Anti-symmetric periodic solutions for the third order differential systems

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ABSTRACT

The existence of periodic solutions with anti-symmetries for third order pendulum-like differential systems has been dealt with by using the topological degree. The results obtained enrich the relative works.

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1. Introduction

Afuwape et al. [1] studied the existence of solutions of the periodic boundary value problems for the third order differential equation

\[ x''' + ax'' + bx' + g(t, x) = e(t), \]
\[ x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \]

where \( a < 0 (> 0) \), \( g(t, x) \in C([0, 2\pi] \times R) \) and

\[ xg(t, x) > 0 \quad ( < 0), \quad t \in [0, 2\pi], \quad |x| > d. \]

Andres [2] also studied the existence of periodic solutions for analogue equation (1). These kinds of problems have been applied in various fields, such as nonlinear oscillations [3], electronic theory [4], biological model and other models [5, 6]. However, from the mathematical point of view, the study for third order differential equations is more difficult and complicated than for second equations. This is because some classical tools with respect to second order equations can not be applied in third order equations; third order equation has not variational construct, and the geometric characters of third order derived function are less explicit than of second order derived function. Just as above, in the past few decades, the study for third order differential equation has been paid attention to by many scholars, and many results relative to the periodic boundary value problems for the third order differential equations have been obtained (see [6–8] and references therein).

Anti-symmetric periodic solutions, as a special type of periodic solutions, have been investigated by many researchers. Mawhin and other people studied the existence of anti-periodic solutions for the second order pendulum-type equations (see [9–12]). Okochi [13] discussed the existence of anti-periodic solutions for equation

\[ x'(t) + \varphi(t) \ni f(t), \quad a.e. \ t \in R \]
where \( f(t + \pi) = -f(t), \partial \phi(t) \) is sub-differential of an even function \( \phi \) on a real Hilbert space. [14,15] also discussed the solvability of anti-periodic solution for the first-order evolution equations. However, it seems that many results of anti-periodic solutions are only related to the first or second order differential equations, works for the third order differential equations are very few.

The purpose of this paper is to study the solvability of anti-periodic solutions for the third order differential systems

\[
x'' + Ax'' + \frac{d}{dt} \nabla F(x) + G(t, x) = E(t),
\]

where \( A \) is a real number \((n \times n)\) matrix, \( F(x) \in C^2(\mathbb{R}^n, \mathbb{R}), \nabla F(x) \) is gradient with respect to \( x = (x_1, x_2, \ldots, x_n) \), \( G(t, x) \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n), E(t) \in C(\mathbb{R}^+, \mathbb{R}^n). \) That is, we will prove the Eq. (3) has a solution \( x(t) \) satisfying anti-symmetry

\[
x(t + \pi) = -x(t), \quad t \in \mathbb{R}^+.
\]

Obviously, if (4) holds, then \( x(t + 2\pi) = x(t) \) for \( t \in \mathbb{R}^+ \), i.e. \( x(t) \) is also a \( 2\pi \)-periodic solution of Eq. (3).

2. Main results

We assume that \( A \) is positive definite or negative definite \((n \times n)\) matrix of real numbers, and might as well assume \( A \) is a diagonal matrix written as \( A = \text{diag}(a_1, a_2, \ldots, a_n) \), where \( a_i > 0 (<0) \) \((i = 1, 2, \ldots, n) \). Otherwise, we may take a homothetic transformation to turn \( A \) into diagonal matrix. For convenience, we introduce some notations as follows

\[
C^k_{2\pi} = \{ x(t) \in C^k(\mathbb{R}^+, \mathbb{R}^n) : x(t + 2\pi) = x(t) \},
\]

\[
C^0_{2\pi} = \{ x(t) \in C^0_{2\pi} : x(t + \pi) = -x(t) \},
\]

\[
|x|_\infty = \max_{t \in [0, 2\pi]} |x(t)|, \quad x(t) \in C^0_{2\pi},
\]

\[
|\cdot|_p \text{ denotes norm in } L^p[0, 2\pi], \quad |\cdot, \cdot| \text{ and } |\cdot| \text{ denote inner product and vector norm in } \mathbb{R}^n, \text{ respectively.}
\]

For making use of Leray-Schauder degree to prove that the Eq. (3) has a solution satisfying (4), we first consider homotopic systems with (3)

\[
Ax'' = \lambda \left( -x'' - \frac{d}{dt} \nabla F(x) - G(t, x) + E(t) \right) \quad (0 \leq \lambda \leq 1).
\]

**Lemma 1.** Suppose that

1. \( A = \text{diag}(a_1, a_2, \ldots, a_n) \), \( a_i > 0 \), \( i = 1, 2, \ldots, n \)
2. \( G(t + 2\pi, x) = G(t, x) \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n) \), there exists a positive number \( d, \) such that for \( |x| > d \)

\[
|G(t, x)| \leq L(|a_i|_1, |E|_1) d \leq K(|a_i|_1, |E|_1)
\]

(3) \( E(t + 2\pi) = E(t) \in C(\mathbb{R}^+, \mathbb{R}^n). \)

Then \( 2\pi \)-periodic solution \( x(t) \) of Eq. (5) satisfies

\[
|x|_2^2 \leq L(|a_i|_1, |E|_1) \bar{x} + K(|a_i|_1, |E|_1)
\]

(6)

where \( a_d(t) \in L^1[0, 2\pi] : |G(t, x)| \leq a_d(t) \) for \( |x| \leq d, K, L \) are positive numbers only dependent of \( |a_i|_1, |E|_1, \bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t)dt \)

**Proof.** Define a function \( r : [0, 2\pi] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)

\[
r(t, x) = \begin{cases} 
|\frac{G(t, x)}{d}| \frac{d}{dt} \left( t, \frac{d}{|x|} x \right) & \text{if } |x| \geq d \\
0 & \text{if } 0 < |x| < d \\
|x| & \text{if } |x| = 0.
\end{cases}
\]

Letting

\[
h(t, x) = G(t, x) - r(t, x),
\]

we see that

\[
|\langle r(t, x), x \rangle| = 0, \quad |h(t, x)| = |x| \leq 2a_d(t)
\]

(7)

for any \( x \in \mathbb{R}^n, \ t \in [0, 2\pi] \). Multiplying the both sides of Eq. (5) by \( x(t) \), integrating from 0 to \( 2\pi \), and noting

\[
\int_0^{2\pi} \langle x''(t), x(t) \rangle dt = 0, \quad \int_0^{2\pi} \frac{d}{dt} \nabla F(x(t)), x(t) \rangle dt = 0,
\]
we have
\[
\int_0^{2\pi} \langle Ax(t), x'(t) \rangle dt - \lambda \int_0^{2\pi} \langle r(t, x(t)), x(t) \rangle dt - \lambda \int_0^{2\pi} \langle h(t, x(t)), x(t) \rangle dt = -\lambda \int_0^{2\pi} \langle E(t), x(t) \rangle dt.
\]
Writing \( \tilde{a} = \min_{i=1,2,...,n}(a_i) \), \( x(t) = \tilde{x} + \bar{x}(t) \), we obtain from (7) that
\[
\tilde{a} \int_0^{2\pi} |x'(t)|^2 dt \leq \lambda \int_0^{2\pi} \langle h(t, x(t)) - E(t), \bar{x}(t) + \tilde{x} \rangle dt \leq \int_0^{2\pi} (2|a_d(t)| + |E(t)|)(|\tilde{x}| + |\bar{x}(t)|) dt.
\]
(8)

Notice that for any \( \mu > 0 \)
\[
\int_0^{2\pi} (2|a_d(t)| + |E(t)|)|\tilde{x}(t)| dt \leq |\tilde{x}|_\infty (2|a_d|_1 + |E|_1)
= \frac{2}{2\mu} (2|a_d|_1 + |E|_1)
\leq \mu^2 |\tilde{x}|_\infty^2 + \frac{1}{4\mu^2} (2|a_d|_1 + |E|_1)^2
\leq \frac{\pi}{6} \mu^2 |x'|_2^2 + \frac{1}{4\mu^2} (2|a_d|_1 + |E|_1)^2,
\]
here using of Sobolev inequality
\[
|\tilde{x}|_\infty^2 \leq \frac{\pi}{6} |\tilde{x}|_2^2 = \frac{\pi}{6} |x'|_2^2.
\]

Obviously, if taking \( \mu \) such as \( c_1^2 := \tilde{a} - \frac{\pi}{6} \mu^2 > 0 \), we get that
\[
|x'|_2^2 \leq \frac{2}{c_1^2} |a_d|_1 + |E|_1 |\tilde{x}| + \frac{1}{4c_1^2 \mu^2} (2|a_d|_1 + |E|_1)^2
:= L(|a_d|_1, |E|_1) |\tilde{x}| + K(|a_d|_1, |E|_1).
\]
(9)
The proof is complete. \( \Box \)

**Lemma 2.** If \( x(t) \in C_0 \), i.e. \( x(t + \pi) = -x(t) \), then \( \int_0^{2\pi} x(t) dt = 0 \) or \( \bar{x} = 0 \).

**Proof.** Because
\[
\int_0^{2\pi} x(t) dt = \int_0^\pi x(t) dt + \int_\pi^{2\pi} x(t) dt,
\]
\[
\int_\pi^{2\pi} x(t) dt = \int_0^\pi x(t + \pi) dt = -\int_0^\pi x(t) dt,
\]
the Lemma is true. \( \Box \)

**Lemma 3.** Let the conditions in Lemma 1 hold. If \( x(t) \) is a \( 2\pi \)-periodic solution of Eq. (5) satisfying (4), then there are positive numbers \( K_1, K_2 \) independent of \( \lambda \), such that
\[
|x|_\infty \leq K_1, \quad |x'|_\infty \leq K_2.
\]
(10)

**Proof.** Combining inequality (6) and Lemma 2, we have
\[
|x'|_2^2 \leq K(|a_d|_1, |E|_1) := K_0^2,
\]
which yields from Sobolev inequality
\[
|\tilde{x}|_\infty^2 \leq \frac{\pi}{6} |\tilde{x}|_2^2 = \frac{\pi}{6} |x'|_2^2.
\]
that
\[ |x|_\infty \leq |\tilde{x}| + |\tilde{\tilde{x}}| \leq \sqrt{\frac{\pi}{6}} K_0 := K_1. \] (12)

To prove the second inequality in (10), multiplying (5) by \( x'(t) \) and integrating on \([0, 2\pi]\), we obtain
\[
\int_0^{2\pi} \langle Ax''(t), x'(t) \rangle \, dt = \lambda \int_0^{2\pi} \left( -G(t, x(t)) - \left( \frac{\partial^2 F(x)}{\partial x_i \partial x_j} \right) x' + E(t), x'(t) \right) \, dt,
\]

based on the definitions of \( A \) and \( \tilde{a} \) as well as Schwarzine inequality, it shows that
\[
\tilde{a} \int_0^{2\pi} |x'(t)|^2 \, dt \leq \left[ \left( \int_0^{2\pi} |G(t, x)|^2 \right)^\frac{1}{2} + \left( \int_0^{2\pi} |\frac{\partial^2 F(x)}{\partial x_i \partial x_j} x'|^2 \right)^\frac{1}{2} + \left( \int_0^{2\pi} |E(t)|^2 \right)^\frac{1}{2} \right] \left( \int_0^{2\pi} |x'(t)|^2 \, dt \right)^\frac{1}{2}.
\]

Therefore,
\[
|x'|_2 \leq \frac{1}{\tilde{a}} (\sqrt{2\pi} G + \tilde{F} K_0 + |E|_2) := K_3,
\] (13)

where \( G = \max_{t \in [0, 2\pi], \|x\| \leq K_1} |G(t, x)|, \frac{\partial^2 F(x)}{\partial x_i \partial x_j} x'(t) = \frac{d}{dt} \nabla F(x), \tilde{F} = \max_{\|x\| \leq K_1} \frac{\partial^2 F(x)}{\partial x_i \partial x_j} \). From (11), there exists \( t_0 \in [0, 2\pi] \) such that \( |x'(t_0)| \leq (\sqrt{2\pi})^{-1} K_0 \), this concludes that
\[
|x|_\infty \leq (\sqrt{2\pi})^{-1} K_0 + \sqrt{2\pi} K_3 := K_2. \quad \Box
\]

**Theorem 1.** Suppose that

(I) \( A = \text{diag}(a_1, a_2, \ldots, a_n) \), \( a_i > 0 \), \( i = 1, 2, \ldots, n \);

(II) \( G(t, x) \in C(R^+ \times R^n, R^n), G(t + \pi, -x) = -G(t, x) \), and there exists \( d > 0 \) such that for \( |x| > d \)
\[
G(t, x) \leq 0, \quad t \in R^+, x \in R^n;
\]

(III) \( E(t) \in C(R^+, R^n), E(t + \pi) = -E(t), F(x) \in C^2(R^n, R), F(-x) = F(x) \).

Then Eq. (3) has a 2\( \pi \)-periodic solution \( x(t) \) satisfying symmetry (4).

**Proof.** Notice that \( x(t) \in C^1_0 \) may be written as Fourier series
\[
x(t) = \sum_{i=0}^{\infty} [a_{2i+1} \cos(2i + 1)t + b_{2i+1} \sin(2i + 1)t],
\]
where \( a_{2i+1}, b_{2i+1} \in R^n \). We define a mapping \( T : \Omega \mapsto C^1_0 \) by
\[
Tx(t) = \sum_{i=0}^{\infty} \frac{1}{2i + 1} [a_{2i+1} \sin(2i + 1)t - b_{2i+1} \cos(2i + 1)t]
\]
\[
= \int_0^t x(s) \, ds - \frac{b_{2i+1}}{2i + 1}, \quad (14)
\]

where
\[
\Omega = \{ x(t) \in C^1_0 : |x|_\infty < K_1 + 1, |x'|_\infty < K_2 + 1, |x''|_2 < K_3 + 1 \}.
\]

Obviously, when \( x(t) \in C^1_0 \)
\[
E(t + \pi) - G(t + \pi, x(t + \pi)) - \left( \frac{\partial^2 F(x(t + \pi))}{\partial x_i \partial x_j} \right) x'(t + \pi)
\]
\[
= -E(t) - G(t + \pi, -x(t)) + \left( \frac{\partial^2 F(x(t))}{\partial x_i \partial x_j} \right) x'(t)
\]
\[
= - \left[ E(t) - G(t, x(t)) - \left( \frac{\partial^2 F(x(t))}{\partial x_i \partial x_j} \right) x'(t) \right],
\]

thus \( E(t) - G(t, x(t)) - \left( \frac{\partial^2 F(x(t))}{\partial x_i \partial x_j} \right) x'(t) \in C^1_0 \). It is easy to check that Eq. (3) is equivalent to integral equation
\[
x(t) = T^2 \left\{ A^{-1} \left[ E(t) - G(t, x(t)) - \left( \frac{\partial^2 F(x)}{\partial x_i \partial x_j} \right) x'(t) \right] \right\} - A^{-1} x'(t)
\]
We consider the third order differential system

\[ Fx(t) = T^2 \left\{ A^{-1} \left[ E(t) - G(t, x(t)) - \left( \frac{\partial^2 F(x)}{\partial x_i \partial x_j} \right) x'(t) \right] \right\} - A^{-1} x'(t) := F_1 x(t) - F_2 x(t) \]

has a fixed point on \( \Omega \), where

\[ F_1 x(t) = T^2 \left\{ A^{-1} \left[ E(t) - G(t, x(t)) - \left( \frac{\partial^2 F(x)}{\partial x_i \partial x_j} \right) x'(t) \right] \right\}, \quad F_2 x(t) = A^{-1} x'(t). \]

Based on the definition of \( T \), we see that the mapping \( F_1 \) is continuous for \( x(t) \), moreover is completely continuous for

\[
(F_1x(t))' = T \left\{ A^{-1} \left[ E(t) - G(t, x(t)) - \left( \frac{\partial^2 F(x)}{\partial x_i \partial x_j} \right) x'(t) \right] \right\}
\]

is bounded on \( \Omega \). On the other hand, we observe that the mapping \( F_2 \) is continuous for \( x(t) \) and uniformly bounded in \( \Omega \), meanwhile, for any \( t_1, t_2 \in [0, 2\pi] \)

\[
|F_2 x(t_2) - F_2 x(t_1)| = |A^{-1}(x'(t_2) - x'(t_1))|
\]

\[
= |A^{-1} \int_{t_1}^{t_2} x''(t)dt| 
\]

\[
\leq \|A^{-1}\| \left| \int_{t_1}^{t_2} |x''(t)|dt \right| 
\]

\[
\leq \|A^{-1}\| |x''|_2 \sqrt{|t_2 - t_1|} 
\]

\[
\leq \|A^{-1}\| K_3 \sqrt{|t_2 - t_1|} 
\]

where \( \|A^{-1}\| \) is norm of matrix \( A^{-1} \), which yields the mapping \( F_2 x(t) \) is equicontinuous, furthermore is completely continuous by Arzela–Ascoli theorem. Thus, the mapping

\[ F_1 x(t) := \lambda F_1 x(t) - \lambda F_2 x(t) \]

is completely continuous on \([0, 1] \times \Omega \). Obviously, the fixed point of mapping \( F_1 \) is the solution of Eq. (5) associated with condition (4), we know by the prior estimates (10) and (13) and the definition of \( \Omega \) that for any \((\lambda, x(t)) \in [0, 1] \times \partial \Omega \)

\[ x(t) - F_1 x(t) = x(t) - \lambda Fx(t) \neq 0, \]

Leray–Schauder degree

\[ \text{deg}(I - \lambda F, \Omega, \theta) = \text{deg}(I - F, \Omega, \theta) = \text{deg}(I, \Omega, \theta) = 1. \]

So the mapping \( F \) has a fixed point on \( \Omega \), that is, Eq. (3) has a \( 2\pi \)-periodic solution \( x(t) \) satisfying (4). \qed

Similarly, we can obtain following theorem.

**Theorem 2.** Suppose that

(1) \( A = \text{diag}(a_1, a_2, \ldots, a_n), a_i < 0, i = 1, 2, \ldots, n; \)

(2) \( G(t, x) \in C(R^+ \times R^n, R^n), G(t + \pi, -x) = -G(t, x), \) and there exists \( d > 0, \) such that for \( |x| > d \)

\[ \langle G(t, x), x \rangle \geq 0, \quad t \in R^+, x \in R^n; \]

(3) \( E(t) \in C(R^+, R^n), E(t + \pi) = -E(t) \); \( F(x) \in C^2(R^n, R), F(-x) = F(x). \)

Then Eq. (3) has a \( 2\pi \)-periodic solution \( x(t) \) satisfying anti-symmetrical condition (4).

**Example.** We consider the third order differential system

\[
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}''' + \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \frac{d}{dt} \nabla \begin{pmatrix} x_1^2 x_2^4 \\ (\sin t)^2 x_1^3 x_2^2 \end{pmatrix} = \begin{pmatrix} c \sin t \\ d \cos t \end{pmatrix}
\]

where \( a_1, a_2 > 0, c, d \in R. \) It is easy to check that the above system satisfies the conditions of Theorem 1, so it has a \( 2\pi \)-periodic solution with anti-symmetry (4).

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