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Torsion-freeness and non-singularity over right p.p.-rings

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Abstract

A right R-module M is non-singular if $xI \neq 0$ for all non-zero $x \in M$ and all essential right ideals I of R. The module M is torsion-free if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for all $r \in R$. This paper shows that, for a ring R, the classes of torsion-free and non-singular right R-modules coincide if and only if R is a right Utumi-p.p.-ring with no infinite set of orthogonal idempotents. Several examples and applications of this result are presented. Special emphasis is given to the case where the maximal right ring of quotients of R is a perfect left localization of R.

1. Introduction

There are several ways to extend the concept of torsion-freeness from modules over integral domains to arbitrary non-commutative rings. The most straightforward approach towards such a generalization is to call a right module M over a ring R *torsion-free in the classical sense* if, for all non-zero $x \in M$ and all regular $c \in R$, one has $xc \neq 0$. Here $c \in R$ is *regular* if it is neither a right nor a left zero-divisor. However, this approach has only

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limited applicability since, for instance, the set of all elements of a module annihilated by regular elements need not be a submodule unless R is a right Ore-ring [7, Problem 3.D.16]. To overcome the inherent limitations of this generalization, Goodearl introduced the notion of non-singularity in [7]. For a right R-module M, *the singular submodule of* M is $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of $R\}$; this takes the place of the torsion submodule in the general setting. The module M is called *non-singular* if $Z(M) = 0$, and *singular* if $M = Z(M)$, while the *right singular ideal of* R is $Z_r(R) = Z(R_R)$. The ring R is *right non-singular* if it is non-singular as a right R-module. In case R is right non-singular, the singular and the non-singular modules are the elements of, respectively, the torsion and torsion-free classes of the *Goldie torsion theory* of R.

A different approach to define torsion-freeness for modules over arbitrary rings is motivated by homological properties of torsion-free modules over an integral domain. Hattori used this method in [8] when he defined a right R-module M to be *torsion-free* if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for all $r \in R$. While all flat modules are torsion-free, the converse fails in general. Dauns and Fuchs continued Hattori's work in [5], and developed a theory for torsion-free modules. The ring R is *a torsion-free ring* if all its right ideals are torsion-free as R-modules. This is the notion of torsion-freeness used in this paper. The property that R is a torsion-free ring is right–left-symmetric. However, torsion-freeness is not Morita-invariant in contrast to flatness and non-singularity as is shown in [5, Example 5.4], and as we will see in Theorem 5.1. In our discussion as well as in [5] and [8], p.p.-rings play an important role: a ring R is *a right p.p.-ring* if every principal right ideal of R is projective, or equivalently, the right annihilator of every element of R is generated by an idempotent. If R is a ring without an infinite family of orthogonal idempotents, then the property that R is a p.p.-ring is right–left-symmetric by [3, Lemma 8.4].

The goal of this paper is to investigate the relationship between torsion-freeness and non-singularity. We show that the classes of torsion-free and non-singular right R-modules coincide if and only if R is a right Utumi-p.p.-ring that contains no infinite set of orthogonal idempotents (Theorem 3.7). Here, a right non-singular ring R is *a right Utumi-ring* if every $\mathcal{S}$-closed right ideal of R is a right annihilator (see [11, Chapter XII, Section 4]). Here, a submodule U of an R-module M is *$\mathcal{S}$-closed* if M/U is non-singular. By [11, Chapter XII, Proposition 4.7], a right non-singular ring R is a right Utumi-ring if and only if every non-essential right ideal has a non-zero left annihilator. The right and left Utumi-rings are exactly the right and left non-singular rings for which $Q^r = Q^\ell$ [7, Theorem 2.38]. Here Q^r denotes the maximal right ring of quotients of R; see Section 2 below.

Several applications of Theorem 3.7 are discussed. Particular attention is given to the case when R is a non-singular ring for which Q^r is *a perfect left localization of* R (i.e., Q^r is a flat right R-module, and the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism; see [7] and [11] for details on these rings). Goodearl showed in [7, Theorem 5.17] that the right non-singular rings R whose maximal right ring of quotients Q^r is a perfect left localization of R are the rings for which the finitely generated non-singular right modules are precisely the finitely generated submodules of free modules. We shall call such a ring *right strongly non-singular*. For these rings, the property that the classes of non-singular and torsion-free right R-modules coincide is preserved by Morita equivalence if and only if R is a right semi-hereditary ring without an infinite set of orthogonal idempotents (Theorem 5.1). Furthermore, the latter are exactly the right strongly non-singular rings for which the classes of torsion-free, non-singular and flat modules coincide (Theorem 5.2).

Finally, we would like to mention that the term *domain* denotes a ring without zero-divisors, while *integral domain* indicates a commutative domain.

2. When non-singular modules are torsion-free

Since a ring R is always torsion-free as an R-module, any ring for which the classes of torsion-free and non-singular modules coincide has to be right non-singular. The maximal right ring Q^r of quotients of such an R is (von Neumann) regular and right self-injective [7]. We begin our discussion with a technical result.

Lemma 2.1. *Let R be a left p.p.-ring. The following conditions are equivalent for a right R-module M:*

(a) *M is torsion-free.*
(b) *xR is torsion-free for all $x \in M$.*
(c) *If $xr = 0$ ($x \in M$, $r \in R$) and $e \in R$ is the idempotent with $\mathrm{ann}_\ell(r) = Re$, then $x = xe$.*

Proof. (a) $\Rightarrow$ (b) holds since a left p.p.-ring is a torsion-free ring, and submodules of torsion-free modules over such a ring are torsion-free [5].

(b) $\Rightarrow$ (c). Suppose $x \in M$ such that $xr = 0$ for some $r \in R$. Let e be the idempotent of R with $\mathrm{ann}_\ell(r) = Re$. Since xR is torsion-free, there are elements $b_1, \ldots, b_m \in R$ and $s_1, \ldots, s_m \in R$ such that $x = xb_1s_1e + \cdots + xb_ms_me$ by [5, Lemma 1.1]. Thus, $x = xe$.

(c) $\Rightarrow$ (a) is a direct consequence of [5, Lemma 1.1]. □

Note that [5, Lemma 1.1] also shows that torsion-free right modules are torsion-free in the classical sense: if c is a regular element of the ring, then $xc = 0$ implies $x = 0$.

The next result describes the right non-singular rings for which all non-singular right modules are torsion-free. It strongly resembles [7, Proposition 5.16] which characterizes the rings for which all non-singular right R-modules are flat:

Proposition 2.2. *The following are equivalent for a right non-singular ring R:*

(a) *R is a left p.p.-ring such that Q^r is a torsion-free right R-module.*
(b) *All non-singular right R-modules are torsion-free.*

Proof. (a) $\Rightarrow$ (b). Let M be a non-singular right R-module, and consider $x \in M$. Since R is right non-singular, there is a monomorphism $xR \to \bigoplus_n Q^r$ for some $n < \omega$ by [11, Chapter XII, Proposition 7.2]. As a left p.p.-ring, R is a torsion-free ring. Hence, submodules of torsion-free modules are torsion-free (see [5, Proposition 3.6]), and, therefore, xR is torsion-free. By Lemma 2.1, M is torsion-free.

(b) $\Rightarrow$ (a). Since R is right non-singular, every right ideal of R is torsion-free by (b), and R is a torsion-free ring. To see that R is a left p.p.-ring, it suffices by [5, Theorem 4.5] to show that the left annihilator of each $r \in R$ is finitely generated. Let $\mathrm{ann}_\ell(r) = \{s_i \mid i \in I\}$; then the element $x = (s_i)_{i\in I}$ of R^I satisfies $xr = (s_i r)_{i\in I} = 0$. Since R^I is non-singular, and hence torsion-free by (b), for some $m < \omega$ one can find $b_1, \dots, b_m \in R^I$ and $u_1, \dots, u_m \in \mathrm{ann}_\ell(r)$ such that $x = b_1u_1 + \cdots + b_m u_m$. Write $b_j = (r_{ij})_{i\in I}$ for $j = 1, \dots, m$ to obtain $s_i = r_{i1}u_1 + \cdots + r_{im}u_m$ for all $i \in I$. Therefore, $\mathrm{ann}_\ell(r)$ is generated by $u_1, \dots, u_m$. Finally, Q^r is a non-singular R-module, and hence torsion-free. □

Turning to the commutative case, observe that a commutative ring R is non-singular if and only if it is semi-prime [3, Lemma 1.3].

Corollary 2.3. *The following are equivalent for a commutative ring R*:

(a) *R is a p.p.-ring.*
(b) *R is a semi-prime ring such that every non-singular R-module is torsion-free.*

Proof. (a) ⇒ (b). In view of Proposition 2.2, it remains to show that the maximal ring of quotients, Q, of R is torsion-free as a R-module. Let $q \in Q$ and $r \in R$ be such that $qr = 0$. There exists an idempotent e of R such that $\mathrm{ann}(r) = Re$, and an essential ideal I of R such that $qI \subseteq R$. For every $x \in I$, one has $qxr = qrx = 0$, whence $qx = qxe$, since R is a torsion-free R-module. Hence, $(q - qe)I = 0$. Since Q is a non-singular R-module, one obtains that $q - qe = 0$; and Q is a torsion-free R-module.

(b) ⇒ (a) is a direct consequence of Proposition 2.2. □

Proposition 2.2 applies in particular in case Q^r is flat as a right R-module:

Corollary 2.4. *The following conditions are equivalent for a right non-singular ring R for which Q^r is flat as a right R-module*:

(a) *R is a left p.p.-ring.*
(b) *All non-singular right R-modules are torsion-free.*

However, even torsion-free modules over a right and left semi-hereditary, right and left strongly non-singular ring need not be non-singular as the following result shows. In its proof, we use the fact that the class of torsion-free modules is closed with respect to quotients modulo relatively divisible submodules [5, Lemma 1.3]. Here, a submodule U of a right R-module M is *relatively divisible* or *an RD-submodule* if $U \cap Mr = Ur$ for all $r \in R$.

Proposition 2.5. *Let $\{R_i \mid i \in I\}$ be an infinite family of right non-singular rings. If R is a subring of $\prod_{i\in I} R_i$ which contains $\bigoplus_{i\in I} R_i$, then R is a right non-singular ring for which there exists a torsion-free singular R-module M. Moreover, the R_i's and R can be chosen in such a way that R is a right and left semi-hereditary, right and left strongly non-singular ring.*

Proof. Since R is a right essential product of the R_i's, it is a right non-singular ring whose maximal right ring of quotients is $\prod_{i\in I} Q_i^r$ where Q_i^r is the maximal right ring of quotients of R_i; see [7, Proposition 4.15]. Since $U = \bigoplus_{i\in I} R_i$ is an essential right ideal of R, the module $M = R/U$ is singular. On the other hand, the direct sum is always (pure and hence) relatively divisible in the direct product. Thus U is a relatively divisible submodule of R. By [5, Lemma 1.3], M is a torsion-free R-module.

If each R_i is chosen to be a regular right and left self-injective ring, then $Q_i^r = R_i = Q_i^\ell$. Therefore, $R = \prod_{i\in I} R_i$ is its own maximal right and left ring of quotients. Clearly, R is a perfect right and left localization of itself, and therefore right and left strongly non-singular. □

In the discussion of maximal rings of quotients, the multiplication map $Q^r \otimes_R Q^r \to Q^r$ plays a central role. It is an isomorphism if and only if the embedding $R \subseteq Q^r$ is an epimorphism of rings by [11, Chapter XI, Proposition 1.2]. Proposition 2.5 allows us to construct a right non-singular ring R for which all non-singular right modules are torsion-free, but the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is not an isomorphism: for a field F, set $R_n = F$ for all $n < \omega$, and consider the subring $R = \bigoplus_{n<\omega} R_n + F1$ of $\prod_{n<\omega} R_n$. By [7, Example 3.11], R is non-singular, and Q^r is flat as a right and left R-module, but the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is not an isomorphism. It is easy to see that R is a p.p.-ring. Therefore, all non-singular R-modules are torsion-free.

3. Semi-perfect left localizations

We begin the discussion of this section by considering rings R for which all torsion-free right modules are non-singular.

Lemma 3.1. *Let R be a left p.p.-ring such that all torsion-free right R-modules are non-singular. Then we have*:

(a) *Direct sums of torsion-free injective right R-modules are injective.*
(b) *Every torsion-free injective right R-module is the direct sum of indecomposable injectives.*

Proof. (a) Let M_i $(i \in I)$ be torsion-free injective right modules. Since R is left p.p., their Cartesian product $\prod_{i\in I} M_i$ is again a torsion-free (by [5, Corollary 4.6]) injective right module. The same is true for the injective hull $E(\bigoplus_{i\in I} M_i)$ as a summand of $\prod_{i\in I} M_i$. As $\bigoplus_{i\in I} M_i$ is an RD-submodule of $\prod_{i\in I} M_i$, the factor module $[\prod_{i\in I} M_i]/[\bigoplus_{i\in I} M_i]$ is torsion-free. Now $E(\bigoplus_{i\in I} M_i)/[\bigoplus_{i\in I}]M_i$ is both torsion-free and singular, so the hypothesis implies that it must be 0. Thus $\bigoplus_{i\in I} M_i$ is injective.

(b) From (a) it follows at once that the union of a continuous well-ordered ascending chain of torsion-free injective right R-modules is again torsion-free injective. Let M be any torsion-free injective right R-module, and let $0 \neq a \in M$. By Zorn's lemma, there is an injective submodule N of M that is maximal with respect to $a \notin N$. Then $M = N \oplus A$ for some injective submodule A of M. By maximality, A is indecomposable. This shows that any torsion-free injective right R-module $\neq 0$ contains indecomposable injective submodules. It is now routine to verify that M is a direct sum of indecomposable injective submodules. □

Proposition 3.2. *Let R be a left p.p.-ring such that Q^r is torsion-free as a right R-module. If all torsion-free right R-modules are non-singular, then every set of orthogonal idempotents in R is finite. In particular, R is a Baer-ring.*

Proof. Observe that R is a right non-singular ring, and hence an essential submodule of Q^r. By Lemma 3.1, Q^r is the direct sum of indecomposable injectives. This sum must be finite since $1 \in R$ can have but a finite number of non-zero coordinates. Then, R cannot contain infinitely many orthogonal idempotents. By [3, Lemma 8.4], R is a right and left p.p.-ring which satisfies the ascending chain condition for right and left annihilators. Because of [5, Theorem 4.9], R is a Baer-ring. □

Proposition 3.3. *Let R be a p.p.-ring with no infinite set of orthogonal idempotents. Then, the annihilator of an element in a torsion-free right R-module is a principal right ideal generated by an idempotent.*

Proof. Let M be a torsion-free right R-module, $0 \neq x \in M$, and $A = \operatorname{ann} x$. If $r_0 \in R$ satisfies $xr_0 = 0$, then, by Lemma 2.1(c), we have $x(1 - e_0) = 0$ for the idempotent e_0 with $Re_0 = \operatorname{ann}_\ell r_0$. If there is an $r_1 \in A \setminus (1 - e_0)R$, then $e_0 r_1 \neq 0$, but $xe_0 r_1 = 0$, so the idempotent $f \in R$ with $\operatorname{ann}_\ell(e_0 r_1) = R(1 - f)$ satisfies $x = x(1 - f)$ and $(1 - f)e_0 r_1 = 0$.

Define $e_1 = (1 - f)e_0$. Thus $e_1 r_1 = 0$ and $1 - e_0 \in \operatorname{ann}_\ell(e_0 r_1) = R(1 - f)$, which shows that $(1 - e_0)f = 0$, $e_0 f = f$. Hence $e_1^2 = (1 - f)e_0(1 - f)e_0 = (1 - f)(e_0 - f)e_0 = (1 - f)e_0 = e_1$, i.e., e_1 is an idempotent. Furthermore, $e_1 e_0 = e_1$ implies $(1 - e_0)R \subseteq (1 - e_1)R$. This inclusion must be proper, since $0 \neq e_0 r_1 \notin (1 - e_0)R$, but $e_0 r_1 \in (1 - e_1)R$ because of $e_1(e_0 r_1) = e_1 r_1 = 0$. If there is an $r_2 \in A \setminus (1 - e_1)R$, then we continue this process and keep going to obtain a properly ascending chain $(1 - e_0)R \subset (1 - e_1)R \subset (1 - e_2)R \subset \cdots$. This chain has to terminate, say, at $(1 - e)R$ (where $e^2 = e$), since R contains no infinite set of orthogonal idempotents. We conclude that $A = (1 - e)R$, as claimed. □

Corollary 3.4. *Let R be a p.p.-ring with no infinite set of orthogonal idempotents. The cyclic submodules of a torsion-free right R-module are projective. In particular, all torsion-free* (*left and right*) *R-modules are non-singular.*

Proof. Because of the right–left symmetry of the hypothesis, it suffices to consider right modules only. By Proposition 3.3, the annihilator of an element x in a torsion-free right R-module M is of the form eR for some idempotent e of R. This is an essential right ideal in R only if $eR = R$, so $Z(M) = 0$. □

Observe that the hypothesis that R has no infinite set of orthogonal idempotents cannot be omitted in Proposition 3.3. Indeed, in the notation of Proposition 2.5 and its proof,

R/U is torsion-free, and the annihilator ideal U of $1 + U \in R/U$ is not generated by an idempotent. Note that U is not $\mathcal{S}$-closed in R.

The maximal right ring Q^r of quotients of a right non-singular ring R is a *semi-perfect left localization of* R if Q^r is torsion-free as a right R-module, and the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism. Furthermore, a ring R has *finite right Goldie-dimension* ($G\text{-dim}\, R_R < \infty$) if, in R, every direct sum of non-zero right ideals has only a finite number of summands. A ring of finite right Goldie-dimension is a *right Goldie-ring* if it satisfies the ascending chain condition (ACC) for right annihilators. The semi-prime right Goldie-rings are the rings which admit a semi-simple Artinian classical right ring of quotients. Finally, by a *right extending* ring we mean a ring R in which every right ideal is contained in a summand of R_R as an essential submodule.

Lemma 3.5. *The following are equivalent for a ring R in which every set of orthogonal idempotents is finite*:

(a) *R is a Baer-ring such that Q^r is a torsion-free right R-module.*
(b) *R is a right Utumi-ring that is also a Baer-ring.*
(c) *R is a right non-singular right extending ring.*
(d) *R is right non-singular, and every $\mathcal{S}$-closed right ideal is generated by an idempotent.*

Proof. For (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d), see 12.1 and 12.2 in [6].

(a) $\Rightarrow$ (b). Baer-rings are non-singular. Let $I \neq R$ be an $\mathcal{S}$-closed right ideal of R. Then the right module $M = R/I$ is non-singular, so by [11, Chapter XII, Proposition 7.2] M can be embedded in $\bigoplus_n Q^r$ for some $n < \omega$. Since Q^r is torsion-free by assumption, Proposition 3.3 implies that I is generated by an idempotent e of R.

(d) $\Rightarrow$ (a). Let $x \in Q^r$ and $A = \operatorname{ann}_r x$. By (d), R is right non-singular, and hence Q^r is non-singular as a right R-module. Therefore A is an $\mathcal{S}$-closed right ideal in R. Again by (d), A is generated by an idempotent. This means that xR is torsion-free, and hence the torsion-freeness of Q^r follows.

The same argument applied to R leads to the conclusion that R is a p.p.-ring. Proposition 3.2 implies that R is a Baer-ring. $\square$

Let us point out that condition (a) in the preceding lemma might be satisfied by rings with infinite sets of orthogonal idempotents, as is shown by the following example.

Example 3.6. Let F_i $(i < \omega)$ be copies of the prime field of characteristic 2, and let R be the ring of the eventually constant vectors in $F = \prod_{i<\omega} F_i$. Then $F = E(R) = Q^r$. Now R is a Baer-ring with identity $1 = (1_1, \ldots, 1_i, \ldots)$ and orthogonal idempotents $e_i = 1_i \in F_i \subset R$ $(i < \omega)$. To see that Q^r is torsion-free, let $sr = 0$ with $s \in Q^r$, $r \in R$. Evidently, $u = 1 - r \in R$ satisfies $su = s$ and $ur = 0$.

Theorem 3.7. *The following are equivalent for a ring R*:

(a) *R is a right Goldie- right p.p.-ring, and Q^r is a semi-perfect left localization of R.*
(b) *R is a right Utumi-p.p.-ring without an infinite set of orthogonal idempotents.*

(c) *R is a right non-singular ring without an infinite set of orthogonal idempotents, and all finitely generated non-singular right R-modules are torsion-free.*
(d) *A right R-module is torsion-free if and only if it is non-singular.*
(e) *A submodule of a torsion-free module is relatively divisible if and only if it is $\mathcal{S}$-closed.*

Furthermore, each ring R satisfying condition (a) *is a Baer-ring and has a semi-simple Artinian maximal right ring of quotients.*

Proof. (a) ⇒ (b). By Corollary 3.4, every torsion-free right R-module is non-singular. Since Q^r is a semi-perfect localization of R, the ring R is Baer by Proposition 3.2. Because of Lemma 3.5, R is a right Utumi-ring.

(b) ⇒ (d). Since R has no infinite set of orthogonal idempotents, it is a right and left p.p.-ring if it is right or left p.p.-ring by [3, Lemma 8.4]. In view of Corollary 3.4, every torsion-free R-module is non-singular. To see the converse, observe that [5, Theorem 4.9] yields that R is a Baer-ring. Hence, Q^r is a torsion-free right R-module because of Lemma 3.5. By Proposition 2.2, all non-singular right R-modules are torsion-free.

(d) ⇒ (e). Let M be a torsion-free R-module, and U a submodule of M. By [5, Lemma 1.3], U is a relatively divisible submodule of M if and only if M/U is a torsion-free R-module. Since the classes of torsion-free and non-singular right R-modules coincide, the latter occurs exactly if M/U is non-singular, i.e., if and only if U is $\mathcal{S}$-closed in M.

(e) ⇒ (c). R is torsion-free as a right R-module and $\{0\}$ is always a relatively divisible submodule. Therefore, by (e), $\{0\}$ is $\mathcal{S}$-closed in R, which is equivalent to $Z_r(R) = 0$.

For a right R-module M, consider an exact sequence $0 \to U \to F \to M \to 0$ where F is free. By [5], F is a torsion-free module. Then, M is non-singular if and only if U is an $\mathcal{S}$-closed submodule of M. By (e), this occurs exactly if it is relatively divisible. Because of [5, Lemma 1.3], U is an RD-submodule of F if and only if M is torsion-free. Hence the classes of torsion-free and non-singular right R-modules coincide.

To see that R does not contain any infinite set of orthogonal idempotents it suffices to show that R has finite right Goldie-dimension. By [11, Chapter XIII, Proposition 3.3], this holds exactly if $D = \bigoplus_{i\in I} Q^r$ is injective as a right R-module. This is true by virtue of Lemma 3.1 and the preceding paragraph.

(c) ⇒ (a). Let M be a non-singular right R-module. Since non-singularity is inherited by submodules, every finitely generated submodule M is torsion-free by (c), and the same holds for M itself. By Proposition 2.2, R is a left p.p.-ring, and Q^r is torsion-free as a right R-module. However, a left p.p.-ring without an infinite set of orthogonal idempotents is right p.p. by [3, Lemma 8.4]. Therefore, R satisfies (b).

By what has been shown so far, conditions (b) and (e) are equivalent, and any ring which satisfies condition (e) has finite right Goldie-dimension. The ring R also has the ACC for right annihilators by [3, Lemma 8.4], and therefore it is a right Goldie-ring. By [11, Chapter XII, Theorem 2.5], a right non-singular ring R has finite right Goldie-dimension if and only if its maximal right ring of quotients is semi-simple Artinian. Moreover, [11, Chapter XII, Corollary 2.6] shows that in this case Q^r is a perfect right localization. In particular, the map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism, and Q^r is a semi-perfect localization of R.

Finally, a right p.p.-ring which satisfies the ACC for right annihilators is a Baer-ring; see [5, Theorem 4.9]. □

Since a left or right Goldie-ring does not contain an infinite set of orthogonal idempotents, we see that the class of rings satisfying Theorem 3.7 are the right p.p.-, right Goldie-, right Utumi-rings. By [6, Proposition 12.3], these are the right p.p.-, right Goldie-, right extending rings. Moreover, the right non-singular rings without an infinite set of orthogonal idempotents satisfying the conditions of the last theorem can also be described by various closure properties of the class of torsion-free right R-modules:

Corollary 3.8. *The following are equivalent for a right non-singular ring R without an infinite set of orthogonal idempotents*:

(a) *All finitely generated non-singular right R-modules are torsion-free.*
(b) *The class of torsion-free modules is the torsion-free class of a hereditary torsion theory.*
(c) *Submodules and essential extensions of torsion-free right R-modules are torsion-free.*
(d) *Submodules and the injective hull of a torsion-free right R-module are torsion-free.*

Proof. (a) ⇒ (b). As before, all non-singular right R-modules are torsion-free. By Theorem 3.7, R is a right p.p.-ring, and hence both a torsion-free and a right non-singular ring. Therefore, submodules of torsion-free modules are torsion-free. Since R is also left p.p., the class of torsion-free modules is closed under products and extensions by [5]. Consequently, it is the torsion-free class of a torsion theory of right R-modules. To establish that this torsion theory is hereditary, it suffices to show that the class of torsion-free modules is closed under essential extensions [11, Chapter VI, Proposition 3.2]. Let M be a right R-module containing an essential submodule U which is torsion-free. By Theorem 3.7, U is non-singular, and the same holds for M since the class of non-singular modules is closed with respect to essential extensions. Another appeal to Theorem 3.7 yields that M is torsion-free.

(b) ⇒ (c). Since the class of torsion-free modules is the torsion-free class of a torsion theory, it is closed with respect to submodules. Furthermore, if M is a torsion-free module, then the same holds for its injective hull $E(M)$, since the torsion theory is hereditary. However, any essential extension of M is isomorphic to a submodule of $E(M)$, and hence torsion-free.

(c) ⇒ (d) is obvious.

(d) ⇒ (a). Let M be a finitely generated non-singular right R-module. By [11, Chapter XII, Proposition 7.2], M can be embedded into $\bigoplus_n Q^r$ for some $n < \omega$. However, Q^r is the injective hull of R as a right R-module, and hence torsion-free by (d). Another application of (d) yields that M is torsion-free as a submodule of a torsion-free module. □

In particular, the Goldie torsion theory is the only hereditary torsion theory over a right non-singular ring without an infinite family of orthogonal idempotents whose torsion-free class is the class of torsion-free modules.

An important class of rings arising in the discussion of non-singular rings is the class of the reduced rings [11, Chapter XII, Section 5]. A ring R is *reduced* if it does not contain any nilpotent elements. By [11, Lemma 5.1], every reduced ring is right and left non-singular.

Corollary 3.9.

(a) *The following are equivalent for a ring* R:
  (i) *R is a reduced ring for which the classes of torsion-free and non-singular right R-modules coincide.*
  (ii) *$R = R_1 \times \cdots \times R_n$ where each R_i is a domain which has right Goldie-dimension* 1.

(b) *In case R is a domain, the classes of torsion-free and non-singular right R-modules coincide if and only if G-*$\dim R_R = 1$.

Proof. (a) (i) $\Rightarrow$ (ii). Since the classes of torsion-free and non-singular right R-modules coincide, R is a right p.p.-ring without an infinite family of orthogonal idempotents. Hence, there are primitive idempotents $e_1, \ldots, e_n \in R$ such that $1_R = e_1 + \cdots + e_n$. By [11, Chapter I, Lemma 12.2], every idempotent of R is central, and $R = R_1 \times \cdots \times R_n$ where $R_i = e_i R e_i$. Suppose $r, s \in R_i$ with $rs = 0$. There is a central idempotent $e \in R$ such that $\mathrm{ann}_r(r) = eR$. Since e_i is central, $e_i e$ and $e_i - e_i e$ are orthogonal idempotents in $e_i R e_i$ with $e_i = e_i e + (e_i - e_i e)$. Because e_i is primitive, either $e_i e = e_i$ or $e_i e = 0$. In the first case, $e_i \in \mathrm{ann}_r(r)$ and $r = e_i r e_i = 0$. On the other hand, if $e_i e = 0$, then $s = es$ yields $s = e_i s e_i = e_i e s e_i = 0$. Therefore, R_i is a domain. Since by Theorem 3.7 R has finite right Goldie-dimension, the same holds for each of the R_i's. However, a domain with finite right Goldie-dimension has right Goldie-dimension 1.

(ii) $\Rightarrow$ (i). Clearly, R cannot have any nilpotent elements. Since R_i is a domain with finite right Goldie-dimension, it is a right Ore-domain, and its classical ring of quotients is a division algebra Q_i. Then, Q_i is the maximal right ring of quotients of R_i, and is torsion-free in the classical sense. By [5, Remark (2)], Q_i is torsion-free, and the same holds for $Q = Q_1 \times \cdots \times Q_n$. It is easy to see that Q is the maximal right ring of quotients of R. In particular, Q is a torsion-free R-module. Furthermore, R obviously is a reduced right and left p.p.-ring. By Theorem 3.7, the classes of torsion-free and non-singular right R-modules coincide.

(b) is a direct consequence of (a). $\square$

In particular, every reduced ring R for which the classes of torsion-free and non-singular right R-modules coincide is a right Utumi-ring. However, Q^r need not be flat as a right R-module even in this case. For instance, let R be a domain which has right Goldie-dimension 1, but infinite left Goldie-dimension (see, e.g., [3]). Then, R has a classical right ring of quotients, Q^r, which is a division algebra, and the classes of torsion-free and non-singular right R-modules coincide by Corollary 3.9. On the other hand, Q^r is not equal to the maximal left ring Q^ℓ of quotients of R, which cannot consequently be torsion-free as a left R-module by Theorem 3.7 (or [5, Theorem 7.1]). Furthermore, Q^r is not flat as a right R-module for this ring R. Indeed, if it were, then R would be a right strongly non-singular ring. As we will see in Corollary 4.3, the classes of non-singular and torsion-free left R-modules will also coincide in this setting, which results in a contradiction.

In particular, a ring for which the classes of torsion-free and non-singular right R-modules coincide need not be right strongly non-singular. However, such rings do not exist in the commutative setting:

Corollary 3.10. *The following are equivalent for a commutative ring R:*

(a) *The classes of torsion-free and non-singular R-modules coincide.*
(b) *R is a strongly non-singular p.p.-ring without an infinite set of orthogonal idempotents.*
(c) *$R = R_1 \times \cdots \times R_n$ where each R_i is an integral domain.*

Proof. (a) $\Rightarrow$ (c). Since R is torsion-free as an R-module, it is non-singular. However, the notions of being reduced, non-singular or semi-prime coincide for commutative rings. By Corollary 3.9, R is the finite product of (integral) domains.

(c) $\Rightarrow$ (b). It is easily checked that R is a p.p.-ring of finite Goldie-dimension. Moreover, each finitely generated non-singular R-module M is of the form $M = M_1 \oplus \cdots \oplus M_n$ where each M_i is a non-singular R_i-module. Thus, each M_i can be embedded into a free R_i-module. Consequently, M is isomorphic to a submodule of a free R-module, and R is a strongly non-singular ring.

(b) $\Rightarrow$ (a). In view of Theorem 3.7, it remains to show that the maximal ring of quotients, Q, of R is a torsion-free R-module. However, this holds since, by Corollary 2.3, all non-singular R-modules are torsion-free. □

4. Perfect left localizations

We now consider the right strongly non-singular rings R such that the classes of torsion-free and non-singular right R-modules coincide. Our first result describes when the right and left maximal ring of quotients coincide and are semi-simple Artinian. Rings with this property will be the central focus of this and the following section.

Proposition 4.1. *The following are equivalent for a right and left non-singular ring R:*

(a) *$Q^r = Q^\ell$, and Q^r is a semi-simple Artinian ring.*
(b) *$G\text{-dim}_R R = G\text{-dim}\, R_R < \infty$ and $G\text{-dim}_R Q^r = G\text{-dim}\, Q^r_R$.*

Proof. (a) $\Rightarrow$ (b). Observe that Q^r is always a rational, and hence an essential extension of R, so $G\text{-dim}\, Q^r_{Q^r} = G\text{-dim}\, Q^r_R = G\text{-dim}\, R_R$. A similar result holds for Q^ℓ.

Since Q^r is semi-simple Artinian and $Q^r = Q^\ell = Q$, [11, Chapter XII, Theorem 2.5] yields that $Q = \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r)$ where $D_1, \ldots, D_r$ are division algebras. Then $n_1 + \cdots + n_r$ is both the right and left Goldie-dimension of Q as a Q-module. The remark in the preceding paragraph implies that all the four indicated Goldie-dimensions are equal.

(b) $\Rightarrow$ (a). Since R has finite right Goldie-dimension, Q^r is a semi-simple Artinian ring [11, Chapter XII, Theorem 2.5]. Setting $Q = Q^r$, we will show that $Q^\ell = Q$ as well. As

R_R is an essential submodule of Q^r, their Goldie-dimensions are equal, and one has

$$G\text{-}\dim_R R = G\text{-}\dim R_R = G\text{-}\dim Q^r_R = G\text{-}\dim_R Q^r < \infty.$$

Consequently, R is an essential submodule of Q also as a left R-module. Since R is left non-singular, Q is a rational extension of R on the left, so it is a left quotient ring of R. Since Q is semi-simple Artinian also as a left Q-module, it has no proper essential extensions. Hence, by [7, Theorem 2.30], Q is the maximal left ring of quotients of R, and $Q = Q^\ell$. □

Theorem 4.2. *The following are equivalent for a ring R:*

(a) *R is a right and left non-singular ring without an infinite set of orthogonal idempotents such that every $\mathcal{S}$-closed one-sided ideal is generated by an idempotent.*
(b) *R is a right or left p.p.-ring such that $Q^\ell = Q^r$ is semi-simple Artinian.*
(c) *R is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents.*
(d) *R is a right strongly non-singular ring for which the classes of torsion-free and non-singular right R-modules coincide.*
(e) *The following are equivalent for a right R-module M:*
  (i) *M is torsion-free.*
  (ii) *M is non-singular.*
  (iii) *The injective hull $E(M)$ of M is flat.*

Proof. (a) ⇒ (b). Let I be an $\mathcal{S}$-closed right (left) ideal. Since I is generated by an idempotent e, it is the right (left) annihilator of $1 - e$. Hence, R is a right and left Utumi-p.p.-ring without an infinite set of orthogonal idempotents. Therefore, $Q^r = Q^\ell$ by [7, Theorem 2.38]. Moreover, Q^r is torsion-free, and hence Q^r is semi-simple Artinian by Theorem 3.7.

(b) ⇒ (c). Without loss of generality, R is a right p.p.-ring. By [11, Chapter XII, Theorem 2.5], the fact that Q^r is semi-simple Artinian yields that R has finite right Goldie-dimension, and therefore has no infinite set of orthogonal idempotents. Because of [3, Lemma 8.4], R is a left p.p.-ring.

In order to verify that R is right strongly non-singular it remains to show that Q^r is a perfect left localization of R. By [11, Chapter XII, Corollary 2.6], Q^r is a perfect right localization of R since it is semi-simple Artinian. Therefore, the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism. Furthermore, since Q^r is semi-simple Artinian, $Q^r = Q^\ell$ yields that Q^r is flat as a right R-module; cf. [11, Chapter XI, Proposition 5.4]. Therefore, Q^r is a perfect left localization of R.

(c) ⇒ (d). Since R is right strongly non-singular, Q^r is a flat right R-module, and hence torsion-free. Now apply Theorem 3.7.

(d) ⇒ (e). It remains to show the equivalence of (ii) and (iii). Because the classes of non-singular and torsion-free right R-modules coincide, the ring Q^r is semi-simple Artinian by Theorem 3.7. Therefore, the injective hull of a non-singular right R-module M is a right Q^r-module; see [11, Chapter XII, Corollary 2.8]. Consequently, $E(M)$ is isomorphic to a direct summand of $\bigoplus_I Q^r$ for some index-set I since Q^r is semi-simple Artinian. As R is strongly right non-singular, the latter module is flat, and the same holds for $E(M)$. Conversely, since R is a right p.p.-ring by Theorem 3.7, it is torsion-free. Thus, submodules of flat modules are torsion-free.

(e) $\Rightarrow$ (a). Because of the equivalence of (i) and (ii), Theorem 3.7 yields that R is a right p.p.-ring without an infinite set of orthogonal idempotents such that Q^r is semi-simple Artinian. As the injective hull of R_R, Q^r is flat by (d). Hence, [11, Chapter XI, Proposition 5.4] yields $Q^r = Q^\ell$. By [7, Theorem 2.38], R is a right and left Utumi-ring. Thus every $\mathcal{S}$-closed one-sided ideal I of R is an annihilator, since R is also a left p.p.-ring. By Theorem 3.7, R is a Baer-ring, and therefore I is generated by an idempotent. □

Since conditions (a) and (b) in Theorem 4.2 are right–left symmetric, it is of course also equivalent to the left module version of conditions (c), (d), and (e). In particular, one obtains

Corollary 4.3. *The following are equivalent for a ring R:*

(a) *R is a right strongly non-singular ring for which the classes of torsion-free and non-singular right R-modules coincide.*
(b) *R is a left strongly non-singular ring for which the classes of torsion-free and non-singular left R-modules coincide.*
(c) *The classes of torsion-free and non-singular modules coincide for right and left R-modules.*

Proof. It remains to show that (c) implies (a). By Theorem 3.7, R is a right and left Utumi-p.p.-ring such that Q^r is semi-simple Artinian. Now apply Theorem 4.2. □

Furthermore, if R is a right strongly non-singular p.p.-ring R which does not contain an infinite set of orthogonal idempotents, then one obtains the following characterization of finitely generated torsion-free modules which resembles the one for finitely generated non-singular modules in [7, Theorem 5.17]: a finitely generated right R-module is torsion-free if and only if it is isomorphic to a submodule of a free module.

Let R be a semi-prime right and left Goldie-ring. By [7, Theorems 3.35 and 3.37], R is a right and left non-singular ring, and there exists a semi-simple Artinian ring Q which is the right and left classical rings of quotients of R as well as the maximal right and left ring of quotients of R. By [11, Chapter XI, Proposition 5.4], Q is flat as a right and left R-module. Because of [11, Chapter XII, Corollary 2.6], Q is a perfect right and left localization of R, i.e., R is a right and left strongly non-singular ring. Furthermore, every essential right (left) ideal of R contains a regular element c of R [3,7]. Since cR and Rc are essential in R [7], an R-module is non-singular if and only if it is torsion-free in the classical sense. Furthermore, these rings are the only right Ore-rings for which the concepts of non-singularity and classical torsion-freeness coincide (e.g., see [7, Problem 3.D.17]).

By virtue of the above, torsion-free singular modules do not exist over semi-prime Goldie-rings, because—as we have pointed out after Lemma 2.1—torsion-free right R-modules are torsion-free in the classical sense.

Theorem 4.4. *The following are equivalent for a semi-prime ring R:*

(a) *R is a right and left Goldie-ring which is a p.p.-ring.*
(b) *The following are equivalent for a right or left R-module M:*
  (i) *M is non-singular.*
  (ii) *M is torsion-free.*
(c) *R is a right p.p.-ring without an infinite set of orthogonal idempotents such that every finitely generated submodule of Q^r is contained in a cyclic free submodule.*
(d) *R is a right p.p.-ring without an infinite family of orthogonal idempotents such that Q^r is flat as a right R-module.*

Proof. (a) $\Rightarrow$ (c). Let U be a finitely generated submodule of Q^r. Without loss of generality, it suffices to consider the case where U is generated by two elements, a and b. Because R is a left Ore-domain, there are $r, s \in R$ as well as regular elements c and d of R such that $a = c^{-1}r$ and $b = d^{-1}s$. Select regular elements c_1 and d_1 in R such that $d_1c = c_1d$. Then, $a = c^{-1}d_1^{-1}d_1r = (d_1c)^{-1}d_1r$ and $b = d^{-1}c_1^{-1}c_1s = (c_1d)^{-1}c_1s$. Therefore, $aR + bR \subseteq (c_1d)^{-1}R$, and the latter module is free.

(c) $\Rightarrow$ (b). Since R has no infinite set of orthogonal idempotents, R is a left p.p.-ring. Moreover, R is a torsion-free ring, and submodules of torsion-free modules are torsion-free. Therefore, all finitely generated submodules of Q^r are torsion-free. By Lemma 2.1, Q^r is a torsion-free right R-module. Because of Theorem 3.7, the classes of non-singular and torsion-free right R-modules coincide. Another application of Theorem 3.7 yields that R is a right Goldie-ring. Since R is semi-prime, it has classical right ring of quotients, Q, which is also the maximal right ring of quotients of R. The ring Q is semi-simple Artinian, so it is flat as a left R-module; see [11, Corollary 2.6]. Once we have established that Q is also the classical left ring of quotients, (b) follows by symmetry. Since classical right and left rings of quotients coincide if they exist, it remains to show that R is a left Ore-ring.

To see this, let $r, c \in R$ with c regular, and consider the submodule of Q^r generated by rc^{-1} and c^{-1}. There exist $q \in Q^r$ and $s_1, s_2 \in R$ such that $r = qs_1$ and $c^{-1} = qs_2$. Then $1 = cqs_2$ yields that s_2 is a right regular element of R. Since R is a semi-prime right Goldie-ring, right regular elements are regular. Hence, s_2 is invertible in Q, and $q = c^{-1}s_2^{-1} = (s_2c)^{-1}$. Therefore, $rc^{-1} = qs_1 = (s_2c)^{-1}s_1$, and R satisfies the left Ore-condition.

To verify (b) $\Rightarrow$ (a), observe that R is a right and left Goldie-ring by Theorem 3.7 and its analogue for left modules.

Because of Theorem 4.2, R is a right strongly non-singular ring, and (b) $\Rightarrow$ (d) holds. To prove that (d) implies (b), by Corollary 4.3 it remains to show that Q^r is a perfect left localization of R. For this, it suffices to check that the multiplication map $\lambda : Q^r \otimes_R Q^r \to Q^r$ is a monomorphism. Let $q_{1j}, q_{2j} \in Q^r$ for $j = 1, \ldots, n$ such that $\sum_{j=1}^n q_{1j}q_{2j} = 0$. Observe that a right p.p.-ring without an infinite family of orthogonal idempotents such that Q^r is flat as right R-module is a right Goldie-ring by Theorem 3.7. Therefore, Q^r is the classical right ring of quotients of R. Hence, there are $r_{ij} \in R$ for $i = 1, 2$ and $j = 1, \ldots, n$ and a regular $t \in R$ with $q_{ij} = r_{ij}t^{-1}$. Choose $s_{2j} \in R$ and a regular $t_1 \in R$ such that $r_{2j}t_1 = ts_{2j}$. Thus $\sum_{j=1}^n q_{1j} \otimes q_{2j} = \sum_{j=1}^n r_{1j}t^{-1} \otimes r_{2j}t^{-1} = \sum_{j=1}^n r_{1j}t^{-1} \otimes tt^{-1}r_{2j}t^{-1} = \sum_{j=1}^n r_{1j} \otimes s_{2j}t_1^{-1}t^{-1} = \sum_{j=1}^n 1 \otimes r_{1j}s_{2j}t_1^{-1}t^{-1} = 1 \otimes q$

where $q = \sum_{j=1}^{n} r_{1j} s_{2j} t_1^{-1} t^{-1} \in Q^r$. Then $0 = \lambda(\sum_{j=1}^{n} q_{1j} \otimes q_{2j}) = \lambda(1 \otimes q) = q$, and λ is one-to-one. □

For reduced rings, we obtain the following result:

Corollary 4.5.

(a) *The following are equivalent for a ring R:*
  (i) *R is a reduced right strongly non-singular ring for which the classes of torsion-free and non-singular right R-modules coincide.*
  (ii) *$R = R_1 \times \cdots \times R_n$ where each ring R_i has right and left Goldie-dimension* 1.

(b) *Let R be a domain. Then, R is strongly non-singular, and the classes of torsion-free and non-singular right R-modules coincide if and only if $G\text{-dim}\, R_R = 1$ and $G\text{-dim}_R\, R = 1$.*

Proof. Combine Theorems 4.2 and 4.4 with Corollary 3.9. □

5. Torsion-freeness and Morita-equivalence

Let $S = \mathbb{Z}[x]$ be the ring of polynomials over $\mathbb{Z}$, and F its field of quotients. Then, $R = \mathrm{Mat}_2(S)$ is a semi-prime right and left Goldie-ring. However, R is not a right or left p.p.-ring, because $\mathbb{Z}[x]$ contains a non-projective ideal generated by two elements [3, Theorem 8.17]. Therefore, all torsion-free R-modules are non-singular, but in view of Theorem 3.7 there exists a non-singular R-module which is not torsion-free. This example also shows that the equality of the classes of non-singular and torsion-free right R-modules is not preserved under Morita-equivalence even if R is a right strongly non-singular ring.

Theorem 5.1. *The following are equivalent for a ring R:*

(a) *R is a right strongly non-singular right semi-hereditary ring without an infinite set of orthogonal idempotents.*

(b) *Every ring S Morita-equivalent to R is strongly non-singular, and the classes of torsion-free and non-singular right S-modules coincide.*

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{F} : M_R \to M_S$ and $\mathcal{G} : M_S \to M_R$ be an equivalence. By Theorem 3.7, R is a right Goldie-ring, and $Q^r(R)$ is a semi-simple Artinian ring. Because of [11, Chapter X, Proposition 3.2], the rings $Q^r(S)$ and $Q^r(R)$ are Morita-equivalent. Hence, $Q^r(S)$ is semi-simple Artinian too. Observe that S is non-singular if and only if $Q^r(S)$ is a regular ring (cf. [11, Chapter XII, Proposition 2.2]), and the latter holds for $Q^r(S)$. Moreover, S has finite right Goldie-dimension, since it is a right non-singular ring with a semi-simple Artinian right ring of quotients. It remains to show that S is strongly right non-singular. Once this has been established, Theorem 4.2 will guarantee that the classes of non-singular and torsion-free right S-modules coincide.

Let M be a finitely generated non-singular S-module. Since S is a right non-singular ring, M is isomorphic to a submodule of $\bigoplus_n Q^r(S)$ for some $n < \omega$, and $\mathcal{G}(M)$ is isomorphic to a submodule of $\mathcal{G}(\bigoplus_n Q^r(S))$. But $Q^r(S)$ being the injective hull of S as an S-module yields that $\mathcal{G}(Q^r(S))$ is the injective hull of $\mathcal{G}(S)$ by [2, Proposition 21.6]. Since $\mathcal{G}(S)$ is a projective R-module, it is non-singular, and the same holds for $\mathcal{G}(Q^r(S))$. Consequently, $\mathcal{G}(M)$ is a non-singular R-module which is in view of [2, Proposition 21.6] finitely generated. Since R is right strongly non-singular, there is a projective right R-module P such that $\mathcal{G}(M)$ is isomorphic to a submodule of P. Then $M \cong \mathcal{F}\mathcal{G}(M)$ is isomorphic to a submodule of $\mathcal{F}(P)$ which is a projective S-module.

(b) $\Rightarrow$ (a). Obviously, R is a strongly non-singular ring for which the classes of torsion-free and non-singular right R-modules coincide. Thus R has finite right Goldie-dimension, and contains no infinite set of orthogonal idempotents. It remains to show that R is right semi-hereditary. Observe that $\mathrm{Mat}_n(R)$ is Morita-equivalent to R for all $0 < n < \omega$. Since the classes of non-singular and torsion-free right $\mathrm{Mat}_n(R)$-modules coincide, $\mathrm{Mat}_n(R)$ is a right p.p.-ring for all $n < \omega$ by Theorem 3.7. Hence, for each such n, every right ideal of R which is generated by at most n elements is projective by [3, Theorem 8.17], i.e., R is right semi-hereditary. □

In [4], Chatters and Khuri investigated right non-singular right Goldie-rings for which every finitely generated non-singular right module is projective. In view of the previous results of this paper, these are exactly the rings discussed in previous theorem.

We now turn to additional characterizations of the rings discussed in Theorem 5.1. In particular, we show that they are the rings for which the classes of (torsion-free,) non-singular and flat modules coincide, thus completing the discussion in [7, Proposition 5.16] and [1, Theorem 1].

Theorem 5.2. *The following are equivalent for a ring R:*

(a) *R is right and left semi-hereditary such that $Q^r = Q^\ell$ is semi-simple Artinian.*
(b) *R is a right strongly non-singular right semi-hereditary ring without an infinite set of orthogonal idempotents.*
(c) *R is a left semi-hereditary ring without an infinite set of orthogonal idempotents such that Q^r is flat as a right R-module.*
(d) *The following are equivalent for a right R-module M:*
   (i) *M is torsion-free.*
   (ii) *M is non-singular.*
   (iii) *M is flat.*
(e) *The following are equivalent for a submodule U of a torsion-free right R-module M:*
   (i) *U is relatively divisible in M.*
   (ii) *U is $\mathcal{S}$-closed in M.*
   (iii) *The sequence $0 \to U \to M \to M/U \to 0$ is pure-exact.*
(f) *The classes of flat and non-singular right R-modules coincide.*

Proof. (a) $\Rightarrow$ (b). By Theorem 4.2, R is right strongly non-singular and has no infinite set of orthogonal idempotents.

(b) $\Rightarrow$ (c). Clearly, R is right non-singular. Since Q^r is a left perfect localization of R, the right R-module Q^r is flat. In view of [11, Chapter XII, Corollary 7.4], R is left semi-hereditary.

(c) ⇒ (d). Since R has no infinite set of orthogonal idempotents, it is a right p.p.-ring by [3, Lemma 8.4]. Because of Theorem 3.7, the classes of torsion-free and non-singular modules coincide. Since every flat module is torsion-free, it remains to show that non-singular modules are flat. However, since R is right non-singular, this is a direct consequence of [7, Proposition 5.16].

(d) ⇒ (e). Let M be a torsion-free right R-module, and U a submodule of M. To see (i) ⇒ (ii), assume that U is relatively divisible in M. By [5], M/U is torsion-free, and hence non-singular by (d), i.e., U is $\mathcal{S}$-closed in M.

(ii) ⇒ (iii). If U is $\mathcal{S}$-closed in M, then M/U is non-singular. By (d), it is flat, and the sequence $0 \to U \to M \to M/U \to 0$ is pure.

(iii) ⇒ (i). If the given sequence is pure, then observe that M is flat because of (d). Therefore, M/U is flat. But flat modules are torsion-free, and hence U is a relatively divisible submodule of M.

(e) ⇒ (a). Since a submodule of a torsion-free module is $\mathcal{S}$-closed if and only if it is relatively divisible, Theorem 3.7 implies that R is a right p.p.-ring with no infinite set of orthogonal idempotents. Moreover, R is right and left non-singular, and Q^r is semi-simple Artinian. By [11, Chapter XII, Corollary 2.6], Q^r is a perfect right localization of R. In particular, the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism.

Let M be a non-singular right R-module, and consider an exact sequence $0 \to U \to F \to M \to 0$ where F is free. Since U is an $\mathcal{S}$-closed submodule of M, and F is flat, the sequence is pure-exact by (e), and M is flat. [7, Proposition 5.16] yields that R is a left semi-hereditary ring and Q^r is flat as a right R-module. Hence from [11, Chapter XI, Proposition 5.4] we conclude that $Q^r = Q^\ell$, since Q^r is semi-simple Artinian. Thus, Q^ℓ is a perfect right localization of R. However, by [11, Chapter XII, Corollary 7.4], a left semi-hereditary ring R for which Q^ℓ is a right perfect localization of R is right semi-hereditary.

Since (d) ⇒ (f) is obvious, it remains to show (f) ⇒ (c). Since R_R is flat, R is a right non-singular ring. By [7, Proposition 5.16], R is left semi-hereditary, and Q^r is flat as a right R-module. We need only to show that R has no infinite set of orthogonal idempotents. For this, let I be an infinite set, and consider the right R-module $M = \bigoplus_{i \in I} Q^r$. Denote its injective hull by E. Clearly, M is pure in E, since the direct sum of the Q^r is pure in their direct product in which E is a submodule. Next observe that E is non-singular, since Q^r is a non-singular module. Then E is flat by (f), and the same holds for E/M. Using (f) once more yields that E/M is non-singular. Since M is essential in E, this is possible only if $E = M$. By [11, Chapter XIII, Proposition 3.3], R has finite right Goldie-dimension, and therefore contains no infinite set of orthogonal idempotents. □

Of course the right–left symmetry observed in Section 4 exists in this setting too. Furthermore, using Corollary 3.10, one obtains that the commutative rings satisfying Theorem 5.2 are the finite products of Prüfer-domains.

Corollary 5.3. *The following are equivalent for a right non-singular ring R without an infinite set of idempotents*:

(a) *R is a right strongly non-singular right semi-hereditary ring.*
(b) *The class of flat right R-modules is the torsion-free class of a hereditary torsion theory.*

Proof. (a) ⇒ (b). By Theorem 5.2, the class of flat right R-modules coincides with the class of non-singular modules which is the torsion-free class of the hereditary Goldie torsion theory.

(b) ⇒ (a). Since the class of flat modules is closed with respect to submodules, every right ideal of R is flat, and R is a torsion-free ring. If M is a non-singular right R-module, then every finitely generated submodule U of M is isomorphic to a submodule of $\bigoplus_n Q^r$ for some $n < \omega$ by [11, Chapter XII, Proposition 7.2]. As Q^r is the injective hull of the flat module R, it is also flat since the class of flat modules forms the torsion-free class of a hereditary torsion theory [11, Chapter VI, Proposition 3.2]. Thus, U is flat. However, a module is flat whenever all its finitely generated submodules are flat. Hence, all non-singular right R-modules are flat. By [7, Proposition 5.16], R is left semi-hereditary, and Q^r is flat as a right R-module. Because of Theorem 5.2, R is also right semi-hereditary and right strongly non-singular. □

Theorem 5.2 applies in particular if R is a semi-prime right and left Goldie-ring. Semi-prime semi-hereditary Goldie-rings were discussed [9].

Corollary 5.4. *The following are equivalent for a semi-prime ring R:*

(a) *R is a right and left semi-hereditary right and left Goldie-ring.*
(b) *The following are equivalent for every right or left R-module M:*
  (i) *M is torsion-free.*
  (ii) *M is non-singular.*
  (iii) *M is flat.*
(c) *R is a right semi-hereditary ring without an infinite set of orthogonal idempotents such that every finitely generated submodule of Q^r is contained in a cyclic free submodule.*

Proof. Combine Theorems 5.2 and 4.4. □

This section concludes with an example of a right and left strongly non-singular hereditary ring R without an infinite set of orthogonal idempotents which is not semi-prime.

Theorem 5.5. *The following are equivalent for a right and left Noetherian ring R:*

(a) *$R = R_1 \times \cdots \times R_n$ where each R_i is either a prime right and left hereditary ring, or is Morita-equivalent to a lower triangular matrix ring over a division algebra.*
(b) *The following are equivalent for each right R-module M:*
  (i) *M is torsion-free.*
  (ii) *M is non-singular.*
  (iii) *M is flat.*

Proof. (a) ⇒ (b). If R_i is a prime ring, then (b) holds for R_i because of Corollary 5.4. On the other hand, if R_i is Morita-equivalent to a lower triangular matrix ring over a division algebra, then R_i is right non-singular ring, and all non-singular right R-modules are projective by [7, Proposition 5.22]. In particular, $Q^r(R_i)$ is flat as a right R_i-module.

Furthermore, [7, Theorem 5.21] implies that R_i is right and left semi-hereditary. Since R_i has no infinite set of orthogonal idempotents, Theorem 5.2 yields that (b) holds for R_i. Thus R is the product of rings satisfying (b), and therefore it also satisfies (b).

(b) ⇒ (a). By Theorem 5.2, R is a right semi-hereditary ring. Since R is right and left Noetherian, it is right hereditary, and left hereditary by [3, Corollary 8.18]. Consequently, $R = R_1 \times \cdots \times R_n$ where each R_i is prime or right and left Artinian by [3, Theorem 8.22]. Without loss of generality, we may assume that each R_i is indecomposable as a ring. As R_i also satisfies (b), it remains to show that a right and left Artinian indecomposable ring R for which the classes of torsion-free, flat, and non-singular right modules coincide is Morita-equivalent to a lower triangular matrix ring over a division algebra. By Theorem 5.2, the maximal right and left rings of quotients of R are equal. Hence, [7, Theorem 5.23] yields that R is a right non-singular ring for which all non-singular modules are projective. Since R is indecomposable, it is Morita-equivalent to a lower triangular matrix ring over a division algebra because of [7, Theorem 5.27]. □

In particular, every lower triangular matrix ring over a division algebra is an example of a ring which is not semi-prime, but for which the classes of torsion-free, flat, and non-singular modules coincide.

6. Applications

In this section, we show that modules over rings satisfying the equivalent conditions of Theorem 3.7 behave in many ways similar to those over integral domains (e.g., see [10, Chapter 8]).

Hattori [8] calls a right R-module M *divisible* if $\mathrm{Ext}^1_R(R/rR, M) = 0$ for every $r \in R$. Injective modules are obviously divisible.

Proposition 6.1. *Let R be a right Utumi-p.p.-ring without an infinite set of orthogonal idempotents.*

(a) *Every torsion-free divisible right R-module is injective.*
(b) *A right R-module is torsion-free if and only if it is an essential extension of a projective module.*

Proof. (a) Let M be a divisible torsion-free module. Its injective hull $E(M)$ is torsion-free by Corollary 3.8. Since M is divisible, it is a relatively divisible submodule of $E(M)$. Therefore, $E(M)/M$ is torsion-free. Theorem 3.7 yields that $E(M)/M$ is non-singular. Since M is essential in $E(M)$, this is possible only if $M = E(M)$.

(b) Let M be an essential extension of a projective module P. Since P is torsion-free, the same holds for M by Theorem 3.7. Conversely, let M be a torsion-free module. By Theorem 3.7, M is non-singular. Consider the set $\Im = \{S \subseteq M \mid \sum_{x \in S} xR$ is a direct sum$\}$. Since forming a direct sum is of finite character, there is a maximal subset $S_0 \in \Im$. Let V be the $\mathcal{S}$-closure of $\bigoplus_{S_0} xR$ in the non-singular module M. It remains to show that $M = V$.

If $M \neq V$, then let $x \in M \setminus V$, and consider $I = \{r \in R \mid xr \in V\}$. Since M/V is non-singular, I is not an essential right ideal of R, and there is a non-zero right ideal J of R with $I \cap J = 0$. Then, $xJ \cap V = 0$, but $xJ \neq 0$. If y is a non-zero element of xJ, then $S_0 \cup \{y\} \in \Im$, a contradiction. The conclusion is that injective hulls of torsion-free modules are torsion-free. □

Both R and Q^r are R-R-bimodules, and the embedding $R \to Q^r$ is a bimodule map. Therefore, Q^r/R carries a natural bimodule structure that makes both $- \otimes_R Q^r$ and $\mathrm{Tor}_1^R(-, Q^r/R)$ right R-modules.

If R is a right Utumi-p.p-ring without an infinite set of orthogonal idempotents, then the torsion-free modules form the torsion-free class of a hereditary torsion theory by Corollary 3.8. The associated class of *torsion modules* has to be the class of singular modules since the torsion-free class of this torsion-theory coincides with the class of non-singular modules by Theorem 3.7.

Lemma 6.2. *Let R be a right Utumi-p.p-ring without an infinite set of orthogonal idempotents. A right R-module M is torsion if and only if $M \otimes_R Q^r = 0$.*

Proof. Let M be a singular right R-module, and consider an element of the form $x \otimes q$ for some $x \in M$ and $q \in Q^r$. There is an essential right ideal I of R such that $xI = 0$. From [7, Proposition 2.32] it follows that IQ^r is an essential right ideal of Q^r. However, Theorem 3.7 shows that Q^r is semi-simple Artinian. This is possible only if $IQ^r = Q^r$. Hence, there are $i_1, \dots, i_k \in I$ and $q_1, \dots, q_k \in Q^r$ satisfying $q = i_1q_1 + \dots + i_kq_k$. Consequently, $x \otimes q = x \otimes (i_1q_1 + \dots + i_kq_k) = xi_1 \otimes q_1 + \dots + xi_k \otimes q_k = 0$.

Conversely, let M be a right R-module with $M \otimes_R Q^r = 0$. We obtain the exact sequence $0 = M \otimes_R Q^r \to (M/Z(M)) \otimes_R Q^r \to 0$ from which $M/Z(M) \otimes_R Q^r = 0$ follows. Let E be the injective hull of $\overline{M} = M/Z(M)$. We obtain the exact sequence $0 = \overline{M} \otimes_R Q^r \to E \otimes_R Q^r \to E/\overline{M} \otimes_R Q^r = 0$ where the last term vanishes by what has been shown in the first paragraph. Hence, $E \otimes_R Q^r = 0$. There is a natural epimorphism $E \otimes_R Q^r \to E$ defined by $x \otimes q = xq$ since E is a right Q^r-module by [11, Chapter XII, Corollary 2.8]. Hence, $E = 0$, and the same holds for $\overline{M}$. Consequently, M is singular. □

We now turn to the functor Tor_1^R.

Theorem 6.3. *Let R be a right Utumi-p.p-ring without an infinite set of orthogonal idempotents. Then $\mathrm{Tor}_1^R(M, Q^r/R) \cong Z(M)$ for all right R-modules M.*

Proof. By Theorem 3.7, R has finite Goldie-dimension. From [11, Chapter XII, Theorem 2.5] we derive that Q^r is a semi-simple Artinian ring. By [11, Chapter XII, Corollary 2.6], Q^r is a perfect right localization of R, and hence flat as a left R-module.

Let M be a non-singular right R-module, and let consider the exact sequence $0 = \mathrm{Tor}_1^R(M, Q^r) \to \mathrm{Tor}_1^R(M, Q^r/R) \to M \otimes_R R \to M \otimes_R Q^r$ in which the first term vanishes, since Q^r is a flat left R-module. If $E(M)$ denotes the injective hull of M, then the last map in the sequence fits into the top-row of the commutative diagram

$$\begin{array}{ccc} M \otimes_R R & \longrightarrow & M \otimes_R Q^r \\ \downarrow & & \downarrow \\ E(M) \otimes_R R & \xrightarrow{\sigma} & E(M) \otimes_R Q^r \end{array}$$

where the vertical maps are the monomorphisms induced by the inclusion $M \subseteq E(M)$. If $\alpha : E(M) \otimes_R R \to E(M)$ and $\beta : E(M) \otimes_R Q^r \to E(M)$ are the multiplication maps, then $\alpha = \beta\sigma$. Since α is an isomorphism, σ is one-to-one. But then, the top map in the diagram has to be a monomorphism too. In particular, $\operatorname{Tor}_1^R(M, Q^r/R) = 0$.

Now let M be an arbitrary right R-module. Since Q^r/R has flat dimension at most 1 as a left R-module, we obtain the exact sequence

$$\begin{aligned} 0 = \operatorname{Tor}_2^R\big(M/Z(M), Q^r/R\big) &\to \operatorname{Tor}_1^R\big(Z(M), Q^r/R\big) \\ &\to \operatorname{Tor}_1^R\big(M, Q^r/R\big) \to \operatorname{Tor}_1^R\big(M/Z(M), Q^r/R\big) = 0 \end{aligned}$$

in which the last term vanishes by what has been shown in the first paragraph of this proof. Therefore, $\operatorname{Tor}_1^R(M, Q^r/R) \cong \operatorname{Tor}_1^R(Z(M), Q^r/R)$. On the other hand, there is an exact sequence $0 = \operatorname{Tor}_1^R(Z(M), Q^r) \to \operatorname{Tor}_1^R(Z(M), Q^r/R) \to Z(M) \otimes_R R \to Z(M) \otimes_R Q^r = 0$ where the last term vanishes by Lemma 6.2, and the first term by the flatness of Q^r as a left R-module. Hence,

$$\operatorname{Tor}_1^R\big(M, Q^r/R\big) \cong \operatorname{Tor}_1^R\big(Z(M), Q^r/R\big) \cong Z(M) \otimes_R R \cong Z(M). \qquad \square$$

Corollary 6.4. *Let R be a right Utumi-p.p.-ring with no infinite set of orthogonal idempotents. A right R-module M is non-singular if and only if* $\operatorname{Tor}_1^R(M, Q^r/R) = 0$.

We conclude with an example showing that Theorem 6.3 may fail over arbitrary right semi-hereditary rings. Let R be a regular ring which is not semi-simple Artinian. Since R is regular, all right R-modules are flat, but there exists a proper essential right ideal I of R. Then, R/I is a non-zero singular module, but $\operatorname{Tor}_1^R(R/I, Q^r/R) = 0$.

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