathbf{1} \otimes \mathbf{1})
\]

Hence, being:

\[
((\tilde{\varepsilon}_H^{-1})^{-1}) = (2G^* + h_{\text{kin}}) \left[ \mathbf{1} \otimes \mathbf{1} \right] + \frac{\lambda^*}{2G^* + h_{\text{kin}} + \lambda^*} \left( \mathbf{1} \otimes \mathbf{1} \right)
\]

a direct comparison with \((59)\) shows that the modified elastic constants that allow the equality

\[
((\tilde{\varepsilon}_H^{-1})^{-1}) = \tilde{\varepsilon}_H
\]

to be fulfilled, are given by

\[
G^* = G; \quad \lambda^* = \frac{(2G^* + h_{\text{kin}})\lambda_p}{2G^* + h_{\text{kin}} - \lambda_p} - \frac{2G\lambda(2G^* + h_{\text{kin}})}{4G^2 + 2Gh_{\text{kin}} + 2h_{\text{kin}}}
\]

(73)

**Remark 5.3.** It is worth noting that the need for the modified Lamé moduli \(G^*\) and \(\lambda^*\) basically arises from the fact that we wish to relate the general 3D stress computation algorithm to the plane stress one, the key ingredient of both being the tangent compliance tensor \((G_{H2}^0\) for 3D and \(G_{H2}^0\) for plane stress). We emphasize that this does not mean at all that we change material data since use of \((73)\) serves only to ensure the fulfilment of \((71)\), see also Appendix B for a discussion on this point from a different perspective.

Assuming that the coefficients \(g_i\) of the representation formula \((20)\) have been computed using \((73)\), use of the mapping \((38)\) and of the composition rules \((A.3)\) yields:

\[
\hat{G}_H = g_1(\mathbf{1} \otimes \mathbf{1}) + g_2(\mathbf{1} \otimes \mathbf{1}) + g_3(\mathbf{1} \otimes \mathbf{1}) + g_4(\mathbf{1} \otimes \mathbf{1}) + g_5(\mathbf{1} \otimes \mathbf{1}) + g_6(\mathbf{1} \otimes \mathbf{1}) + g_7(\mathbf{1} \otimes \mathbf{1})
\]

(74)

where the identity \(\tilde{\mathbf{s}}^2 = \mathbf{s}^2\), stemming from the plane stress assumption, has been taken into account. Substitution into \((74)\) of the expression of \(\mathbf{s}^2\) from the 2D Cayley–Hamilton theorem:

\[
\tilde{\mathbf{s}}^2 = l_5 \hat{\mathbf{s}} - l_5^2 \mathbf{1}
\]

(75)

and use of Rivlin’s identity for second-order 2D tensor polynomials \(\text{(Rivlin, 1955)}\):

\[
\mathbf{S} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{S} = (\mathbf{S} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{S}) + l_5(\mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{1})
\]

(76)

yields the representation formula:

\[
\hat{G}_H = \hat{g}_1(\mathbf{1} \otimes \mathbf{1}) + \hat{g}_2(\mathbf{1} \otimes \mathbf{1}) + \hat{g}_3(\mathbf{1} \otimes \mathbf{1}) + \hat{g}_4(\mathbf{1} \otimes \mathbf{1})
\]

(77)

where

\[
\hat{g}_1 = g_1 + l_5^2 g_5
\]

\[
\hat{g}_2 = g_2 + g_4 \quad \hat{g}_3 = g_2 + g_4 + l_5^2 g_5
\]

\[
\hat{g}_4 = g_2 + g_4 + l_5^2 g_5
\]

\[
l_5 \quad \text{and} \quad l_5^2 \text{ being the principal invariants of the 2D tensor} \hat{\mathbf{s}}.
\]

6. Consistent tangent tensor for plane stress

Objective of this section is that of evaluating the consistent tangent tensor \((\text{Simo and Taylor, 1985})\), i.e., the 2D rank-four tensor:

\[
\tilde{\varepsilon}_\text{tan} = d_4 \sigma
\]

consistent with the stress computation algorithm discussed in Section 5. To this end we shall follow the approach detailed in Section 3 and start from considering the linearization of the 2D elastic law and of the constitutive equation \((60)\) for the in-plane relative stress:
\[ d_\varepsilon \sigma = \hat{\varepsilon} - \hat{\varepsilon} d_\varepsilon \mathbf{p}; \quad d_\tau = \hat{\tau} - \hat{\tau} d_\tau \mathbf{p} \]
to get by comparison:
\[ \hat{\varepsilon}_{\text{tan}} = \hat{\varepsilon} - \hat{\varepsilon} \hat{\varepsilon}_{\text{H}}^{-1} \hat{\varepsilon} + \hat{\varepsilon} \hat{\varepsilon}_{\text{H}}^{-1} d_\varepsilon \hat{\tau} \]
In perfect analogy with the 3D case, the expression of the derivative \( d_\varepsilon \hat{\tau} \) can be obtained from the linearization with respect to the driving variable \( \hat{\tau} \) of the residual equations at the local converged state as
\[ d_\varepsilon \hat{\tau} = \hat{\Delta}_{\text{tan}} \hat{\varepsilon}_{\text{H}}^{-1} \hat{\varepsilon} \]
for
\[ \hat{\Delta}_{\text{tan}} = \hat{G}_{\text{H}}^{-1} - \frac{\hat{G}_{\text{H}}^{-1} \mathbf{n}_H \otimes \hat{G}_{\text{H}}^{-1} \mathbf{n}_H}{\mathbf{n}_H \cdot \hat{\mathbf{n}}_H + \Theta + H_{\text{iso}}} \]
whence the expression of the plane stress consistent tangent is obtained as
\[ \hat{\varepsilon}_{\text{tan}} = \hat{\varepsilon} - \hat{\varepsilon} \hat{\varepsilon}_{\text{H}}^{-1} \hat{\varepsilon} + \left( \hat{\varepsilon} \hat{\varepsilon}_{\text{H}}^{-1} \mathbf{D} \hat{\varepsilon}_{\text{H}}^{-1} \right) \hat{\Delta}_{\text{tan}} \]
where
\[ \hat{\varepsilon}_{\text{H}}^{-1} = \hat{\varepsilon}_{\text{H}}^{-1} \hat{\varepsilon} = \hat{m}_1 (\hat{\mathbf{1}} \otimes \hat{\mathbf{1}}) + \hat{m}_2 (\hat{\mathbf{1}} \otimes \hat{\mathbf{1}}) \]
\( \hat{m}_1 \) and \( \hat{m}_2 \) being given by
\[ \hat{m}_1 = \frac{2G}{2G + \hat{h}_{\text{kin}}}; \quad \hat{m}_2 = \frac{\hat{\lambda} \hat{h}_{\text{lin}}}{(2G + \hat{h}_{\text{kin}})(2G + \hat{h}_{\text{kin}} + 2\hat{\lambda})} \]
Using the matrix representation, the evaluation of \( \hat{G}_{\text{H}}^{-1} \) simply requires the inversion of a \( 3 \times 3 \) matrix. However, in the second part of this paper, it is shown that \( \hat{G}_{\text{H}}^{-1} \) can be given an explicit (inverse-free) representation in terms of tensor products of the 2D mappings \( \mathbf{1} \) and \( \mathbf{S} \) of the 3D rank-two tensors \( \mathbf{1} \) and \( \mathbf{S} \) onto the plane stress subspace. Furthermore, this approach allows one to obtain the coefficients of the representation of \( \hat{G}_{\text{H}}^{-1} \) and \( \hat{\varepsilon}_{\text{tan}} \) directly from those of the relevant 3D tensors by following the same procedure presented in Palazzo et al. (2001) for the 3D case.

7. Concluding remarks

The projected algorithm for general three-invariant elastoplastic models under plane stress conditions is consistently derived from the general return mapping scheme. This has been obtained without introducing any special assumption or treatment as done for other solution algorithms which, on the contrary, are specifically tailored for very particular cases. The proposed approach crucially relies upon the developments carried out in the paper illustrating how the tensor relationships entering the 3D constitutive algorithm can be specialized to a dimensionally reduced ambient space.

The perfect analogy between the fully 3D case and the 2D plane stress one is given further evidence by a novel expression of the consistent tangent tensor derived in the paper. Actually, besides being more compact and effective with respect to earlier expressions, the presented form of the 3D consistent tangent is amenable to an immediate specialization to the plane stress case.

A full detail of the actual implementation of the devised solution strategy is addressed in the second part of the paper (Valoroso and Rosati, 2008).

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Appendix A. Some tensor algebra

In this section we recall some basic properties of tensor calculus involving products between second-order tensors defined on a \( n \)-dimensional \( (n = 3 \) or \( n = 2 \) \) inner product space \( \mathcal{V} \). We shall also denote by \( \text{Lin} \) (Lin) the space of all second-(fourth-) order tensors on \( \mathcal{V} \).

The dyadic product of two elements \( \mathbf{A}, \mathbf{B} \in \text{Lin} \) is defined as (Gurtin, 1981):
\[ (\mathbf{A} \otimes \mathbf{B}) \mathbf{C} = (\mathbf{B} \cdot \mathbf{C}) \mathbf{A} = \text{tr}(\mathbf{B}^T \mathbf{C}) \mathbf{A} \quad \forall \mathbf{C} \in \text{Lin} \quad (A.1) \]
where \( \text{tr} \) is the trace operator and the superscript \( T \) denotes the transpose.

According to the definition given by Del Piero (1979) the so-called square tensor product is defined as
\[ (\mathbf{A} \otimes \mathbf{B}) \mathbf{C} = \mathbf{A} \mathbf{B}^T \quad \forall \mathbf{C} \in \text{Lin} \quad (A.2) \]
Hence, denoting by \( \mathbf{1} \) the identity tensor in Lin, it turns out to be:
\[
\mathbf{1} = \mathbf{I} \otimes \mathbf{I}
\]
where \( \mathbf{1} \) is the identity tensor in Lin.

The component form of dyadic and square tensor products in a Cartesian frame are provided by
\[
(\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_i B_j \quad (\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ik} B_{jl} \quad \forall \, \mathbf{A}, \mathbf{B} \in \text{Lin}
\]
and can be easily derived from the definitions (A.1) and (A.2).

Given \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \text{Lin} \) and invoking (A.1) and (A.2), the following composition rules
\[
(\mathbf{A} \otimes \mathbf{B}) \otimes (\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A} \otimes \mathbf{C}) \otimes (\mathbf{B} \otimes \mathbf{D})
\]
\[
(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})
\]
\[
(\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{A} \otimes \mathbf{B})\mathbf{C}
\]
can be shown to hold.

In order to represent tensor quantities in the finite element implementation and to detail the mappings introduced in (30) and (38), it is customary to represent second-order tensors as vectors and rank-four tensors as matrices. This can be done according to the following vector representation of symmetric stress (\( \mathbf{T} \)) and strain (\( \mathbf{U} \)) tensors:
\[
\mathbf{T} = [T_{11}, T_{22}, T_{12}, T_{33}, T_{23}, T_{13}]^T
\]
\[
\mathbf{U} = [U_{11}, U_{22}, 2U_{12}, U_{33}, 2U_{23}, 2U_{13}]^T
\]
yielding the representation of 2D second-order tensors as
\[
\hat{\mathbf{T}} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix} \quad \hat{\mathbf{U}} = \begin{bmatrix} U_{11} \\ U_{22} \\ 2U_{12} \end{bmatrix}
\]

Appendix B. Inversion and static condensation

Let \( \mathbf{A} \) be a symmetric invertible rank-four tensor. The matrix representation of the inverse \( \mathbf{A}^{-1} \) in the form (37):
\[
[\mathbf{A}^{-1}] = \begin{bmatrix} ([\mathbf{A}^{-1}]_{pp}) & ([\mathbf{A}^{-1}]_{pc}) \\ \vdots & \ddots \end{bmatrix}
\]
follows from enforcement of the condition \( \mathbf{A}[\mathbf{A}^{-1}] = \mathbf{I} \), that yields:
\[
([\mathbf{A}^{-1}]_{pp}) = ([\mathbf{A}]_{pp} - [\mathbf{A}]_{pc})^{-1}
\]
\[
([\mathbf{A}^{-1}]_{pc}) = -([\mathbf{A}]_{pp} - [\mathbf{A}]_{pc})^{-1}([\mathbf{A}]_{cc} - ([\mathbf{A}]_{cp})^{-1}([\mathbf{A}]_{pp} - [\mathbf{A}]_{pc})^{-1}
\]
\[
([\mathbf{A}^{-1}]_{cp}) = -([\mathbf{A}]_{cc} - ([\mathbf{A}]_{cp})^{-1}([\mathbf{A}]_{pp} - [\mathbf{A}]_{pc})^{-1}
\]
\[
([\mathbf{A}^{-1}]_{cc}) = ([\mathbf{A}]_{cc} - ([\mathbf{A}]_{cp})^{-1}([\mathbf{A}]_{pp} - [\mathbf{A}]_{pc})^{-1}
\]
Note that all terms in (B.2) are well-defined since \([\mathbf{A}]_{cc}\) and \([\mathbf{A}]_{pp}\) are principal minors of the matrix form of \( \mathbf{A} \) that is invertible by assumption.

Consider now the tensor equation in the unknown \( \mathbf{X} \):
\[
\mathbf{A} \mathbf{X} = \mathbf{Y}
\]
where \( \mathbf{X} \) and \( \mathbf{Y} \) are rank-two symmetric tensors. Using (29), (30) and (37) the matrix form of (B.3) is obtained as
\[
\begin{bmatrix} [\mathbf{A}]_{pp} & [\mathbf{A}]_{pc} \\ [\mathbf{A}]_{cp} & [\mathbf{A}]_{cc} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{Y}} \\ \hat{\mathbf{Y}} \end{bmatrix}
\]
whose solution can be represented in the form:
\[
\begin{align*}
\hat{\mathbf{X}} &= ([\mathbf{A}]_{pp} - [\mathbf{A}]_{pc})^{-1}([\mathbf{Y}] - [\mathbf{A}]_{pp} [\mathbf{A}]_{cc}^{-1} [\mathbf{Y}]) \\
\hat{\mathbf{X}} &= ([\mathbf{A}]_{cc} - [\mathbf{A}]_{cp})^{-1}([\mathbf{Y}] - [\mathbf{A}]_{pc} [\mathbf{A}]_{pp}^{-1} [\mathbf{Y}])
\end{align*}
\]
Hence, for \( \mathbf{Y} = \{0\} \), one has
\[
\hat{\mathbf{X}} = ([A_{pp} - A_{pc} A_{cc}^{-1} A_{cp}])^{-1} \mathbf{Y} = ([A^{-1}]_{pp} \hat{\mathbf{Y}}
\] (B.5)
as the comparison between (B.4) and (B.2) shows.

Introducing the 2D tensor \( \hat{A}^{-1} \) via the mapping:
\[
(A^{-1})_{pp} = p \hat{A}^{-1} p^T = \hat{A}^{-1}
\]
relationship (B.5) states that the matrix form of the tensor \( \hat{A}^{-1} \) coincides with the inverse of the matrix obtained from the static condensation of \( [A] \) with respect to the upper-left block, i.e.
\[
[A^{-1}] = ([A_{pp} - A_{pc} A_{cc}^{-1} A_{cp}])^{-1} = [\hat{A}_{ulk}]^{-1}
\] (B.6)
Hence, if (B.3) is specialized to the elastic relationship (2), the above equation provides the plane elasticity matrix:
\[
[\hat{E}] = [E_{pp} - E_{pc} E_{cc}^{-1} E_{cp}]
\] (B.7)
that is, the matrix form of (46).

Remark B.1. Unlike (B.7), the matrix representation of \( \hat{E}_H^{-1} \) cannot be obtained via the static condensation of the matrix associated with the 3D operator \( E_H \). This can be easily checked by considering the 2D tensor \( \hat{E}_H \) defined by the mapping:
\[
p E_H p^T = \hat{E}_H = (2G + h_{kin})(1 \otimes 1) + \lambda (1 \otimes 1)
\]
and comparing its inverse:
\[
(\hat{E}_H)^{-1} = \frac{1}{2G + h_{kin}} \left[ (1 \otimes 1) - \frac{\lambda}{2G + h_{kin} + 2\lambda} (1 \otimes 1) \right]
\] (B.8)
with (72). It is then apparent that (72) can be obtained from (B.8) by setting \( \lambda = \lambda_p \).

On the other hand, inverting both sides of Eq. (B.6) and setting \( A = E_H^{-1} \) yields
\[
[\hat{E}_H^{-1}] = [(E_H^{-1})_{ulk}]
\]
so that we ultimately infer:
\[
[\hat{E}_H^{-1}] \vert_{\lambda = \lambda_p} = [(E_H^{-1})_{ulk}] \vert_{\lambda = \lambda_p}
\]

The above relationship states that the matrix representation of \( \hat{E}_H^{-1} \) can be obtained by substituting \( \lambda = \lambda_p \) in the matrix obtained by statically condensing the matrix form of the 3D operator \( E_H^{-1} \) with respect to its upper-left block. Clearly, this is not equivalent to first setting \( \lambda = \lambda_p \) in the 3D elastic operator \( E \) and then statically condensing the sum \( E_H = E + h_{kin} \) owing to the nonlinearity of the static condensation operation.

Appendix C. A proof of the positive-definiteness of \( \hat{G}_H \)

Let \( \hat{B}_H \) be the rank-four 2D tensor:
\[
\hat{B}_H = \gamma d_{vi}^2 \phi - (\gamma d_{vi}^2 \phi)(\hat{A}_H)^{-1}(\gamma d_{vi}^2 \phi)
\] (C.1)
so as to write:
\[
\hat{G}_H = \hat{E}_H^{-1} + \hat{B}_H
\]
In order to show that \( \hat{G}_H \) is positive-definite, and hence invertible, we shall prove that both \( \hat{E}_H^{-1} \) and \( \hat{B}_H \) fulfill the same property. The tensor \( \hat{E}_H^{-1} \) turns out to be positive-definite since it is the inverse of the sum of two positive-definite tensors:
\[
\hat{E}_H = \hat{E} + \hat{h}_{kin}
\]
Actually, \( \hat{E} \) is the plane elastic operator and \( \hat{h}_{kin} \) is the rank-four two-dimensional tensor whose associated matrix reads:
\[
[[h_{kin}]] = \begin{bmatrix}
[\mathbb{H}_{kin}] & [\mathbb{O}] \\
[\mathbb{O}] & [\mathbb{H}_{kin}]
\end{bmatrix}
\]
see also (37); hence, \([\mathbb{H}_{kin}]\) and \([\mathbb{H}_{kin}]\), being principal minors of a positive-definite matrix, turn out to be positive-definite according to a well-known theorem of linear algebra (Golub and Van Loan, 1989).
Further, the strict convexity of $\phi$ ensures that
\[ \tilde{A}_H = \tilde{H}_L^{-1} + \gamma \mathbf{d}_T \phi \]
is positive-definite as well. In order to prove the positive-definiteness of $\tilde{H}_H$, consider the rank-four 3D tensor $\mathbf{I}$, whose associated matrix is given by
\[ [\mathbf{L}] = \begin{bmatrix} \mathbf{I}^\top \mathbf{I} & \mathbf{I} \mathbf{I} \end{bmatrix} + \gamma \begin{bmatrix} \mathbf{d}_T^2 \phi \mathbf{I} : \mathbf{d}_T^2 \phi \mathbf{I} \end{bmatrix} = \begin{bmatrix} [\mathbf{L}_{pp}] : [\mathbf{L}_{pc}] \\ [\mathbf{L}_{pc}] : [\mathbf{L}_{cc}] \end{bmatrix} \] (C.2)

On account of (B.2) and (C.2) one has then:
\[ [\tilde{H}_H]^{-1} = ([L]^{-1})_{pp} = ([L]_{pp} - [L]_{pc} [L]_{cc}^{-1} [L]_{pc})^{-1} \] (C.3)

Since $\mathbf{I}$ is positive-definite, such is $L^{-1}$; hence, the positive definiteness of $\tilde{H}_H$ follows from that of $((L^{-1})_{pp})$

**Appendix D. Iterative algorithms for plane stress**

Alternative to the intrinsic approach presented in Section 5, the zero-stress condition:
\[ \mathbf{P}_c \mathbf{\sigma} = \mathbf{0} \] (D.1)
can be incorporated within a 3D formulation based on a local or a global iterative algorithm that allows one to numerically obtain the appropriate modification to the constitutive equations. Two of these algorithms are briefly revisited in the following.

In the local iteration method developed by Klinkel and Govindjee (2002) the stress constraint is enforced via Lagrange multipliers; in particular, use is made of the linearization of (D.1) in the form:
\[ \mathbf{P}_c \mathbf{\sigma}^{(k+1)} = \mathbf{P}_c \mathbf{\sigma}^{(k)} + \mathbf{P}_c \mathbf{d}_c \mathbf{\sigma}^{(k)} \mathbf{P}_c^T \mathbf{\delta}^{(k+1)} + \mathbf{P}_c \mathbf{d}_c \mathbf{\sigma}^{(k)} \mathbf{P}_c^T \mathbf{\delta}^{(k+1)} = \mathbf{0} \] (D.2)

The superscript $(k)$ being a local iteration counter.

Since the local return mapping takes place for given in-plane total strain $\mathbf{P} \varepsilon$ (i.e., $\mathbf{P} \varepsilon \mathbf{\delta}^{(k+1)} = \mathbf{0}$) and the constraint equation (D.1) has to hold at solution, one infers from the previous relationship:
\[ \mathbf{P}_c \mathbf{\sigma}^{(k)} + \mathbf{P}_c \mathbf{d}_c \mathbf{\sigma}^{(k)} \mathbf{P}_c^T \mathbf{\delta}^{(k)} = \mathbf{0} \] (D.3)

whose mapping to the 2D subspace complementary of the plane stress subspace gives:
\[ \mathbf{\sigma}^{(k)} + \mathbf{\tilde{E}}_{\text{tan}} \mathbf{\delta}^{(k+1)} = \mathbf{0} \] (D.4)

whereby one obtains the updated out-of-plane total strain:
\[ \mathbf{\tilde{\varepsilon}}^{(k+1)} = \mathbf{\tilde{\varepsilon}}^{(k)} - \left( \mathbf{E}_{\text{tan}} \right)^{-1} \mathbf{\tilde{\varepsilon}}^{(k)} \] (D.5)
to be used for computing the solution at next iteration of the plane constitutive problem via the 3D algorithm.

Iterations on (D.4) are performed until a suitable norm of the out-of-plane stress is zero up to the desired tolerance; at this stage (D.1) is fulfilled and the material tangent of the zero-stress model needed for the global Newton iteration has to be computed.

To this end consider the partitioned form of the the tangent relationship (23):
\[ \begin{align*}
\mathbf{P} \mathbf{d} \mathbf{\varepsilon} &= \mathbf{P}_c \mathbf{d}_c \mathbf{\varepsilon} + \mathbf{P}_c \mathbf{d}_c \mathbf{\varepsilon} \mathbf{P}_c^T \mathbf{d} \mathbf{\varepsilon} \\
\mathbf{P}_c \mathbf{d} \mathbf{\varepsilon} &= \mathbf{P}_c \mathbf{d}_c \mathbf{\varepsilon} \mathbf{P}_c^T \mathbf{d} \mathbf{\varepsilon} + \mathbf{P}_c \mathbf{d}_c \mathbf{\varepsilon} \mathbf{P}_c^T \mathbf{d} \mathbf{\varepsilon}
\end{align*} \] (D.6)

Owing to the zero-stress condition, it turns out to be:
\[ \mathbf{P}_c \mathbf{d} \mathbf{\varepsilon} = \mathbf{0} \] (D.7)

however, since the out-of-plane strain increment is in general non-zero, i.e.
\[ \mathbf{P}_c^T \mathbf{d} \mathbf{\varepsilon} \neq \mathbf{0} \] (D.8)
the in-plane material tangent moduli depend upon the variation of the full strain state. Accordingly, in order to obtain the reduced tangent matrix one has to use first the static condensation of the 3D tangent and then map the result onto the plane stress subspace to get the plane stress tangent matrix:
\[ [\mathbf{E}_{\text{tan}}] = [\mathbf{E}_{\text{tan,pp}}] - [\mathbf{E}_{\text{tan,pc}}] [\mathbf{E}_{\text{tan,cc}}]^{-1} [\mathbf{E}_{\text{tan,cp}}] \] (D.9)

see also Appendix B.
The local iteration procedure is entirely implemented at the Gauss point level and only requires the treatment of the Lagrange multipliers (out-of-plane strains) as additional internal variables that, as such, are updated in the same way as the plastic variables, i.e. at the end of each converged equilibrium step. This is in contrast with the global iteration algorithm based on the procedure proposed by de Borst (1991), where the out-of-plane strains are updated iteratively during global equilibrium iterations and can no longer be interpreted as Lagrange multipliers.

The starting point for the development of the global iteration algorithm is the linear expansion of the actual stress tensor in the form:

$$\sigma^{(j+1)} = \sigma^{(j)} + E^{(j)} \dashv \delta \varepsilon^{(j+1)}$$  \hspace{1cm} (D.10)

where the superscript \( \langle j \rangle \) denotes the global equilibrium iteration counter.

Using the partitioned form (D.6) for Eq. (D.10) and enforcing the zero-stress condition (D.1) at iteration \( j + 1 \) one arrives at:

$$[\varepsilon^{(j+1)}] = [\varepsilon^{(j)}] - [E^{(j)}]^{-1} ([E^{(j)}] \dashv [\delta \varepsilon^{(j+1)}]) \hspace{1cm} (D.11)$$

and

$$[\delta \varepsilon^{(j+1)}] = [E^{(j)}] \dashv [\delta \varepsilon^{(j+1)}] + [\delta \varepsilon^{(j)}] - [E^{(j)}] \dashv [E^{(j)}]^{-1} [\sigma^{(j)}] \hspace{1cm} (D.12)$$

where \([E^{(j)}] \dashv \delta \varepsilon^{(j+1)}\) is the condensed tangent given by (D.9). The above equations provide in turn the updating formula for the out-of-plane strain, that has to be computed before entering the 3D return mapping algorithm, and the stress terms relevant to the consistent elastoplastic moduli matrix and to the stress divergence vector.

It is worth noting that, though very different from a conceptual standpoint, the local iteration algorithm and the global iteration method look very similar and can be both implemented at the Gauss point level. In our experience no difference has been found in terms of computational efficiency between the two methods; the only practical differences lie in the different storage requirements and updating procedures for the auxiliary variables that are needed to compute the solution in the two cases.

References


