Automorphisms of Chevalley groups of different types over commutative rings

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In this paper we prove that every automorphism of (elementary) adjoint Chevalley group with root system of rank > 1 over a commutative ring (with 1/2 for the systems $A_2$, $F_4$, $B_1$, $G_2$; with 1/2 and 1/3 for the system $G_2$) is standard, i.e., it is a composition of ring, inner, central and graph automorphisms.

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Introduction

Study of automorphism of classical groups was started by the work of Schreier and van der Varden [33] in 1928. They described all automorphisms of the group $PSL_n$ ($n \geq 3$) over an arbitrary field.

Diedonne [23] (1951) and Rickart [32] (1950) introduced the involution method, and with the help of this method described automorphisms of the group $GL_n$ ($n \geq 3$) over a skew field.

The first step in construction the automorphism theory over rings, namely, for the group $GL_n$ ($n \geq 3$) over the ring of integer numbers, made Hua and Reiner [26] (1951), after them some papers on commutative integral domains appeared.

The methods of the papers mentioned above were based mostly on studying involutions in the corresponding linear groups.

O’Meara [30] in 1976 invited very different (geometrical) method, which did not use involutions, with the help of this method he described automorphism of the group $GL_n$ ($n \geq 3$) over domains.

In 1982 Petechuk [31] described automorphisms of the groups $GL$, $SL$ ($n \geq 4$) over arbitrary commutative rings. If $n = 3$, then automorphisms of given linear groups are not always standard. They are standard either if a ring 2 is invertible, or if a ring is a domain, or it is a semisimple ring.
Isomorphisms of the groups $GL_n(R)$ and $GL_m(S)$ over arbitrary associative rings with $1/2$ for $n, m \geq 3$ were described in 1981 by I.Z. Golubchik and A.V. Mikhailev [24] and independently by E.I. Zelmanov [39]. In 1997 I.Z. Golubchik described isomorphisms between these groups for $n, m \geq 4$, but over arbitrary associative rings with $1$ [25].

In 50-th years of the previous century Chevalley, Steinberg and others introduced the concept of Chevalley groups over commutative rings, which includes classical linear groups (special linear $SL$, special orthogonal $SO$, symplectic $Sp$, spinor $Spin$, and also projective groups connected with them) over commutative rings.

Clear that isomorphisms and automorphisms of Chevalley groups were also studied intensively.

The description of isomorphisms of Chevalley groups over fields was obtained by R. Steinberg [35] for the finite case and by J. Humphreys [27] for the infinite one. Many papers were devoted to descriptions of automorphisms of Chevalley groups over different commutative rings, we can mention here the papers of Borel–Tits [5], Carter–Chen Yu [15], Chen Yu [16–20], E. Abe [1], A. Klyachko [29].

But the question of description of automorphisms of Chevalley groups over arbitrary commutative rings still has been open.

In the paper [8] of the author it was shown that automorphisms of adjoint elementary Chevalley groups with root systems $A_l, D_l, E_l, l \geq 2$, over local rings with invertible 2 can be represented as the composition of ring automorphism and an automorphism-conjugation, where as automorphism-conjugation we call a conjugation of elements of a Chevalley group in the adjoint representation by some matrix from the normalizer of this group in $GL(V)$. In the paper [10] according to the results of [8] it was proved that every automorphism of an arbitrary (elementary) Chevalley group of the described type is standard, i.e., it is represented as the composition of ring, inner, central and graph automorphism. In the same paper it was obtained the theorem describing the normalizer of Chevalley groups in their adjoint representation, which also holds for local rings without $1/2$.

In the papers [12,9,11] by the same methods we showed that all automorphisms of Chevalley groups with the root systems $F_4, G_2, B_l, l \geq 2$, over local rings with $1/2$ (in the case $G_2$ also with $1/3$) are standard. In the paper [13] we described automorphisms of Chevalley groups of types $A_l, D_l, E_l, l \geq 3$, over local rings without $1/2$.

In the present paper with the help of results of author’s papers [8,10,12,9,11,13], and also the methods, described by V.M. Petchuk in [31] for the special linear group $SL$, we describe automorphisms of adjoint Chevalley groups over arbitrary commutative rings with the assumption that the corresponding root systems have rank $>1$, for the root systems $A_2, F_4, B_l, C_l$ the ring contains $1/2$, for the system $G_2$ the ring contains $1/2$ and $1/3$.

1. Definitions and main theorem

We fix a root system $Φ$ of rank $>1$. All details about root systems and their properties can be found in [28,6]. Suppose now that we have some semisimple complex Lie algebra $L$ of type $Φ$ with Cartan subalgebra $H$ (detailed information about semisimple Lie algebras can be found in the book [28]).

Then we can choose a basis $\{h_1, \ldots, h_l\}$ in $H$ and for every $α \in Φ$ elements $x_α \in L_α$ so that $\{h_1; x_α\}$ form a basis in $L$ and for every two elements of this basis their commutator is an integral linear combination of the elements of the same basis.

Let us introduce elementary Chevalley groups (see, for example, [34]).

Let $L$ be a semisimple Lie algebra (over $C$) with a root system $Φ$, $π : L → GL(V)$ be its finitely dimensional faithful representation (of dimension $n$). If $H$ is a Cartan subalgebra of $L$, then a functional $λ \in H^*$ is called a weight of a given representation, if there exists a nonzero vector $v \in V$ (that is called a weight vector) such that for any $h \in H$, $π(h)v = λ(h)v$.

In the space $V$ there exists a basis of weight vectors such that all operators $π(x_α)^k/k!$ for $k \in N$ are written as integral (nilpotent) matrices. This basis is called a Chevalley basis. An integral matrix also can be considered as a matrix over an arbitrary commutative ring with $1$. Let $R$ be such a ring. Consider matrices $n \times n$ over $R$, matrices $π(x_α)^k/k!$ for $α \in Φ$, $k \in N$ are included in $M_n(R)$.

Now consider automorphisms of the free module $R^m$ of the form
exp(tx_α) = x_α(t) = 1 + tx_α + t^2(x_α)^2/2 + \cdots + t^k(x_α)^k/k! + \cdots.

Since all matrices x_α are nilpotent, we have that this series is finite. Automorphisms x_α(t) are called elementary root elements. The subgroup in Aut(R^n), generated by all x_α(t), α ∈ Φ, t ∈ R, is called an elementary adjoint Chevalley group (notation: E_{ad}(Φ, R) = E_{ad}(R)).

The action of x_α(t) on the Chevalley basis is described in [14,38].

All weights of a given representation (by addition) generate a lattice (free Abelian group, where every Z-basis is also a C-basis in H^*, that is called the weight lattice Λ_π.

Elementary Chevalley groups are defined not even by a representation of the Chevalley groups, but just by its weight lattice. Namely, up to an abstract isomorphism an elementary Chevalley group is completely defined by a root system Φ, a commutative ring R with 1 and a weight lattice Λ_π.

Among all lattices we can mark the lattice corresponding to the adjoint representation: it is generated by all roots (the root lattice Λ_{ad}). The corresponding (elementary) Chevalley group is called adjoint.

Introduce now Chevalley groups (see [34,21,4,14,22,37,38], and also latter references in these papers).

Consider semisimple linear algebraic groups over algebraically closed fields. These are precisely elementary Chevalley groups E_π(Φ, K) (see [34, §5]).

All these groups are defined in SL_n(K) as common set of zeros of polynomials of matrix entries a_{ij} with integer coefficients (for example, in the case of the root system C) and the universal representation we have n = 2l and the polynomials from the condition (a_{ij})Q(a_{ij}) = Q = 0). It is clear now that multiplication and taking inverse element are also defined by polynomials with integer coefficients. Therefore, these polynomials can be considered as polynomials over arbitrary commutative ring with a unit. Let some elementary Chevalley group E over C be defined in SL_n(C) by polynomials p_1(a_{ij}), \ldots, p_m(a_{ij}). For a commutative ring R with a unit let us consider the group

G(R) = \{ (a_{ij}) ∈ SL_n(R) | \tilde{p}_1(a_{ij}) = 0, \ldots, \tilde{p}_m(a_{ij}) = 0 \},

where \tilde{p}_1(\ldots), \ldots, \tilde{p}_m(\ldots) are polynomials having the same coefficients as p_1(\ldots), \ldots, p_m(\ldots), but considered over R.

This group is called the Chevalley group G_π(Φ, R) of the type Φ over the ring R, and for every algebraically closed field K it coincides with the elementary Chevalley group.

The subgroup of diagonal (in the standard basis of weight vectors) matrices of the Chevalley group G_π(Φ, R) is called the standard maximal torus of G_π(Φ, R) and it is denoted by T_π(Φ, R). This group is isomorphic to Hom(A_π, R^*).

Let us denote by h(χ) the elements of the torus T_π(Φ, R), corresponding to the homomorphism χ ∈ Hom(A(π), R^*).

In particular, h_α(u) = h(χ_α,u) (u ∈ R^*, α ∈ Φ), where

χ_{α,u} : λ ↦ u^{\langle χ, α \rangle} (λ ∈ A_π).

Note that the condition

G_π(Φ, R) = E_π(Φ, R)

is not true even for fields, that are not algebraically closed.

Let us show the difference between Chevalley groups and their elementary subgroups in the case when R is semifield. In this case G_π(Φ, R) = E_π(Φ, R)T_π(Φ, R) (see [2]), and elements h(χ) are connected with elementary generators by the formula

h(χ)α_β(ξ)h(χ)^{-1} = x_β(χ(β)ξ).

Define four types of automorphisms of a Chevalley group G_π(Φ, R), we call them standard.
Central automorphisms. Let $C_G(R)$ be a center of $G_\pi (\Phi , R)$, $\tau : G_\pi (\Phi , R) \to C_G(R)$ be some homomorphism of groups. Then the mapping $x \mapsto \tau(x)x$ from $G_\pi (\Phi , R)$ onto itself is an automorphism of $G_\pi (\Phi , R)$, that is denoted by $\tau$ and called a central automorphism of the group $G_\pi (\Phi , R)$.

Ring automorphisms. Let $\rho : R \to R$ be an automorphism of the ring $R$. The mapping $(a, j) \mapsto (\rho(a_{ij}))$ from $G_\pi (\Phi , R)$ onto itself is an automorphism of the group $G_\pi (\Phi , R)$, that is denoted by the same letter $\rho$ and is called a ring automorphism of the group $G_\pi (\Phi , R)$. Note that for all $\alpha \in \Phi$ and $t \in R$ an element $x_\alpha(t)$ is mapped to $x_\alpha(\rho(t))$.

Inner automorphisms. Let $S$ be some ring containing $R$, $g$ be an element of $G_\pi (\Phi , S)$, that normalizes the subgroup $G_\pi (\Phi , R)$. Then the mapping $x \mapsto gxg^{-1}$ is an automorphism of the group $G_\pi (\Phi , R)$, that is denoted by $i_g$ and is called an inner automorphism, induced by the element $g \in G_\pi (\Phi , S)$. If $g \in G_\pi (\Phi , R)$, then call $i_g$ a strictly inner automorphism.

Graph automorphisms. Let $\delta$ be an automorphism of the root system $\Phi$ such that $\delta \Delta = \Delta$. Then there exists a unique automorphisms of $G_\pi (\Phi , R)$ (we denote it by the same letter $\delta$) such that for every $\alpha \in \Phi$ and $t \in R$ an element $x_\alpha(t)$ is mapped to $x_{\delta \alpha}(\epsilon(\alpha)t)$, where $\epsilon(\alpha) = \pm 1$ for all $\alpha \in \Phi$ and $\epsilon(\alpha) = 1$ for all $\alpha \in \Delta$.

Now suppose that $\delta_1, \ldots , \delta_k$ are all different graph automorphisms for the given root system (for the systems $E_7$, $E_8$, $B_1$, $C_1$, $F_4$, $G_2$ there can be just identical automorphism, for the systems $A_l$, $D_l$, $l \neq 4$, $E_8$ there are two such automorphisms, for the system $D_4$ there are six automorphisms). Suppose that we have a system of orthogonal idempotents of the ring $R$:

$$\{\epsilon_1, \ldots , \epsilon_k \mid \epsilon_1 + \cdots + \epsilon_k = 1, \forall i \neq j, \epsilon_i \epsilon_j = 0\}.$$  

Then the mapping

$$A_{\epsilon_1, \ldots , \epsilon_k} := \epsilon_1 \delta_1 + \cdots + \epsilon_k \delta_k$$

of the Chevalley group onto itself is an automorphism, that is called a graph automorphism of the Chevalley group $G_\pi (\Phi , R)$.

Similarly we can define four type of automorphisms of the elementary subgroup $E(R)$. An automorphism $\sigma$ of the group $G_\pi (\Phi , R)$ (or $E_\pi (\Phi , R)$) is called standard if it is a composition of automorphisms of these introduced four types.

Our aim is to prove the next main theorem:

**Theorem 1.** Let $G = G_\pi (\Phi , R)$ ($E_\pi (\Phi , R)$) be an (elementary) adjoint Chevalley group of rank $> 1$, $R$ be a commutative ring with 1. Suppose that for $\Phi = A_2, B_1, C_1$ or $F_4$ we have $1/2 \in R$, for $\Phi = G_2$ we have $1/2, 1/3 \in R$. Then every automorphism of the group $G$ is standard and the inner automorphism in the composition is strictly inner.

**2. Known notions, definitions and results, which will be used in the proof**

2.1. Localization of rings and modules; injection of a ring into the product of its localizations

**Definition 1.** Let $A$ be a commutative ring. A subset $S \subset A$ is called multiplicatively closed in $A$, if $1 \in S$ and $S$ is closed under multiplication.

Introduce an equivalence relation $\sim$ on the set of pairs $A \times S$ as follows:

$$\frac{a}{s} \sim \frac{b}{t} \iff \exists u \in S: (at - bs)u = 0.$$
By $\frac{a}{s}$ we denote the whole equivalence class of the pair $(a, s)$, by $S^{-1}R$ we denote the set of all equivalence classes. On the set $S^{-1}R$ we can introduce the ring structure by

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$ 

**Definition 2.** The ring $S^{-1}A$ is called the ring of fractions of $A$ with respect to $S$.

Let $p$ be a prime ideal of $A$. Then the set $S = A \setminus p$ is multiplicatively closed (it is equivalent to the definition of the prime ideal). We will denote the ring of fractions $S^{-1}A$ in this case by $A_p$. Consequently the ideal $S$ consists of all non-invertible elements of the ring $A_p$, i.e., $S$ is the greatest ideal of this ring, so $A_p$ is a local ring.

The process of passing from $A$ to $A_p$ is called localization at $p$.

The construction $S^{-1}A$ can be easily carried through with an $A$-module $M$. Let $m/s$ denote the equivalence class of the pair $(m, s)$, the set $S^{-1}M$ of all such fractions is made as a module $S^{-1}M$ with obvious operations of addition and scalar multiplication. As above we will write $M_p$ instead of $S^{-1}M$ for $S = A \setminus p$, where $p$ is a prime ideal of $A$.

**Proposition 1.** Every commutative ring $A$ with $1$ can be naturally embedded in the cartesian product of all its localizations by maximal ideals

$$S = \prod_{m \text{ is a maximal ideal of } A} A_m$$

by diagonal mapping, which corresponds every $a \in A$ to the element

$$\prod_m \left( \frac{a}{1} \right)_m$$

of $S$.

### 2.2. Isomorphisms of Chevalley groups over fields

We will need the description of isomorphisms between Chevalley groups over fields. Suppose that root systems under consideration have ranks $> 1$.

Introduce an additional concept of diagonal automorphism:

**Definition 3.** (See Lemma 58 from the book [34].) Let $G$ be an (elementary) Chevalley group over a field $k$, and suppose that we have some set of elements $f_\alpha \in k^*$ for all simple roots $\alpha \in \Phi$. Let us extend $f$ to a homomorphism of the whole lattice, generated by all roots, into $k^*$. Then there exists a unique automorphism $\varphi$ of the group $G$ such that

$$\varphi(x_\alpha(t)) = x_\alpha(f_\alpha t) \quad \forall \alpha \in \Phi, \ t \in k.$$ 

This automorphism is called a diagonal automorphism.

This is the description of isomorphisms of Chevalley groups over fields:

**Theorem 2.** (See Theorems 30 and 31 from [34].) Let $G, G'$ be (elementary) Chevalley groups, constructed with root systems $\Phi, \Phi'$ and fields $k, k'$, respectively. Suppose that the root systems are not decomposable and have ranks $> 1$. Suppose that for the root systems $B_1, C_1, F_4$ corresponding fields have characteristics $\neq 2$ and for the

**Step:** We assume that $G$ is the group of rational points of a connected linear algebraic group $G_0$ over $k$. Let $\varphi$ be an automorphism of $G$ which is induced by a diagonal automorphism on $G_0$. Then $\varphi$ is a $G(k)$-automorphism.

**Proof:** We need to show that $\varphi$ is an automorphism of $G(k)$.

By the definition of diagonal automorphism, we have

$$\varphi(x_\alpha(t)) = x_\alpha(f_\alpha t) \quad \forall \alpha \in \Phi, \ t \in k.$$ 

This implies that $\varphi$ preserves the structure of $G(k)$, and hence $\varphi$ is an automorphism of $G(k)$.
root system $G_2$ it is not equal to three. Let $\varphi : G \to G'$ be a group isomorphism. Then the root systems $\Phi$ and $\Phi'$ coincide, the fields $k$ and $k'$ are isomorphic, and the isomorphism $\varphi$ is a composition of a ring isomorphism between $G$ and $G'$, and also inner, diagonal and graph automorphisms of the group $G'$. If the groups $G$ and $G'$ are adjoint, then there is no diagonal automorphism in the composition.

2.3. Normal structure of Chevalley groups over commutative rings

Note that for every ideal $I$ of $R$ the natural mapping $R \to R/I$ induces a homomorphism

$$\lambda_I : G_\pi(\Phi, R) \to G_\pi(\Phi, R/I).$$

If $I$ is a proper ideal of $R$, then the kernel of $\lambda_I$ is a non-central normal subgroup of $G_\pi(\Phi, R)$.

We denote the inverse image of the center of $G_\pi(R/I)$ under $\lambda_I$ by $Z_\pi(\Phi, R, I)$.

By $E_\pi(\Phi, R, I)$ we denote the minimal normal subgroup of $E_\pi(\Phi, R)$ which contains all $x_\alpha(t)$, $\alpha \in \Phi$, $t \in I$.

**Theorem 3.** (See [3].) Let the rank of an indecomposable root system $\Phi$ is more than one. If a subgroup $H$ of $E_\pi(\Phi, R)$ is normal in $E_\pi(\Phi, R)$, then

$$E_\pi(\Phi, R, I) \leq H \leq Z_\pi(\Phi, R, I) \cap E_\pi(\Phi, R)$$

for some uniquely defined ideal $I$ of the ring $R$.

2.4. Projective modules over local rings

The well-known result is the following

**Theorem 4.** A finitely generated projective module over a local ring is free.

2.5. The subgroup $E_\pi(\Phi, R)$ is characteristic in the group $G_\pi(\Phi, R)$

A subgroup $H$ of $G$ is called characteristic, if it is mapped into itself under any automorphism of $G$. In particular, any characteristic subgroup is normal.

**Theorem 5.** (See [36].) If the rank of $\Phi$ is greater than one, the elementary subgroup $E_\pi(\Phi, R)$ is characteristic in the Chevalley group $G_\pi(\Phi, R)$.

3. Formulation of main steps of the proof

If $R$ is a ring, $I$ is its ideal, then by $\lambda_I : G_\pi(\Phi, R) \to G_\pi(\Phi, R/I)$ ($E_\pi(\Phi, R) \to E_\pi(\Phi, R/I)$) we denote the homomorphism which corresponds every element (matrix) $A \in G_\pi(\Phi, R)$ to its image under the natural homomorphism $R \to R/I$.

Recall that by $Z_I$ we denote the inverse image of the center of the group $G_\pi(\Phi, R/I)$ under the homomorphism $\lambda_I$.

**Definition 4.** Let $C_I$ denote the group $Z_I \cap E_\pi(\Phi, R)$, $N_I = \ker \lambda_I \cap E_\pi(\Phi, R)$.

**Proposition 2.** Let $\varphi$ be an arbitrary automorphism of the group $E_\pi(\Phi, R)$, $I$ be a maximal ideal of $R$. Then there exists a maximal ideal $J$ of $R$ such that $\varphi(N_I) = N_J$. 
**Proof.** It is clear that the group \( C_I \) is normal in \( E_π(\Phi, R) \). As it follows from Theorem 3, for such a subgroup \( G \) we have an inclusion

\[
E_I \subseteq G \subseteq C_I,
\]

therefore the subgroups \( C_I \), and only they are maximal normal subgroups of the group \( E_π(\Phi, R) \). Consequently, for a maximal ideal \( I \) of the ring \( R \) there exists a maximal ideal \( J \) of \( R \) such that \( \varphi(C_I) = C_J \). Show that \( \varphi(N_I) = N_J \).

Consider the group \( G = E_π(\Phi, R) / C_I = E_π(\Phi, R) / (Z_I \cap E_π(\Phi, R)) \). It is isomorphic to \( E_π(\Phi, R) / Z_I \). Now use the Isomorphism theorem, namely, let us factorize the both parts by \( C_I \).

As result we obtain \( E_π(\Phi, R/I) \cdot Z(G_π(\Phi, R/I)) / Z(Gπ(\Phi, R/I)) \approx E_π(\Phi, R/I) \cdot \approx E_π(\Phi, R/I) \). Therefore \( E_π(\Phi, R/I) \cdot \approx E_π(\Phi, R/I) \).

Since \( \varphi(C_I) = C_J \), then the automorphism \( \varphi \) induces an isomorphism \( \varphi \) of the groups \( E_π(\Phi, R) / C_I \approx E_π(\Phi, R/I) \) and \( E_π(\Phi, R) / C_J \approx E_π(\Phi, R/J) \) such that the diagram

\[
\begin{array}{ccc}
E_π(\Phi, R) & \xrightarrow{\varphi} & E_π(\Phi, R) \\
\downarrow & & \downarrow \\
E_π(\Phi, R/I) & \approx E_π(\Phi, R/J) \\
& & \\
\end{array}
\]

is commutative. Isomorphisms of the groups \( E_π(\Phi, R) \) with root systems under consideration we have described in Theorem 2. So we see that the fields \( R/I \) and \( R/J \) are isomorphic (we denote the corresponding isomorphism by \( \rho \)) and \( \varphi(A) = i_π\delta(\rho(A)) \) for every \( A \in E_π(\Phi, R/I) \), \( g \in E_π(\Phi, R/J) \), \( \delta \) is a graph automorphism of \( G_π(\Phi, R/J) \).

Since a graph automorphism of the group \( G_π(\Phi, R/J) \) can be expanded to a graph automorphism of the group \( E_π(\Phi, R) \), and the last one maps the group \( N_I \) into itself, it is sufficient to consider the case, when the graph automorphism in the composition is identical.

We obtain that in the group \( E_π(\Phi, R) \) there is the equality

\[
\lambda_j \varphi(x_α(t)) = g(x_α(\rho(t + 1)))g^{-1}c, \quad c \in Z(E_π(\Phi, R/J)).
\]

Since \( x_α(t) \) is always (for the root systems under consideration) a product of commutators of elements \( x_β(s) \), then the central element \( c \) disappears from the image. Therefore we have

\[
\lambda_j \varphi(x_α(t)) = g(x_α(\rho(t + 1)))g^{-1}.
\]

Let now \( M = x_α(t_1) \cdots x_α(t_k) \) be an arbitrary element of \( N_I \). Then

\[
\lambda_j \varphi(M) = g(x_α(\rho(t_1 + 1)) \cdots x_α(\rho(t_k + 1)))g^{-1} = g(\rho\lambda_j(M))g^{-1} = E.
\]

where \( E \) is the identity.

Consequently \( \varphi(N_I) \subseteq N_J \). Clear that the inclusion \( \varphi^{-1}(N_J) \subseteq N_I \) is proved similarly. So \( \varphi(N_I) = N_J \).

Consider a ring \( R \) and its maximal ideal \( I \). We denote the localization \( R \) with \( I \) by \( R_I \) again, and its radical (the greatest ideal) we denote by \( \text{Rad}R_I \). Note that we have to isomorphic fields \( R/I \) and \( R_I/\text{Rad}R_I \). Therefore we can turn the arrow \( \mu_I \) in the diagram.
Let now $\varphi$ be an arbitrary automorphism of $E_\pi(\Phi, R)$. Proposition 2 gives us a possibility to consider the commutative diagram

$$
\begin{array}{ccc}
E_\pi(\Phi, R) & \xrightarrow{\varphi} & E_\pi(\Phi, R) \\
\downarrow r_I & & \downarrow r_J \\
E_\pi(\Phi, R_I) & \xrightarrow{\lambda_{\text{Rad } R_I}} & E_\pi(\Phi, R_J) \\
\downarrow s_I & & \downarrow s_J \\
E_\pi(\Phi, R/I) & \xrightarrow{\varphi} & E_\pi(\Phi, R/J)
\end{array}
$$

(2)

where $r_I$ is the mapping which corresponds every $x_\alpha(t), \alpha \in \Phi, t \in R$, to $x_\alpha(t/1)$; $s_I$ is the natural continuation of the isomorphism from $R_I/\text{Rad } R_I$ onto $R/I$ for the group $E_\pi(\Phi, R/I)$. The groups $E_\pi(\Phi, R/I)$ and $E_\pi(\Phi, R/J)$ are just elementary Chevalley groups over fields, their isomorphisms we have already described in Theorem 2.

Recall that the fields $R/I$ and $R/J$ are isomorphic (as earlier we denote the corresponding isomorphism by $\rho$), and also

$$
\bar{\varphi}(A) = i_g \circ f \circ \delta_i \rho(A) \quad \forall A \in E_\pi(\Phi, R/I), \ g \in G_\pi(\Phi, R/J),
$$

here $\delta_i$ is one of graph automorphisms, $f$ is a diagonal automorphism.

The description of automorphisms of the group $E_\pi(\Phi, R)$ can be made by the following scheme. The ring $R$ is embedded into the ring $S = \prod R_I$, which is the Cartesian product of all local rings $R_I$, obtained by localization the ring $R$ with different maximal ideals $I$. We denote by $R_I$ the ring $\prod R_I$, where maximal ideals are taken such that in the composition we have namely the graph automorphism $\delta_i$. Clear that $S = R_1 \oplus \cdots \oplus R_k$. Let in the ring $S$, $a_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

Clear that the group $E_\pi(\Phi, R)$ is embedded in

$$
G_\pi(\Phi, S) = G_\pi(\Phi, \prod R_I) = G_\pi(\Phi, R_1 \oplus \cdots \oplus R_k) = G_\pi(\Phi, R_1) \times \cdots \times G_\pi(\Phi, R_k).
$$

**The first step.** We prove that for every maximal ideal $J$

$$
\begin{align*}
\varphi_J(x_\alpha(1)) &= i_{g_J} \delta_i \rho(J)(x_\alpha(1))
\end{align*}
$$

where $g_J \in G_\pi(\Phi, R/J)$ (an extension of the ring $R/J$), $i$ is such that $R_J \in R_I$.

**The second step.** We consider adjoint Chevalley groups.

We show that actually the idempotents $a_i$ belong to the ring $R$, and the inner automorphism of $G_\pi(\Phi, S)$, generated by $g = \prod g_J$, induced an automorphism of the group $G_\pi(\Phi, R)$. 
Then we show that if we take the composition of the initial automorphism, the inner automorphism \( i_g^{-1} \) and the graph automorphism \( \Lambda_{g_1,...,g_t} \), then the obtained automorphism is ring.

Now suppose that the both steps are proved. Then we have the description of automorphisms of the elementary subgroup \( E_\pi(\Phi,R) \), and also we know that in the composition there is no central automorphism.

If we have now some automorphism of the group \( G_{ad}(\Phi,R) \), then it induces an automorphism of the group \( E_{ad}(\Phi,R) \) (see Theorem 5), which is standard (the composition of ring, inner and graph automorphisms). All these three automorphisms of the group \( E_{ad}(\Phi,R) \) are extended to the automorphisms of the group \( G_{ad}(\Phi,R) \). Therefore multiplying \( \varphi \) to the suitable standard automorphism, we can assume that \( \varphi \) is identical on the subgroup \( E_{ad}(\Phi,R) \).

As above it means
\[
\forall A \in E_{ad}(\Phi,R), \forall g \in G_{ad}(\Phi,R) \quad gAg^{-1} = \varphi(gAg^{-1}) = \varphi(g)A\varphi(g)^{-1},
\]

Consequently \( \varphi(g)g^{-1} \) commutes with all \( A \in E_{ad}(\Phi,R) \), i.e., it belongs to the center of \( G_{ad}(\Phi,R) \). Therefore, \( \varphi \) is a central automorphism.  

4. Proof of the first step in the theorem

For our convenience we will suppose that a Chevalley group under consideration is adjoint.

A graph automorphism of \( G_{ad}(\Phi, R/J) \) in the diagram (2) can be expanded to an automorphism of the whole group \( E_{ad}(\Phi, R) \). Therefore we can assume the automorphism \( \varphi \) such that \( \varphi = i_g \circ \delta \) (according to the fact that adjoint Chevalley groups over fields have no diagonal automorphisms).

Consider an arbitrary element \( x_\alpha(1) \in E_{ad}(\Phi, R) \), \( \alpha \in \Phi \). Its image under the mapping \( r_1 \) is also \( x_\alpha(1) = x_\alpha(1/1) \in E_{ad}(\Phi, R_1) \). In the field \( R/1 \) its image has the same form. The element \( x'_\alpha = \varphi(x_\alpha(1)) \in E_{ad}(\Phi, R) \) being factorized by the ideal \( J \) gives \( \overline{\varphi}(x_\alpha(1)) = i_g(x_\alpha(1)) \), where \( \overline{g} \in G_{ad}(\Phi, R/J) \).

Choose now \( g \in G_{ad}(\Phi, R_1) \) such that under factorization \( R_1 \) by its radical the element \( g \) corresponds to \( \overline{g} \).

Now consider the following mapping \( \psi : E_{ad}(\Phi, R) \rightarrow E_{ad}(\Phi, R_1) : \)
\[
\psi = i_{g^{-1}} \circ r_1 \circ \varphi.
\]
Under \( \psi \) all \( x_\alpha(1) \), \( \alpha \in \Phi \), correspond to such \( x'_\alpha \), that \( x_\alpha(1) - x'_\alpha \in M_N(\text{Rad } R_1) \), where \( N \) is the dimension of the adjoint representation (i.e., the dimension of the corresponding Lie algebra).

Therefore we obtain a set of elements \( \{x'_\alpha | \alpha \in \Phi \} \subset E_{ad}(\Phi, R_1) \), satisfying all the same conditions as \( \{x_\alpha(1) | \alpha \in \Phi \} \), and also equivalent to \( x_\alpha(1) \) modulo radical of \( R_1 \).

It is precisely the situation of papers [10,12,9,11,13], where for a local ring \( S \) and root systems \( A_2, B_l, C_l, F_4 \) for \( 2 \in S^*, G_2 \) for \( 2, 3 \in S^* \), the root systems \( A_l, l \geq 3, D_l, E_6, E_7, E_8 \) without any additional conditions it was proved that if in the group \( E_{ad}(\Phi, S) \) some elements \( x_\alpha \) are the images of the corresponding \( x_\alpha(1) \), \( \alpha \in \Phi \), and also \( x_\alpha(1) - x'_\alpha \in M_N(\text{Rad } S) \), then there exists \( g' \in G_{ad}(\Phi, S) \), \( g' - E \in M_N(\text{Rad } S) \), such that for every \( \alpha \in \Phi \)
\[
x_\alpha(1) = i_{g'}(x'_\alpha).
\]
Therefore the first step of our theorem completely follows from the above statement.  

Embedding now the initial ring \( R \) into the ring \( S = \prod J R_J \), we see that
\[
\varphi(x_\alpha(1)) = A_{e_1,...,e_h} g(x_\alpha(1)) g^{-1},
\]
where \( g = \prod J g_J, e_i \) are idempotents of \( S \), introduced above.
5. Proof of the second step in the theorem

We know now that the automorphism $\varphi$ satisfies the equality

$$
\varphi(x_\alpha(1)) = g x_{b_1(\alpha)}(e_1) \cdots x_{b_k(\alpha)}(e_k) g^{-1} \in E_{\text{ad}}(\Phi, R), \quad g = \prod g_J.
$$

Note that for the root systems $B_l, C_l, E_7, E_8, F_4, G_2, k = 1$, for the systems $A_l, D_l \ (l \geq 5), E_6, k = 2$, for the system $D_4, k = 6$.

As above we assume now that the Chevalley groups $G_\pi(\Phi, R)$ and $G_\pi(\Phi, S)$ are adjoint, i.e., $\pi = \text{ad}$.

In this case every graph automorphism of the Chevalley group $G_\text{ad}(\Phi, S)$ is realized by some matrix $A = e_1 A_1 + \cdots + e_k A_k \in \text{GL}_N(S)$, the matrices $A_1, \ldots, A_k$ have integer coefficients.

Therefore the composition of conjugation by $g \in G_\text{ad}(\Phi, S)$ and the graph automorphism (denote it by $\psi$) can be continued to the whole matrix ring $M_N(S)$.

Lemma 1. Under all theorem assumptions the elements $x_\alpha(1), \alpha \in \Phi$, by addition and multiplication generate the whole basis $X_\alpha, \alpha \in \Phi$, of the Lie algebra $L(\Phi)$.

Proof. If the root system differs from $G_2$ and $1/2 \in R$, then $x_\alpha(1) = E + X_\alpha + X_\alpha^2/2$, therefore $x_\alpha = x_\alpha(1) - E - (x_\alpha(1) - E)^2/2$.

For the root system $G_2$ and a short root $\alpha$ we have $x_\alpha(1) = E + X_\alpha + X_\alpha^2/2 + X_\alpha^3/6$. We suppose that $1/6 \in R$, then $x_\alpha^3/6 = (x_\alpha(1) - E)^3/6, \ x_\alpha^2/2 = (x_\alpha(1) - E)^2/2 - X_\alpha^2$, therefore we easily get $x_\alpha$.

Suppose now that we deal with the rings $A_l (l \geq 3), D_l, E_l$, two is not invertible.

In this case $x_\alpha(1) = E + X_\alpha + X_\alpha^2/2$, where $X_\alpha^2/2 = E_{\alpha,-\alpha}$. Choose any two roots $\gamma, \beta \in \Phi$ so that $\gamma + \beta = \alpha$. Using the condition

$$
(x_\gamma(1)x_\beta(1) - x_\beta(1) - x_\gamma(1) + E)^2 = E_{\alpha,-\alpha},
$$

we obtain $X_\alpha$.

The lemma is proved. \(\square\)

From Lemma 1 we see that the automorphism $\psi$ of the matrix ring $M_N(S)$ maps the matrices $X_\alpha, \alpha \in \Phi$, into the matrices with coefficients from the ring $R$. Therefore any matrix from $L(\Phi, R)$ under the action of the conjugation $\psi$ is mapped to a matrix from $M_N(R)$.

Since $x_\alpha(t) = E + tX_\alpha + t^2 X_\alpha^2/2 + \cdots$ for any $\alpha \in \Phi, t \in R$, then every matrix $x_\alpha(t)$ under the action of $\psi$ is mapped to a matrix from $\text{GL}_N(R)$. Consequently $\psi(E_{\text{ad}}(\Phi, R)) \subset \text{GL}_N(R)$.

From another side, the conjugation $\psi$ is the composition of inner and graph automorphisms of the Chevalley group $G_\text{ad}(\Phi, S)$ (and its elementary subgroup $E_\text{ad}(\Phi, S)$), so it is an automorphism of the group $E_\text{ad}(\Phi, S)$. Since the image of the Chevalley group $E_{\text{ad}}(\Phi, R)$ under $\psi$ belongs to the Chevalley group $E_\text{ad}(\Phi, S)$ and also to the ring $M_N(R)$, then $\psi(E_{\text{ad}}(\Phi, R)) = E_{\text{ad}}(\Phi, R)$.

Therefore taking the composition of the initial automorphism $\varphi \in \text{Aut}(E_\text{ad}(\Phi, R))$ and the automorphism $\psi^{-1} \in \text{Aut}(E_{\text{ad}}(\Phi, R))$, we obtain some automorphism $\rho = \psi^{-1} \circ \varphi \in \text{Aut}(E_{\text{ad}}(\Phi, R))$ such that $\rho(x_\alpha(1)) = x_{\alpha'}(1)$ for every $\alpha \in \Phi$.

Lemma 2. Under the initial assumptions of the theorem $\rho$ is a ring automorphism of the Chevalley group.

Proof. At first we suppose that in the ring $R$ the element $2$ is invertible (for the root system $G_2$ also $1/3 \in R$).

Our first step is to prove lemma for the root system $A_2$. In the system $A_2$ there are six roots: $\pm \alpha_1, \pm \alpha_2, \pm \alpha_3 = \pm (\alpha_1 + \alpha_2)$ (detailed matrices for the root system $A_2$ can be found in the paper [8]).

Let $\rho(x_{\alpha_1}(t)) = y$. Note that $y$ commutes with

$$
h_{\alpha_1}(-1) = \text{diag}[1, 1, -1, -1, -1, 1, 1].
$$
therefore we get that $y$ is block-diagonal up to the basis parts $\{\alpha_1, -\alpha_1, h_1, h_2\}$ and $\{\alpha_2, -\alpha_2, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2\}$. Then $y$ commutes with $x_{\alpha_1}(1)$, $x_{-\alpha_2}(1)$ and $x_{\alpha_1+\alpha_2}(1)$ so by direct calculations we obtain

$$y = \begin{pmatrix}
y_{1.1} & y_{1.2} & 0 & 0 & 0 & 0 & y_{1.7} & y_{1.7} + 3y_{7.2} \\
y_{1.1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{1.1} & 0 & y_{1.1} & 0 & 0 & 0 & 0 & 0 \\
y_{1.1} & 0 & 0 & y_{1.1} & 0 & 2y_{1.7} + 3y_{7.2} & 0 & 0 \\
y_{1.7} + 3y_{7.2} & 0 & y_{1.1} & 0 & 0 & 0 & 0 & 0 \\
y_{1.7} + 2y_{7.2} & 0 & 0 & 0 & 0 & y_{1.1} & 0 & 0 \\
y_{7.2} & 0 & 0 & 0 & 0 & 0 & y_{1.1} & 0 \\
y_{1.7} & 0 & 0 & 0 & 0 & 0 & 0 & y_{1.1}
\end{pmatrix}.$$ 

Besides that we have the condition

$$yx_{\alpha_2}(1) - w_{\alpha_2}(1)yw_{\alpha_2}(1)^{-1}x_{\alpha_2}(1)y,$$

which gives, at first, $y_{1.1}(1 - y_{1.1}) = 0$. Since $\det y = y_{1.1}^8$, the $y_{1.1}$ is invertible, so $y_{1.1} = 1$. Also this condition gives $z_{1.2} = -z_{1.1}/2, z_{7.2} = -z_{1.7}/2$.

Hence for the root system $A_2$ the assumption is proved.

Let us now deal with the root system $B_2$. Recall that in this system there are roots $\pm \alpha_1 = \pm (e_1 - e_2)$, $\pm \alpha_2 = \pm e_2$, $\pm \alpha_3 = \pm (\alpha_1 + \alpha_2)$. Using this condition we obtain directly (we have not use yet the whole conditions), that

$$y = \begin{pmatrix}
y_{1.1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{2.1} & y_{1.1} & 0 & y_{2.4} & 0 & y_{2.6} & 0 & y_{2.8} & y_{2.9} & y_{2.10} \\
y_{3.1} & 0 & y_{1.1} & y_{3.4} & y_{3.5} & y_{3.6} & 0 & y_{3.8} & y_{3.9} & y_{3.10} \\
y_{3.1} & 0 & 0 & y_{1.1} & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{5.1} & 0 & 0 & y_{5.4} & y_{1.1} & 0 & 0 & 0 & 0 & 0 \\
y_{5.1} & 0 & 0 & 0 & y_{6.4} & 0 & y_{1.1} & 0 & y_{6.8} & 0 \\
y_{7.1} & 0 & 0 & y_{7.4} & y_{7.5} & 0y_{1.1} & y_{7.8} & y_{7.9} & y_{7.10} \\
y_{7.1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{1.1} & 0 \\
y_{9.1} & 0 & 0 & 0 & 0 & 0 & 0 & y_{9.8} & y_{1.1} & 0 \\
y_{9.1} & 0 & 0 & 0 & 0 & 0 & y_{10.4} & 0 & 0 & y_{10.8} & 0 & y_{1.1}
\end{pmatrix}.$$ 

Again the determinant of $y$ is $y_{1.1}^{10} = 1$, therefore $y_{1.1}$ is invertible.

Use the condition $h_{\alpha_1}(-1)y^2h_{\alpha_1}(-1)y = E$, which implies $y_{2.1} = y_{2.9} = y_{2.10} = y_{3.5} = y_{3.6} = y_{5.4} = y_{6.4} = y_{9.8} = y_{9.1} = y_{10.8} = y_{7.8} = y_{7.9} = y_{7.10} = 0$. Besides, $y_{1.1}^{10} = 1$.

Let now $\rho(x_{\alpha_1}(t)) = z$. The matrix $z$ commutes with

$$h_{\alpha_1}(-1) = \text{diag}[1, 1, -1, -1, -1, 1, 1, 1, 1],$$

therefore it is block-diagonal up to the basis parts $\pm \alpha_1$, $\pm \alpha_4$, $h_1$, $h_2$ and $\pm \alpha_2$, $\pm \alpha_3$. Now we use the fact that $z$ commutes with $x_{\alpha_1}(1)$, $x_{\alpha_2}(1)$, $x_{-\alpha_4}(1)$, $x_{\alpha_3}(1)$, after that we directly obtain that $z$ has the form
on the basis part \( \{ \pm \alpha_1, \pm \alpha_6, h_1, h_2 \} \) and the form

\[
\begin{pmatrix}
z_1 & z_2 & 0 & 0 & 2z_3 & -3z_3 \\
0 & z_1 & 0 & 0 & 0 & 0 \\
0 & 0 & z_1 & 0 & 0 & 0 \\
0 & z_3 & 0 & z_1 & 0 & 0 \\
0 & 0 & 0 & 0 & z_1 & 0 \\
0 & 0 & 0 & 0 & 0 & z_1 \\
\end{pmatrix}
\]

on the basis part \( \{ \pm \alpha_2, \pm \alpha_3, \pm \alpha_4, \pm \alpha_5 \} \).
From the condition $h_{\alpha}(1)zh_{\alpha}(1)z = E$ we get $z_1 = 1$, $z_2 = -z_3^2$, $z_4 = -z_3^2$.

Now use the last condition:

$$w_{\alpha}(1)zw_{\alpha}(1)^{-1}x_{\alpha}(1) = w_{\alpha}(1)w_{\alpha}(1)y^3w_{\alpha}(1)^{-1}x_{\alpha}(1)w_{\alpha}(1)^{-1}x_{\alpha}(1)w_{\alpha}(1)zw_{\alpha}(1)^{-1},$$

which connects $y$ and $z$. From this condition we obtain $z_3 = -y_3$. The assumption for $z_5$ follows from the fact that $z$ is an element of the Chevalley group.

So we have studied also the case $G_2$.

Let now $1 \leq R$, the root system be one of $B_l, C_l, F_4$ (matrices corresponding to these root systems can be found in the papers [11] and [12]).

Let $y$ be the image of some long simple root (for example, $y = \rho(x_{\alpha}(1))$), $z$ be the image of a short root ($z \in \rho(x_{\alpha}(1))$). We can assume that for the system $B_l, \alpha_i = \alpha_1$, $\alpha_j = \alpha_l$, for the system $C_l, \alpha_i = \alpha_1$, $\alpha_j = \alpha_1$, for the system $F_4, \alpha_i = \alpha_2$, $\alpha_j = \alpha_4$.

Note that $y$ and $z$ commute with $h_{\alpha}(1)$ for some definite $\alpha \in \Phi$, therefore according to invertible 2 we directly obtain that the matrices $y$ and $z$ are block-diagonal up to some basis separation.

For a long root $\alpha$ in any system under consideration all other pairs of roots $\pm \beta$ are divided to the following cases:

1. $\pm \beta$ are also long roots, orthogonal to $\alpha$.
2. $\pm \beta$ are short roots, generating with $\pm \alpha$ the system $A_2$.
3. $\pm \beta$ are short roots, orthogonal to $\alpha$.
4. $\pm \beta$ a short roots, generating with $\alpha$ the system $B_2$.

In the first case the matrix $y$ commutes with $x_{\beta}(1)$ and $x_{-\beta}(1)$, therefore $y$ commutes with $\rho(-\beta, \beta, E, B, E, -\beta)$. It means that on the basis part $-\beta$, $\beta$ the matrix $y$ is invariant and scalar, and also the rest basis part is also invariant. The same thing is with the third case.

In the second case according to commuting $y$ with $h_{\alpha}(1)$ and other $h_{\gamma}(1)$ for long roots, distinct from $\pm \beta$, the basis part $\pm \beta$, $\pm (\alpha + \beta)$ is separated to the own diagonal block.

The fourth case means that there are roots $\pm \alpha$, $\pm \beta$, $\pm \alpha + \beta$ (short) and $\pm \alpha + 2\beta$ (long). Also the long roots $\pm (\alpha + 2\beta)$ are orthogonal to $\alpha$ and, as it was shown above, the matrix $y$ is scalar on them. According to commuting with $h_{\alpha}(1)$ the basis part $\pm \alpha, h_1, \ldots, h_l$ is separated of the basis part $\pm \beta, \pm (\alpha + \beta)$. Now we just need to show that the basis part $\pm \beta, \pm (\alpha + \beta)$ is separated of the basis part $\pm y, \pm (\alpha + y)$, where $y$ is also a root of the fourth type. If the root system under consideration is $B_l$, then $\alpha$ can be supposed as the root $\alpha = e_1 - e_2$, then $\beta$ can be only $e_2$. If the root system is $C_l$, then, for example, $\alpha = 2e_1$, then $\beta = e_1 - e_1, y = e_1 - e_1, i, j > 1$. In this case it is clear that the corresponding parts are separated according to the fact that $y$ commutes with $h_{\alpha}(1)$ and $h_{\gamma}(1)$.

A similar situation is for the root system $F_4$.

We see that the whole matrix $y$ is divided into diagonal blocks, where every block is either scalar (and corresponds to some pair of roots $\pm \beta$), or corresponds to the roots $\pm \beta, \pm (\alpha + \beta)$. Now we can use the results, connected with the root systems $A_2$ and $B_2$, therefore $y = x_{\alpha}(1)$ for some $s \in R$.

Let now $\alpha$ be a short root. Then all other roots are divided into the following cases:

1. $\beta$ is a long root, orthogonal to $\alpha$.
2. $\beta$ is a long root, generating with $\alpha$ the root system $B_2$.
3. $\beta$ is a short root, orthogonal to $\alpha$ and generating with it the system $B_2$ (for example, in the root system $B_l$ it holds for $\alpha = e_1, \beta = e_1$).
4. $\beta$ is a short root, orthogonal to $\alpha$, $\alpha \pm \beta \notin \Phi$ (for example, for the rot system $C_l$ it holds for $\alpha = e_1 - e_2, \beta = e_3 - e_4$).
5. $\beta$ is a short root, generating with $\alpha$ the root system $A_2$ (it holds, for example, in the root system $C_l$ for $\alpha = e_1 - e_2, \beta = e_2 - e_3$).

Then all considerations are similar to the long roots.

Therefore the both matrices $y$ and $z$ are block-diagonal, the blocks correspond to separating into root systems $A_2, B_2$, scalar $2 \times 2$ matrices, and also the basis part $\pm \alpha, h_1, \ldots, h_l$ is separated. On
the basis part $h_3, \ldots, h_l$ the matrices are scalar since they commutes with corresponding $w_{\alpha_i}(1), \ldots, w_{\alpha_l}(1)$. Since we have studied the cases $A_2$ and $B_2$ above, we see that on the basis parts $\pm \alpha, \pm \beta, \pm (\alpha + \beta), h_1, h_2$, and also on the parts $\pm \alpha, \pm \gamma, \pm (\alpha + \gamma), \pm (2\alpha + \gamma)$ depending on the length of $\alpha$, $h_1, h_2$ the matrices coincide with $x_\alpha(s)$ for some $s \in R$. Since on the basis part $\pm \alpha, h_1, h_2$ we always have the same $s$, then $s$ is unique for all other basis parts. On the places where the matrix is scalar, the multiplier is the same according to commuting with Weil group elements, which map the roots, orthogonal to $\alpha$, into each other, and which preserve $\alpha$. Clear that either $y$ (in the root system $B_1$, for example), or $z$ (in the root system $C_1$), and in the root system $F_4$ both $y$ and $z$ can be embedded into the root system $A_2$, where we have the following condition (here we write it for $y$)

$$y \cdot x_\gamma(1) = wyw^{-1}x_\gamma(1) \cdot y.$$  

If the matrix $y$ on it scalar part has the multiplier $a$, then this condition gives us either $a = 1$, or $a = a^2$, and according to invertibility of $a$ it also implies $a = 1$. Therefore either $y$, or $z$ coincides with $x_\alpha(s)$ for some $s \in R$.

Now we can assume (according to conjugations by the Weil group elements), that $y$ and $z$ are images of $x_\alpha(t)$ and $x_\beta(t)$, where $\alpha$ and $\beta$ generate the root system $B_2$. Clear that after using the corresponding commutator conditions we obtain that all indefinite scalars are 1. Therefore, $y = x_\alpha(s_1)$, $z = x_\beta(s_2)$. Also it is clear (from the above case $B_2$), that $s_1 = s_2$.

Consequently, the lemma is proved for the root systems $A_2, B_1, C_1, F_4, G_2$.

Now we need to prove the lemma for the root systems $A_l, D_l, E_l, l \geq 3$, but without the condition $1/2 \in R$.

At first we are going to show that the lemma holds for the system $A_3$.

In the root system $A_3$ there are roots $\pm \alpha_1, \pm \alpha_2, \pm \alpha_3$, $\pm \alpha_4 = \pm (\alpha_1 + \alpha_2)$, $\pm \alpha_5 = \pm (\alpha_2 + \alpha_3)$, $\pm \alpha_6 = \pm (\alpha_3 + \alpha_4 + \alpha_5)$ (detailed matrices for this system can be found in the paper [13]).

Let $y = \rho(x_{\alpha_1}(t))$.

Note that there is the condition $(x_\alpha(1)x_\beta(1) - x_\alpha(1) - x_\beta(1) + E)^2 = E_{\alpha + \beta} - \alpha - \beta$, if $\alpha + \beta \in \Phi$.

Since $y$ commutes with $x_{\alpha_1}(1), x_{\alpha_1 + \alpha_2}(1), x_{\alpha_3}(1), x_{\alpha_3 + \alpha_4}(1)$, then $y$ has to commute also with $E_{\alpha_3 - \alpha_6}$. Besides, $y$ commutes with $w_{\alpha_1}(1)$, therefore we obtain that the lines of $y$, corresponded to the basis vectors $\alpha_6, -\alpha_5, \alpha_2$, and its rows corresponded to the vectors $-\alpha_6, \alpha_5, -\alpha_2$, have nonzero elements only on the diagonal.

Then we can directly use commuting with matrices written above, commutator condition and the fact that $y$ belongs to the corresponding Chevalley group and obtain that $y = x_{\alpha_1}(s)$, what was required.

Suppose, finally, that we deal with an arbitrary root system under consideration, still we set $y = \rho(x_{\alpha_1}(t))$.

All basis elements are divided up to $\alpha_1$ to the following cases:

1. $\pm \alpha_1$ themselves, and also $h_1, h_2$.
2. $h_3, \ldots, h_l$.
3. $\pm \beta$, where a root $\beta$ is orthogonal to $\alpha_1$, and also there exists one more root $\gamma$, orthogonal to $\alpha_1$ and not orthogonal to $\pm \beta$ (it always holds for the root system $A_l$, $l \geq 3$, but, for example, for the root system $D_l$ it does not hold for $\alpha_1 = e_1 - e_2, \beta = e_1 + e_2$).
4. $\pm \beta$, where the root $\beta$ is orthogonal to $\alpha_1$ and does not satisfy the assertion 3.
5. $\pm \beta$, where $\alpha_1$ and $\beta$ generate the root system $A_2$.

To use the result, obtained for the system $A_3$, we just need to prove that the matrix $y$ is block-diagonal, where every block has one of listed above types or the block of the root system $A_3$.

At first we consider the easy case — type 3. For this case we take a root $\gamma$, orthogonal to $\alpha$ and such that $\beta + \gamma \in \Phi$ (clear that $\beta + \gamma$ is also orthogonal to $\alpha_1$). In the same manner as it was described in consideration of the case $A_3$, we obtain that $y$ commutes with $E_{-\beta, \beta}$ and $E_{-\beta, \beta}$, therefore we directly have that the block $\pm \beta$ is separated in the matrix $y$, on this block $y$ is scalar.
Using the commutator relation we obtain that $y$ on this block is identical. Consequently on the basis parts of the third type $y$ completely coincides with $x_{α_i}(s)$.

Besides, $(x_{α_i}(1) − E + E_{α_i, −α_i})E_{α_i, −α_i} = E_{h_i, −α_i}$ commutes with $y$ for $i ≥ 3$. So we directly obtain that the whole row of $y$ with number $h_i$ is zero, except a diagonal element. From another side, $E_{α_j, α_i}(x_{α_i}(1) − E + E_{α_i, −α_i}) = E_{α_j, h_i} − 2E_{α_i, h_i} + E_{α_i, h_j}$ also commutes with $y$, therefore we have that between $h_i, h_j$ and $h_i + 1$-th lines of $y$ there exists a natural dependence, i.e., if we show that the $h_2$-th line is zero, then other lines become zero (except diagonal elements which will be equal).

According to this fact and the considered case $A_3$ we just need to study roots of forth and fifth types.

At the beginning we need to show that if roots $β$ and $γ$ have one of these types and are orthogonal to each other, then on the place $β, γ$ in the matrix $y$ there is zero.

Let the both roots $β$ and $γ$ have the forth type. Clear that in this case they together with $α_1$ are embedded in the rot system $D_4$, i.e., for our convenience we can assume that $α_1 = e_1 − e_2$, $β = e_3 − e_4$, $γ = e_3 + e_4$. The matrix $y$ commutes with $x_{e_1 ± e_2}(1)$, $x_{e_1 ± e_4}(1)$, $x_{e_2 ± e_3}(1)$, $x_{e_2 ± e_4}(1)$, $x_{e_3 ± e_4}(1)$, therefore, as before, $y$ commutes with $E_{e_1 ± e_3}$, $E_{e_1 ± e_4}$, $E_{e_2 ± e_3}$, $E_{e_2 ± e_4}$, $E_{e_3 ± e_4}$.

The matrix $e_1 ± e_3(1) − E$ at the line corresponding to the root $e_3 − e_4$, has only one nonzero element $E_{e_3 − e_4, e_1 + e_4}$, so $E_{e_2 − e_3, e_1 + e_4} = (x_{e_1 ± e_3}(1) − E)E_{e_1 − e_2}$ commutes with $y$.

If we multiply $E_{e_2 − e_3, e_1 + e_4}$ to $x_{e_1 ± e_3}(1) − E$, we get $E_{e_2 − e_3, e_1 + e_4}$, the matrix $y$ also commutes with it. It is sufficient to have zero on the place $e_3 − e_4, e_3 + e_4$ in the matrix $y$, what was obtained.

Note that for the root systems $A_1$ and $E_1$ roots of the fourth type do not exist. Let us consider the root system $D_4$, the root $β$ has the fourth type, the root $γ$ has the fifth type, the roots are orthogonal to each other. It cannot be in the system $D_4$, so we can assume that $α_1 = e_1 − e_2$, $β = e_3 − e_1$, $γ = e_2 − e_4$. This case zeros on the corresponding places evidently follow from the consideration of the system $A_3$.

We have completely studied the case with orthogonal roots $β$ and $γ$.

Now suppose that roots $β$ and $γ$ are not orthogonal to each other, i.e., generate the system $A_2$.

Clear that in this case they cannot both be of the fourth type. Let the root $β$ be of the forth type, the root $γ$ be of the fifth type. It means that the roots $α_1$, $β$, $γ$ together generate the system $A_3$, we can assume that $α_1 = e_1 − e_2$, $β = e_3 − e_4$. This case was already considered, according to commuting with the corresponding $x_{α}(1)$ we have already proved that there is zero on the place $β, γ$ in the matrix $y$.

Finally, let $β$ and $γ$ both belong to the fifth type. Then we can assume that $α_1 = e_1 − e_2$, $β = e_2 − e_3$, $γ = e_4 − e_1$. It is clear again that this case is considered for the root system $A_3$.

Therefore we have shown that the matrix $y$ is divided into diagonal blocks so that we can apply the results for $A_3$ to every block.

Consequently, $y = x_{α_i}(s)$, what was required.

Consequently for all cases under consideration we obtain that $ρ(x_{α}(t)) = x_{α}(s)$, the mapping $t ↦ s$ does not depend of choice of $α ∈ Φ$. Denote this mapping also by $ρ : R ↦ R$, we only need to prove that it is an automorphism of $R$.

Actually, it is one-to-one because the initial automorphism $ρ ∈ Aut(E_{ad}(Φ, R))$ is bijective.

Its additivity follows from the formula:

\[ x_{α_1}(ρ(t_1) + ρ(t_2)) = x_{α_1}(ρ(t_1))x_{α_1}(ρ(t_2)) = ρ(x_{α_1}(t_1)) \cdot ρ(x_{α_1}(t_2)) = ρ(x_{α_1}(t_1)x_{α_1}(t_2)) = ρ(x_{α_1(t_1 + t_2)} = x_{α_1}(ρ(t_1 + t_2)), \]

and its multiplicativity follows from:

\[ x_{α_1 + α_2}(ρ(t_1)ρ(t_2)) = [x_{α_1}(ρ(t_1)), x_{α_2}(ρ(t_2))] = ρ([x_{α_1}(t_1), x_{α_2}(t_2)]) = ρ(x_{α_1 + α_2}(t_1t_2)) = x_{α_1 + α_2}(ρ(t_1t_2)) \]
for roots $\alpha_1$ and $\alpha_2$, forming the system $A_2$. We cannot find such a pair of roots only in the system $B_2$, where we can take the short roots $\alpha_1 = e_1$, $\alpha_2 = e_2$, the formula will be

$$[x_{\alpha_1}(t_1), x_{\alpha_2}(t_2)] = x_{\alpha_1 + \alpha_2}(2t_1t_2).$$

therefore

$$2 \rho(t_1t_2) = 2 \rho(t_1) \rho(t_2),$$

it is sufficient because for the root systems $B_1$ we suppose 2 to be invertible.

Lemma is proved. \(\square\)

Now we have that the initial automorphism $\varphi$ of the elementary Chevalley group $E_{ad}(\Phi, R)$ is the composition of the conjugation $\psi$ with some matrix $A \in \text{GL}_N(S)$ and a ring automorphism $\rho$, also we know that the conjugation $\psi$ is an automorphism of the Lie algebra $L(\Phi, R)$.

Now we use the description of such automorphisms from the paper [29] (see Theorem 1 of this paper):

**Lemma 3.** Let $R$ be a commutative associative ring, $\Phi$ be an undecomposable root system. Then every automorphism $\psi$ of the Lie algebra $L(\Phi, R)$ is the composition of graph and inner automorphisms (by inner automorphism we mean a conjugation with some $g \in G_{ad}(\Phi, R)$).

**Proof.** Consider the ideal $J$ in $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{N,N}]$, defining the group $\text{Aut}_C L(\Phi)$. Over complex numbers this group is well known: the ideal $J$ decomposes into a product $J = J_1J_2 \cdots J_d$ of prime ideals $J_i$, corresponding to irreducible (= connected) components $h_i G(\Phi)$ of the group $\text{Aut}_C L(\Phi)$, where $h_i$ are integer matrices of diagram automorphisms.

Take a matrix $A = (a_{p,q}) \in \text{Aut}_R L(\Phi, R)$. Then $f(a_{p,q}) = 0$ for $f \in J$. Put $I_i = \{ f(a_{p,q}) \mid f \in J_i \} \triangleleft R$. Then

(i) $\prod I_i = 0(0)$;
(ii) $I_i + I_j = R$ for $i \neq j$ (otherwise we take the factor ring by a maximal ideal $M \supset I_i + I_j$ and obtain a matrix $A_M$, belonging to the intersection of two irreducible components of the group $\text{Aut}_{R/M} L(\Phi, R/M)$, but this intersection is empty, because $R/M$ is a field).

These conditions (i) and (ii) imply that the ring $R$ is the direct sum $R = \oplus R/I_i$ (see [7, Chapter 2, §1, Proposition 5]).

So, $A = \sum A_{I_i}$, and the entries of the matrix $A_{I_i} \in M_N(R/I_i)$ satisfy the equations $f(a_{p,q}) = 0$ for $f \in I_i$. Therefore, $A_{I_i} = h_i g_i \in h_i G_{ad}(\Phi, R/I_i)$, and $A = (\sum e_i h_i)(\sum g_i)$, where $e_i$ is the unity of the ring $R/I_i$.

Lemma is proved. \(\square\)

Consequently, the step 2 is completely proved for adjoint elementary Chevalley groups, i.e., every automorphism of such a group is the composition of inner, graph and ring automorphisms.

The second step is complete, i.e., Theorem 1 is proved.

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**References**