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Cohomology of generalized restricted Lie algebras ☆

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Abstract

In this note, the generalized restricted Lie algebra, which was introduced by Shu Bin in [J. Algebra 194 (1997) 157–177], is studied. By generalizing the concept of restricted subalgebras and the concept of restricted homomorphism, we show that the second generalized restricted cohomology $H^2_{\varphi_L}(L,M)$ is isomorphic to the equivalence classes of those generalized restricted extension of M by L. For any generalized restricted Lie algebra (L, B_L, φ_L) and any generalized restricted L-module M, we show that the sequence

$$0 \to H^1_{\varphi_L}(L,M) \to H^1(L,M) \to \hom_F\big(L, M^L\big) \to H^2_{\varphi_L}(L,M) \to H^2(L,M) \to \hom_F\big(L, H^1(L,M)\big)$$

is exact.

1. Introduction

As is well known, the restricted Lie algebras have played an important role in the theory of modular Lie algebras and their representations. However, many kinds of Lie algebras are not restricted. For instance, the graded Lie algebras of Cartan type $X(n, \mathbf{m})$

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$(X = W, S, H, K)$ may be a restricted Lie algebras only if $\mathbf{m} = (1, \ldots, 1)$. So the concept of generalized restricted Lie algebras is introduced in [5].

Definition 1.1 [5, Definition 1.1]. A Lie algebra L is called a generalized restricted Lie algebra if there are an ordered basis $B_L = \{x_1, \ldots, x_n\}$ of L, an n-tuples natural numbers $\mathbf{s} = (s_1, \ldots, s_n)$, and a map $\varphi_{\mathbf{s}} : B_L \to L$ such that $\operatorname{ad} \varphi_{\mathbf{s}}(x_i) = (\operatorname{ad} x_i)^{p^{s_i}}$. We will denote it by $(L, B_L, \varphi_{\mathbf{s}})$.

It is easily seen that the concept of generalized restricted Lie algebra generalizes the concept of restricted Lie algebra and it is easily shown that the graded Lie algebras of Cartan type are generalized restricted Lie algebras.

Let $x, y \in L$ and t an indeterminate over $U(L)$ (the universal enveloping algebra of L). Then

$$(xt + y)^{p^r} = x^{p^r} t^{p^r} + y^{p^r} + \sum b_i(x, y) t^i.$$

If $r = 1$, then $b_i(x, y) \in L$. However, if $r > 1$, we cannot obtain $b_i(x, y) \in L$. This means that unlike the definition of a restricted Lie algebra, that of a generalized restricted Lie algebra relies on a fixed basis B_L. Therefore, it will meet difficulties to study some important concepts, for instance, the concept of generalized restricted subalgebra of a generalized restricted Lie algebra and the concept of generalized restricted homomorphism.

In this note, we try to refine the theory of generalized restricted Lie algebras. By introducing an equivalence relation on the set consisting of all structures of generalized restricted Lie algebras for a Lie algebra L, we define equality of two structures of generalized restricted Lie algebras for a Lie algebra L. This makes the concept of generalized restricted Lie algebra not rely on a fixed basis. As a result of doing this, we introduce the concept of a generalized restricted subalgebra of a generalized restricted Lie algebra and the concept of a generalized restricted homomorphism. As an application, we study the generalized restricted extension M by L and the cohomology of a generalized restricted Lie algebra. An explanation of the second generalized restricted cohomology is given, and it is shown that the second generalized restricted cohomology $H^2_{\varphi_L}(L, M)$ is isomorphic to the equivalence classes of generalized restricted extensions of M by L. The relation between generalized restricted cohomology and ordinary cohomology is revealed, and it is shown that the sequence

$$\begin{aligned} 0 &\to H^1_{\varphi_L}(L, M) \to H^1(L, M) \to \hom_F\big(L, M^L\big) \\ &\to H^2_{\varphi_L}(L, M) \to H^2(L, M) \to \hom_F\big(L, H^1(L, M)\big) \end{aligned}$$

is exact for any generalized restricted Lie algebra (L, B_L, φ_L) and any generalized restricted L-module M.

Since the graded Lie algebras of Cartan type are generalized restricted Lie algebras, this sequence gives us a way to study the cohomology of the graded Lie algebras of Cartan type.

2. Generalized restricted Lie algebra

Let F be an algebraically closed field of characteristic $p > 0$, L a Lie algebra over F, $U(L)$ the universal enveloping algebra of L, $C(U(L))$ the center of $U(L)$, and $\{U_{(i)} \mid i = 0, 1, 2, \ldots\}$ the canonical filtration of $U(L)$. Then

Lemma 2.1 [6, Lemma 1.9.7]. *Let $\{x_i\}_{i\in I}$ be an ordered basis of L. Assume that there exist a function $k : I \to \mathbb{N}$ and families $\{v_i\}_{i\in I}$ and $\{z_i\}_{i\in I}$ such that for every $i \in I$,*

$$x_i^{k(i)} = v_i + z_i, \quad v_i \in U_{(k(i)-1)},\ z_i \in C\big(U(L)\big).$$

Then $B = \{\mathbf{z}^{\mathbf{r}}\mathbf{x}^{\mathbf{s}} \mid \mathbf{r}, \mathbf{s} \in \mathbb{N}(I),\ s(i) < k(i),\ \forall i \in I\}$ is a basis of $U(L)$.

Let $(L, B_L, \varphi_{\mathbf{s}})$ be a generalized restricted Lie algebra and $B_L = \{x_1, \ldots, x_n\}$ an ordered basis of L. Denote by $I(B_L, \varphi_{\mathbf{s}})$ the ideal of $U(L)$ generated by the elements

$$\varphi_{\mathbf{s}}(x_i) - x_i^{p^{s_i}} \quad (x_i \in B_L).$$

The associative algebra $u_{\varphi_{\mathbf{s}}}(L) := U(L)/I(B_L, \varphi_{\mathbf{s}})$ is called a generalized restricted universal enveloping algebra. Lemma 2.1 implies that $\{\mathbf{x}^{\mathbf{r}} \mid \mathbf{0} \leqslant \mathbf{r} \leqslant \tau_{\mathbf{s}}\}$ is a basis of $u_{\varphi_{\mathbf{s}}}(L)$ and $\dim u_{\varphi_{\mathbf{s}}}(L) = p^{|\mathbf{s}|}$ (where $\tau_{\mathbf{s}} = (p^{s_1} - 1, \ldots, p^{s_n} - 1)$, $|\mathbf{s}| = \sum_{j=1}^n s_j$, and $\mathbf{x}^{\mathbf{s}} = \prod_{j=1}^n x_j^{s_j}$). An L-module M is called a generalized restricted L-module if $x_i^{p^{s_i}} m = \varphi_{\mathbf{s}}(x_i)m$ for $\forall m \in M$, $\forall x_i \in B_L$. It is obvious that a generalized restricted L-module can be identified with $u_{\varphi_{\mathbf{s}}}(L)$-module.

Definition 2.1. Let $(L, B_L, \varphi_{\mathbf{s}})$ and $(L, B'_L, \varphi_{\mathbf{t}})$ be two generalized restricted Lie algebra structures for L. Say that $(L, B_L, \varphi_{\mathbf{s}})$ and $(L, B'_L, \varphi_{\mathbf{t}})$ are equal if $I(B_L, \varphi_{\mathbf{s}}) = I(B'_L, \varphi_{\mathbf{t}})$. Denote this by $(L, B_L, \varphi_{\mathbf{s}}) = (L, B'_L, \varphi_{\mathbf{t}})$.

Let $\mathbf{a} := (a_1, \ldots, a_n)$ be n-tuples of natural numbers. Then we denote $\hat{\mathbf{a}} := (a_{i_1}, \ldots, a_{i_n})$ such that $a_{i_1} \leqslant a_{i_2} \leqslant \cdots \leqslant a_{i_n}$.

Lemma 2.2. *Let $(L, B_L, \varphi_{\mathbf{s}})$ and $(L, B'_L, \varphi_{\mathbf{t}})$ be two generalized restricted Lie algebra structures for L. If $(L, B_L, \varphi_{\mathbf{s}}) = (L, B'_L, \varphi_{\mathbf{t}})$, then*

(1) $\hat{\mathbf{s}} = \hat{\mathbf{t}}$,
(2) $\varphi_{\mathbf{s}}|_{B_L \cap B'_L} = \varphi_{\mathbf{t}}|_{B_L \cap B'_L}$.

Proof. Suppose $B_L = \{x_1, \ldots, x_n\}$ and $B'_L = \{y_1, \ldots, y_n\}$. By rearranging the order of B_L and the order of B'_L, we can assume that $\hat{\mathbf{s}} = (s_1, \ldots, s_n)$ and $\hat{\mathbf{t}} = (t_1, \ldots, t_n)$, i.e., $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$ and $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$.

(1) If $\hat{\mathbf{s}} \neq \hat{\mathbf{t}}$, then there exists a natural number r such that $s_r \neq t_r$ and $s_i = t_i$ for $i < r$. If $s_r < t_r$, then since B'_L is a basis of L, there are $a_{kj} \in F$ $(k = 1, \ldots, r;\ j = 1, \ldots, n)$ such that $x_k = \sum_{j=1}^n a_{kj} y_j$. Since $\{x_1, \ldots, x_r\}$ is linearly independent, there exist $j \geqslant r$

and $k \leqslant r$ such that $a_{kj} \neq 0$. This implies that $B''_L = \{y_1, \ldots, y_{j-1}, x_k, y_{j+1}, \ldots, y_n\}$ is a basis of L. Set $\mathbf{t}' = (t_1, \ldots, t_{j-1}, s_k, t_{j+1}, \ldots, t_n)$. Define a map $\varphi_{\mathbf{t}'}$ on B''_L via

$$\varphi_{\mathbf{t}'}(y_i) := \varphi_{\mathbf{t}}(y_i) \quad \text{for } i \neq j,$$
$$\varphi_{\mathbf{t}'}(x_k) := \varphi_{\mathbf{s}}(x_k).$$

It is easily seen that $(L, B''_L, \varphi_{\mathbf{t}'})$ is a structure of generalized restricted Lie algebra for L and it is obvious that $I(B''_L, \varphi_{\mathbf{t}'}) \subset I(B'_L, \varphi_{\mathbf{t}}) \cup I(B_L, \varphi_{\mathbf{s}}) = I(B'_L, \varphi_{\mathbf{t}})$. This implies $p^{s_k+|\mathbf{t}|-t_j} = p^{|\mathbf{t}'|} = \dim u_{\varphi_{\mathbf{t}'}}(L) \geqslant \dim u_{\varphi_{\mathbf{t}}}(L) = p^{|\mathbf{t}|}$, hence $s_k \geqslant t_j$. This is a contradiction since $s_k \leqslant s_r < t_r \leqslant t_j$. Similarly, we can show that $s_r > t_r$ is impossible. Hence $\hat{\mathbf{s}} = \hat{\mathbf{t}}$.

(2) Let $x_i = y_k \in B_L \cap B'_L$. If $s_i > t_k$, then let I be an ideal of $U(L)$ generated by elements

$$\varphi_{\mathbf{s}}(x_j) - x_j^{p^{s_j}} \quad (j = 1, \ldots, i-1, i+1, \ldots, n) \quad \text{and} \quad \varphi_{\mathbf{t}}(y_k) - y_k^{p^{t_k}}.$$

Lemma 2.1 implies $\dim U(L)/I = p^{|\mathbf{s}|-s_i+t_k} < p^{|\mathbf{s}|} = \dim U(L)/I(B_L, \varphi_{\mathbf{s}})$. This is a contradiction since $I \subset I(B_L, \varphi_{\mathbf{s}}) \cup I(B'_L, \varphi_{\mathbf{t}}) = I(B_L, \varphi_{\mathbf{s}})$. Similarly, we can show that $s_i < t_k$ is impossible. Hence $s_i = t_k$. Now we have

$$\varphi_{\mathbf{s}}(x_i) - \varphi_{\mathbf{t}}(y_k) = \varphi_{\mathbf{s}}(x_i) - x_i^{p^{s_i}} + y_k^{p^{t_k}} - \varphi_{\mathbf{t}}(y_k) \in I(B_L, \varphi_{\mathbf{s}}) \cap L = 0,$$

i.e., $\varphi_{\mathbf{s}}(x_i) = \varphi_{\mathbf{t}}(y_k)$. □

Part (2) of Lemma 2.2 means we can take the union of all $\varphi_{\mathbf{t}}$ for which $(L, B'_L, \varphi_{\mathbf{t}})$ is equal to $(L, B_L, \varphi_{\mathbf{s}})$ a fixed structure of generalized restricted Lie algebra for L, we denote it by φ_L, i.e., φ_L is defined on $\bigcup_{B'_L \in J} B'_L$ (where $J := \{B'_L \mid (L, B'_L, \varphi_{\mathbf{t}}) = (L, B_L, \varphi_{\mathbf{s}})\}$) and

$$\varphi_L|_{B'_L} := \varphi_{\mathbf{t}}. \tag{1}$$

Lemma 2.3. *Let $(L, B'_L, \varphi_{\mathbf{w}})$ and $(L, B_L, \varphi_{\mathbf{s}})$ be two generalized restricted Lie algebra structures for L. If $I(B'_L, \varphi_{\mathbf{w}}) \subset I(B_L, \varphi_{\mathbf{s}})$ and $|\mathbf{s}| = |\mathbf{w}|$, then $(L, B_L, \varphi_{\mathbf{s}}) = (L, B'_L, \varphi_{\mathbf{w}})$.*

Proof. If $I(B'_L, \varphi_{\mathbf{w}}) \neq I(B_L, \varphi_{\mathbf{s}})$, then $I := I(B_L, \varphi_{\mathbf{s}})/I(B'_L, \varphi_{\mathbf{w}}) \neq 0$ is an ideal of $u_{\varphi_{\mathbf{w}}}(L)$ and $u_{\varphi_{\mathbf{w}}}(L)/I \cong u_{\varphi_{\mathbf{s}}}(L)$. This is a contradiction since $\dim u_{\varphi_{\mathbf{w}}}(L) = p^{|\mathbf{w}|} = p^{|\mathbf{s}|} = \dim u_{\varphi_{\mathbf{s}}}(L)$. □

Definition 2.2. K, a subalgebra of L, is called a generalized restricted subalgebra of (L, B_L, φ_L) if there is a generalized restricted Lie algebra (L, B'_L, φ_L) which is equal to (L, B_L, φ_L) such that B'_L contains a basis B_K of K and $\varphi_L(B_K) \subset K$.

ϕ, a homomorphism of L to L', is called a generalized restricted homomorphism of (L, B_L, φ_L) to $(L', B_{L'}, \varphi_{L'})$ if $\tilde{\phi}(I(B_L, \varphi_L)) \subset I(B_{L'}, \varphi_{L'})$, where $\tilde{\phi}$ is the homomorphism of $U(L)$ to $U(L')$ which is uniquely extended by ϕ.

In the special case where $\mathbf{s} = \mathbf{1}$ (in this case, a generalized restricted Lie algebra is a restricted Lie algebra), it is easy to verify that φ_L is identified with a p-mapping, the concept of generalized restricted subalgebra is identified with the concept of a restricted subalgebra and the concept of generalized restricted homomorphism is identified with the concept of restricted homomorphism. Consequently, Definition 2.2 generalizes the concept of a restricted subalgebra and the concept of restricted homomorphism.

If unlike Definition 2.2, we simply define a generalized restricted homomorphism as follows: a homomorphism ϕ of L to L' is called a generalized restricted homomorphism of $(L, B_L, \varphi_{\mathbf{s}})$ to $(L', B_{L'}, \varphi_{\mathbf{t}})$ if $\varphi_{\mathbf{s}}\phi = \phi\varphi_{\mathbf{t}}$, then ϕ must be a map of B_L to $B_{L'}$. Since the concept of restricted homomorphism does not rely on a fixed basis, this definition does not generalize the concept of a restricted homomorphism.

The advantage of Definition 2.2 will be seen in studying the equivalence classes of generalized restricted extensions of M by L.

Example 2.1. Let (L, B_L, φ_L) be a generalized restricted Lie algebra and $B_L = \{x_1, \ldots, x_n\}$. If I is an ideal of L, then there is a subset $\{x_{i_1}, \ldots, x_{i_m}\}$ of B_L such that $B_{L/I} = \{\bar{x}_{i_1}, \ldots, \bar{x}_{i_m}\}$ is a basis of L/I, where $\bar{x}_{i_j}$ is the canonical image of x_{i_j} $(j = 1, \ldots, m)$. Define a map $\varphi_{L/I} : B_{L/I} \to L/I$, via $\bar{x}_{i_j} \to \overline{\varphi_L(x_{i_j})}$. It is easy to verify that $(L/I, B_{L/I}, \varphi_{L/I})$ is a generalized restricted Lie algebra and the canonical homomorphism $\pi : L \to L/I$ is a generalized restricted homomorphism of (L, B_L, φ_L) to $(L/I, B_{L/I}, \varphi_{L/I})$.

This simple example will be used in the sequel.

3. Basic notions of cohomology of Lie algebras

Let L be a Lie algebra, M an L-module. Define the cochain complex $(C(L, M), \partial)$ as follows:

$$C^n(L, M) := \{f : L \times \cdots \times L \to M \mid f \text{ } n\text{-linear, alternating}\}.$$

The coboundary operator $\partial^n : C^n(L, M) \to C^{n+1}(L, M)$ is defined by

$$\begin{aligned}(\partial^n f)(x_0, \ldots, x_n) := {} & \sum_{i=0}^{n} (-1)^i x_i f(x_0, \ldots, \hat{x}_i \ldots, x_n) \\ & + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n).\end{aligned}$$

Let $U(L)^+$ be the ideal of $U(L)$ generated by L. Define the cochain complex $(C(U(L)^+, M), \partial)$ as follows:

$$C^n(U(L)^+, M) := \{f : U(L)^+ \times \cdots \times U(L)^+ \to M \mid f \text{ } n\text{-linear}\}.$$

The coboundary operator $\partial^n : C^n(U(L)^+, M) \to C^{n+1}(U(L)^+, M)$ is defined by

$$\partial^n g(u_0, \ldots, u_n) := u_0 g(u_1, \ldots, u_n) + \sum_{i=1}^{n} (-1)^i g(u_0, \ldots, u_{i-1}u_i, \ldots, u_n).$$

The complexes $(C(U(L)^+, M), \partial)$ and $(C(L, M), \partial)$ are called associative type and Lie type respectively. According to [3, p. 556], we have that

$$H^n(L, M) := \mathrm{Ext}^n_{U(L)}(F, M) \cong H^n\big(C(L, M)\big) \overset{\bar{\alpha}}{\cong} H^n\big(C\big(U(L)^+, M\big)\big)$$

and $\bar{\alpha}$ is induced by $\alpha : C(U(L)^+, M) \to C(L, M)$, which is defined as follows: $\alpha_n : C^n(U(L)^+, M) \to C^n(L, M)$, via

$$(\alpha_n g)(x_1, \ldots, x_n) = \sum_{\sigma} \mathrm{sgn}(\sigma) g(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

where the summation is over all permutations σ of the set $(1, \ldots, n)$ and $\mathrm{sgn}(\sigma)$ is equal to ± 1 according to whether σ is even or odd.

Let (L, B_L, φ_L) be a generalized restricted algebra over F and M be a generalized restricted L-module. Define

$$H^n_{\varphi_L}(L, M) := \mathrm{Ext}^n_{u_{\varphi_L}(L)}(F, M),$$

and call it the nth generalized restricted cohomology group for L with coefficients in M. As above, let $u_{\varphi_L}(L)^+$ be the ideal of $u_{\varphi_L}(L)$ generated by L and define the cochain complex $(C(u_{\varphi_L}(L)^+, M), \partial)$ as follows:

$$C^n\big(u_{\varphi_L}(L)^+, M\big) := \big\{ f : u_{\varphi_L}(L)^+ \times \cdots \times u_{\varphi_L}(L)^+ \to M \mid f \ n\text{-linear} \big\}$$

The coboundary operator $\partial^n : C^n(u_{\varphi_L}(L)^+, M) \to C^{n+1}(u_{\varphi_L}(L)^+, M)$ is defined by

$$\partial^n g(u_0, \ldots, u_n) := u_0 g(u_1, \ldots, u_n) + \sum_{i=1}^{n} (-1)^i g(u_0, \ldots, u_{i-1}u_i, \ldots, u_n).$$

We can similarly show

$$H^n_{\varphi_L}(L, M) \cong H^n\big(C\big(u_{\varphi_L}(L)^+, M\big)\big).$$

Define a mapping: $C^n(u_{\varphi_L}(L)^+, M) \to C^n(U(L)^+, M)$ via $g \to \tilde{g}$ where $\tilde{g}(u_1, \ldots, u_n) = g(\bar{u}_1, \ldots, \bar{u}_n)$ with $u_i \in U(L)^+$ and $\bar{u}_i$ its canonical image in $u_{\varphi_L}(L)^+$. Let $\widetilde{C}(U(L)^+, M) := \{\tilde{g} \mid g \in C(u_{\varphi_L}(L)^+, M)\}$ be the subcomplex of $C(U(L)^+, M)$. Then we have the exact sequence of complexes

$$0 \to \widetilde{C}\big(U(L)^+, M\big) \to C\big(U(L)^+, M\big) \to C\big(U(L)^+, M\big)/\widetilde{C}\big(U(L)^+, M\big) \to 0$$

which gives rise to the usual exact sequence connecting their cohomology groups. Since our map $g \to \tilde{g}$ is injective, we may identify $H^n(\widetilde{C}(U(L)^+, M))$ with $H^n_{\varphi_L}(L, M)$, whence we have the exact sequence

$$\cdots \to H^n_{\varphi_L}(L, M) \to H^n(L, M) \to H^n\big(C\big(U(L)^+, M\big)/\widetilde{C}\big(U(L)^+, M\big)\big)$$
$$\to H^{n+1}_{\varphi_L}(L, M) \to \cdots.$$

It is evident that $H^0(C(U(L)^+, M)/\widetilde{C}(U(L)^+, M)) = 0$, whence we conclude that the canonical homomorphism $H^1_{\varphi_L}(L, M) \to H^1(L, M)$ is injective.

Proposition 3.1. *Let M be a generalized restricted L-module. Then canonical homomorphism of $H^1_{\varphi_L}(L, M)$ into $H^1(L, M)$ maps $H^1_{\varphi_L}(L, M)$ isomorphically onto that subspace of $H^1(L, M)$ whose elements are represented by the Lie 1-cocycles f which satisfy the relation*

$$x_i^{p^{s_i}-1} f(x_i) = f\big(\varphi_L(x_i)\big) \quad \text{for any } x_i \in B_L.$$

Proof. Let f be a Lie 1-cocycle whose cohomology class belongs to $H^1_{\varphi_L}(L, M)$. Then there is an associative 1-cocycle g, defined on $u_{\varphi_L}(L)^+$, such that the cohomology class of $\tilde{g}$ coincides with that of f. This means that there is an element $m \in M$ such that $f(x) = xm + g(x)$. Hence, for $x_i \in B_L$,

$$x_i^{p^{s_i}-1} f(x_i) = x_i^{p^{s_i}} m + x_i^{p^{s_i}-1} g(x_i) = \varphi_L(x_i)m + g\big(\varphi_L(x_i)\big) = f\big(\varphi_L(x_i)\big).$$

Conversely, suppose that f satisfies the relation of the proposition. By the equivalence of the associative cochain complex with the Lie cochain complex it follows that f is the restriction to L of an associative cocycle h defined on $U(L)^+$. We have then for $u \in U(L)$ and $x_i \in B_L$,

$$h\big(u\big(x_i^{p^{s_i}} - \varphi_L(x_i)\big)\big) = ux_i^{p^{s_i}-1}h(x_i) - uh\big(\varphi_L(x_i)\big) = u\big(x_i^{p^{s_i}-1} f(x_i) - f\big(\varphi_L(x_i)\big)\big) = 0.$$

Thus, h vanishes on $I(B_L, \varphi_L)$, and is therefore the natural image $\tilde{g}$ of an associative cocycle g defined on $u_{\varphi_L}(L)^+$. Clearly, the cohomology class of g is mapped into that of f by the canonical map of $H^1_{\varphi_L}(L, M)$ into $H^1(L, M)$. □

4. Generalized restricted extensions

Let M be an abelian Lie algebra, L an arbitrary Lie algebra. An extension of M by L is a pair (E, ϕ), where E is a Lie algebra containing M as an ideal, and ϕ is a homomorphism of E onto L with kernel M. This situation defines on M the structure of an L-module, with L operating on M via E, in the natural fashion. An automorphism of such an extension is an isomorphism α of E onto E which leaves the elements of M fixed and satisfies the relation $\phi\alpha = \phi$. According to [3, p. 564], we have

Lemma 4.1. *α is an automorphism of (E, ϕ) if and only if $\alpha(e) = e + f(\phi(e))$, where f is a Lie 1-cocycle for L in M.*

Suppose that $(L, B_L, \varphi_{\mathbf{s}})$ is a generalized restricted Lie algebra, and M is a generalized restricted L-module together with a p-semilinear map of M into M^L. Then M has the structure of a restricted abelian Lie algebra, $(M, [p])$.

Definition 4.1. (E, ϕ), an extension of M by L, is called a generalized restricted extension if there are an ordered basis B_E of E,$(n+l)$-tuples of natural numbers $\mathbf{w} = (\mathbf{s}, \mathbf{1}) = (s_1, \ldots, s_n, 1, \ldots, 1)$, and a map $\varphi_{\mathbf{w}} : B_E \to E$ such that

(1) $(E, B_E, \varphi_{\mathbf{w}})$ is a generalized restricted Lie algebra;
(2) $(M, [p])$ is a generalized restricted subalgebra of $(E, B_E, \varphi_{\mathbf{w}})$;
(3) ϕ is a generalized restricted homomorphism and $\phi(B_E) \subset B_L \cup \{0\}$.

We denote it by $(E, B_E, \varphi_{\mathbf{w}}, \phi)$.

Lemma 4.2. *Let $(E, B_E, \varphi_{\mathbf{w}}, \phi)$ be a generalized restricted extension of M by L. Suppose $B_L = \{x_1, \ldots, x_n\}$. Then by rearranging the order of B_E, we can assume $B_E = \{e_1, \ldots, e_{n+l}\}$ such that $\phi(e_i) = x_i$ and $\phi\varphi_E(e_i) = \varphi_L\phi(e_i)$ for $i = 1, \ldots, n$.*

Proof. It is obvious that we can harmlessly assume $\mathbf{s} = (s_1, \ldots, s_n)$ and $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$.

For $x_n \in B_L$, the fact that $\phi(B_E) \subset B_L \cup \{0\}$ and ϕ is a surjective implies that there is $e_j \in B_E$ such that $x_n = \phi(e_j)$. The fact that ϕ is a generalized restricted homomorphism implies that

$$x_n^{p^{s_j}} - \phi\varphi_E(e_j) = \tilde{\phi}\big(e_j^{p^{s_j}} - \varphi_E(e_j)\big) \in I(B_L, \varphi_L).$$

Lemma 2.1 implies $s_j \geqslant s_n$, hence $s_j = s_n$. By rearranging the order of B_E (i.e., exchanging the position of e_j in B_E with the position of e_n in B_E), we can assume $e_j = e_n$, i.e., $\phi(e_n) = x_n$,

$$\phi\varphi_E(e_n) - \varphi_L\phi(e_n) = \phi\varphi_E(e_n) - x_n^{p^{s_n}} + x_n^{p^{s_n}} - \varphi_L\phi(e_n) \in I(B_L, \varphi_L) \cap L = 0,$$

hence $\phi\varphi_E(e_n) = \varphi_L\phi(e_n)$.

If we replace n with i $(= n-1, n-2, \ldots, 1$ in this order) in the above discussion, then, similarly, we can obtain $\phi(e_i) = x_i$ and $\phi\varphi_E(e_i) = \varphi_L\phi(e_i)$. □

Let A be an associative algebra. For $x \in A$, let L_x and R_x denote respectively the left and right multiplication by x; $\operatorname{ad} x = \mathrm{L}_x - \mathrm{R}_x$. According to [3, p. 567, formula (1)], we have

$$\sum_{j=0}^{m-1} (-1)^{m-j-1} \binom{m}{j} \mathrm{L}_x^j \mathrm{R}_x^{m-j-1} = \sum_{i=0}^{m-1} \mathrm{L}_x^i (\operatorname{ad} x)^{m-i-1}.$$

Setting $m = p^r$ in this formula, we obtain

$$\sum_{i=0}^{p^r-1} x^i (\operatorname{ad} x)^{p^r-1-i}(c) = cx^{p^r-1} \quad \text{for } c, x \in A. \tag{2}$$

Let $x, y \in A$ and t an indeterminate over A. Then

$$(xt + x)^p = x^p t^p + y^p + \sum_{i=1}^{p-1} b_i(x, y)t^i.$$

According to [6, p. 63], we have

$$\big(\operatorname{ad}(xt + y)\big)^{p-1}(x) = \sum_{i=1}^{p-1} i b_i(x, y)t^{i-1}. \tag{3}$$

Lemma 4.3. *Let (E, ϕ) be an extension of M by L. Then in $U(E)$, we have the formula*

$$(e + m)^{p^k} = e^{p^k} + \sum_{j=0}^{k} \big((\operatorname{ad} e)^{p^j-1} \cdot m\big)^{p^{k-j}} \quad \text{for } m \in M,\ e \in E.$$

Proof. In formula (3), replacing x and y with m and e^{p^r} respectively, we have

$$\big(\operatorname{ad}\big(mt + e^{p^r}\big)\big)^{p-1} \cdot m = \sum_{i=1}^{p-1} i b_i\big(m, e^{p^r}\big)t^{i-1}. \tag{4}$$

Suppose that $\sum_{i=0}^{p-1} t^i c_i(\operatorname{ad} m, \operatorname{ad} e^{p^r})$ is the expansion of $(t \operatorname{ad} m + \operatorname{ad} e^{p^r})^{p-1}$. Then

$$\begin{aligned}\big(\operatorname{ad}\big(mt + e^{p^r}\big)\big)^{p-1} \cdot m &= \big(t \operatorname{ad} m + \operatorname{ad} e^{p^r}\big)^{p-1} \cdot m \\ &= \sum_{i=0}^{p-1} t^i c_i\big(\operatorname{ad} m, \operatorname{ad} e^{p^r}\big) \cdot m.\end{aligned} \tag{5}$$

The fact that $\operatorname{ad} e^{p^r} = (\operatorname{ad} e)^{p^r}$ and M is an abelian ideal of E implies

$$c_i\big(\operatorname{ad} m, \operatorname{ad} e^{p^r}\big) \cdot m = \begin{cases} 0 & i \neq 0, \\ (\operatorname{ad} e^{p^r})^{p-1} & i = 0. \end{cases}$$

By means of a comparison of coefficients of t^i of formulas (4) and (5), we have

$$i b_i\big(m, e^{p^r}\big) = \begin{cases} 0 & i \neq 1, \\ (\operatorname{ad} e^{p^r})^{p-1} & i = 1. \end{cases}$$

Consequently, we obtain

$$\big(e^{p^r}+m\big)^p=\big(e^{p^r}\big)^p+m^p+\big(\operatorname{ad} e^{p^r}\big)^{p-1}\cdot m. \tag{6}$$

Now we show the lemma by induction on k. Setting $r=0$ in formula (6), we obtain $(e+m)^p=(e)^p+m^p+(\operatorname{ad} e)^{p-1}\cdot m$, hence the lemma is valid for $k=1$. According to the induction hypothesis, we have

$$(e+m)^{p^{k+1}}=\left(e^{p^k}+\sum_{j=0}^{k}\big((\operatorname{ad} e)^{p^j-1}\cdot m\big)^{p^{k-j}}\right)^p. \tag{7}$$

The fact that $[m^{p^i},e^{p^j}]=-(\operatorname{ad} e)^{p^j-1}(\operatorname{ad} m)^{p^i}\cdot e=0$ (for $\forall m\in M,\ e\in E,\ i,j\in\mathbb{N}$) implies

$$\left(e^{p^k}+\sum_{j=0}^{k}\big((\operatorname{ad} e)^{p^j-1}\cdot m\big)^{p^{k-j}}\right)^p=\big(e^{p^k}+\big((\operatorname{ad} e)^{p^k-1}\cdot m\big)\big)^p+\sum_{j=0}^{k-1}\big((\operatorname{ad} e)^{p^j-1}\cdot m\big)^{p^{k-j+1}}. \tag{8}$$

Formula (6) ensures

$$\big(e^{p^k}+\big((\operatorname{ad} e)^{p^k-1}\cdot m\big)\big)^p=e^{p^{k+1}}+\big((\operatorname{ad} e)^{p^k-1}\cdot m\big)^p+\big(\operatorname{ad} e^{p^k}\big)^{p-1}\cdot\big((\operatorname{ad} e)^{p^k-1}\cdot m\big). \tag{9}$$

Formulas (7), (8), and (9) provide

$$(e+m)^{p^{k+1}}=e^{p^{k+1}}+\sum_{j=0}^{k+1}\big((\operatorname{ad} e)^{p^j-1}\cdot m\big)^{p^{k+1-j}}. \qquad \square$$

Let (E,B_E,φ_E,ϕ) be a generalized restricted extension of M by L and $B_E=\{e_1,\ldots,e_{n+l}\}$. Suppose that $B'_E=\{e_1+m_1,\ldots,e_{n+l}+m_{n+l}\}$ is another basis of E (where $m_1,\ldots,m_{n+l}\in M$). Define

$$\varphi'(e_i+m_i):=\begin{cases}\varphi_E(e_i)+\sum_{j=0}^{s_i}((\operatorname{ad} e_i)^{p^j-1}\cdot m)^{[p]^{s_i-j}} & i\leqslant n,\\ \varphi_E(e_i)+\sum_{j=0}^{1}((\operatorname{ad} e_i)^{p^j-1}\cdot m)^{[p]^{1-j}} & i>n.\end{cases} \tag{10}$$

Then

$$\begin{aligned}\operatorname{ad}\varphi'(e_i+m_i)&=\operatorname{ad}\left(\varphi_E(e_i)+\sum_{j=0}^{s_i}\left(\operatorname{ad}e_i^{p^j-1}\cdot m_i\right)^{[p]^{s_i-j}}\right)\\&=\operatorname{ad}\left(e_i^{p^{s_i}}+\sum_{i=0}^{s_i}\left((\operatorname{ad}e_i)^{p^j-1}\cdot m_i\right)^{p^{s_i-j}}\right)\\&=\operatorname{ad}(e_i+m_i)^{p^{s_i}}\quad\text{for } i\leqslant n.\end{aligned}$$

Similarly, we have

$$\operatorname{ad}\varphi'(e_i+m_i)=\operatorname{ad}(e_i+m_i)^p\quad\text{for } i>n.$$

This means that (E,B'_E,φ') is a generalized restricted Lie algebra.

Since $(M,[p])$ is a generalized restricted subalgebra of (E,B_E,φ_E), we have

$$m^{p^k}-m^{[p]^k}=\sum_{i=0}^{k-1}\left(\left(m^{[p]^i}\right)^p-\left(m^{[p]^i}\right)^{[p]}\right)^{p^{k-i-1}}\in I(B_E,\varphi_E),\quad\text{for }\forall m\in M,k\in\mathbb{N}. \tag{11}$$

Formula (11) and Lemma 4.3 imply that $I(B'_E,\varphi')\subset I(B_E,\varphi_E)$, Lemma 2.3 ensures $(E,B_E,\varphi_E)=(E,B'_E,\varphi'_E)$. Hence by formula (1) we have $\varphi_E|_{B'_E}=\varphi'$.

In order to avoid a few narrative complications, in the sequel we assume that $\mathbf{s}=(r,\dots,r)$ *in the definition of generalized restricted Lie algebra, but the assumption concerning* $\mathbf{s}$ *is not necessary.*

Definition 4.2. Let (E,B_E,φ_E,ϕ) and $(E',B_{E'},\varphi_{E'},\phi')$ be two generalized restricted extensions of M by L. (E,B_E,φ_E,ϕ) and $(E',B_{E'},\varphi_{E'},\phi')$ are called similar if there exists an ordinary Lie algebra isomorphism γ of E to E' which leaves the elements of M fixed and for which $\phi=\phi'\gamma$. Then (E,B_E,φ_E,ϕ) and $(E',B_{E'},\varphi_{E'},\phi')$ are called equivalent if such a map γ is a generalized restricted homomorphism.

Let γ be a similarity isomorphism of (E,B_E,φ_E,ϕ) to $(E',B_{E'},\varphi_{E'},\phi')$. For $e\in B_E$ and $m\in M$, the relation $\phi'\gamma(e+m)=\phi(e+m)$ implies that there exists $m'\in M$ and $e'\in B_{E'}$ such that $\gamma(e+m)=e'+m'$. According to formula (10), $\varphi_{E'}\gamma(e+m)$ is defined.

Lemma 4.4. *Let* γ *be a similarity isomorphism of* (E,B_E,φ_E,ϕ) *to* $(E',B_{E'},\varphi_{E'},\phi')$. *Then following statements are equivalent*:

(1) γ *is an equivalence isomorphism of* (E,B_E,φ_E,ϕ) *to* $(E',B_{E'},\varphi_{E'},\phi')$.
(2) $\gamma\varphi_E(e+m)=\varphi_{E'}\gamma(e+m)$, *for* $\forall m\in M$ *and* $\forall e\in B_E$.
(3) $\gamma\varphi_E(e)=\varphi_{E'}\gamma(e)$, *for any* $e\in B_E$.

Proof. It is clear that (2) ⇒ (3).

(3) ⇒ (1). Let $\gamma(e) = e' + m'$ (where $e \in B_E$ and $e' \in B_{E'}$) and $\tilde{\gamma}$ be a homomorphism of $U(E)$ to $U(E')$ which is extended by γ. According to Lemma 4.3 and formula (10), we have

$$\begin{aligned}\tilde{\gamma}\big(e^{p^r} - \varphi_E(e)\big) &= \gamma(e)^{p^r} - \gamma\varphi_E(e) = \gamma(e)^{p^r} - \varphi_{E'}\gamma(e) \\ &= \big(e'\big)^{p^r} + \sum_{j=0}^{r}\big(\big(\operatorname{ad} e'\big)^{p^j-1}\cdot m'\big)^{p^{r-j}} - \varphi_{E'}\big(e'\big) \\ &\quad - \sum_{j=0}^{r}\big(\big(\operatorname{ad} e'\big)^{p^j-1}\cdot m'\big)^{[p]^{r-j}}.\end{aligned}$$

Formula (11) implies $\tilde{\gamma}(e^{p^r} - \varphi_E(e)) \in I(B_{E'}, \varphi_{E'})$. Since $I(B_E, \varphi_E)$ is generated by elements $e^{p^r} - \varphi_E(e)$ $(e \in B_E)$, we have $\tilde{\gamma}(I(B_E, \varphi_E)) \subset I(B_{E'}, \varphi_{E'})$, i.e., γ is an equivalence isomorphism of $(E, B_E, \varphi_E, \phi)$ to $(E', B_{E'}, \varphi_{E'}, \phi')$.

(1) ⇒ (2). Let $\gamma(e+m) = e' + m'$ for $e \in B_E$ and $m \in M$. According to Lemma 4.3 and formula (10), we have

$$\begin{aligned}\big(\gamma(e+m)\big)^{p^r} - \varphi_{E'}\gamma(e+m) &= \big(e'\big)^{p^r} + \sum_{j=0}^{r}\big(\big(\operatorname{ad} e'\big)^{p^j-1}\cdot m'\big)^{p^{r-j}} - \varphi_{E'}\big(e'\big) \\ &\quad - \sum_{j=0}^{r}\big(\big(\operatorname{ad} e'\big)^{p^j-1}\cdot m'\big)^{[p]^{r-j}} \in I(B_{E'}, \varphi_{E'}).\end{aligned}$$

The fact that γ is an equivalence isomorphism of $(E, B_E, \varphi_E, \phi)$ to $(E', B_{E'}, \varphi_{E'}, \phi')$ means $\tilde{\gamma}((e+m)^{p^r} - \varphi_E(e+m)) \in I(B_{E'}, \varphi_{E'})$. Hence

$$\begin{aligned}\varphi_{E'}\gamma(e+m) - \gamma\varphi_E(e+m) &= \tilde{\gamma}\big((e+m)^{p^r} - \varphi_E(e+m)\big) \\ &\quad - \big(\big(\gamma(e+m)\big)^{p^r} - \varphi_{E'}\gamma(e+m)\big) \in I(B_{E'}, \varphi_{E'}) \cap E' = (0),\end{aligned}$$

i.e., $\gamma\varphi_E(e+m) = \varphi_{E'}\gamma(e+m)$. □

Lemma 4.5. *Every generalized restricted extension of M by L is similar to one in which M is strongly abelian* (*i.e.*, $[M, M] = 0$ *and* $M^{[p]} = 0$).

Proof. Let $(E, B_E, \varphi_E, \phi)$ be a generalized restricted extension of M by L. According to Lemma 4.2, there are $e_i \in B_E$ and $x_i \in B_L$ $(i = 1, \ldots, n)$ such that $\phi(e_i) = x_i$ and $\phi\varphi_E(e_i) = \varphi_L(x_i)$. Let $B'_E = \{e_1, \ldots, e_n, m_1, \ldots, m_l\}$ where $\{m_1, \ldots, m_l\}$ is a basis of M. Define $\psi_E(e_i) := \varphi_E(e_i)$ and $\psi_E(m_i) := m_i^{[p]}$. It is easy to verify that (E, B'_E, ψ_E) is a generalized restricted Lie algebra and $I(B'_E, \psi_E) \subset I(B_E, \varphi_E)$. According to Lemma 2.3, we have $(E, B_E, \varphi_E) = (E, B'_E, \psi_E)$. This means that we can assume $B_E = \{e_1, \ldots, e_n, m_1, \ldots, m_l\}$. Define $\varphi'_E : B_E \to E$ by $\varphi'_E(e_i) := \varphi_E(e_i)$ and $\varphi'_E(m_i) := 0$. It is easily to verify that $(E, B_E, \varphi'_E, \phi)$ is a generalized restricted extension of M by L which is similar to $(E, B_E, \varphi_E, \phi)$ by the identity map of E onto E. $\square$

Let M be an L-module and $\mathrm{ext}(M, L)$ the set of the similarity classes of those extensions of M by L. If L is a generalized restricted Lie algebra and M is a generalized restricted L-module, we can consider the subset $\mathrm{ext}_s(M, L)$ of $\mathrm{ext}(M, L)$ which consists of the similarity classes of the generalized restricted extensions.

We recall the definition of the vector space structure in $\mathrm{ext}(M, L)$. The subtraction in $\mathrm{ext}(M, L)$ is the map which is induced by the following composition of extensions (E, ϕ) and (E', ϕ'). Let D denote the subalgebra of the direct sum of E and E' which consists of all the elements (e, e') in which $\phi(e) = \phi'(e')$. Let J be the ideal of D consisting of the elements (m, m), with $m \in M$. In D/J, we identify $(M, M)/J$ with M by means of the homomorphism $(m, m') \to m - m'$ whose kernel is exactly J. The homomorphism $(e, e') \to \phi(e)$ induces a homomorphism ψ of D/J onto L, and $(D/J, \psi)$ is the extension of M by L, which represents the difference of the similarity classes of (E, ϕ) and (E', ϕ').

Suppose that $(E, B_E, \varphi_E, \phi)$ and $(E', B_{E'}, \varphi_{E'}, \phi')$ are generalized restricted extensions of M by L. According to the proof of Lemma 4.5, we can assume $B_E = \{e_1, \dots, e_n, m_1, \dots, m_l\}$ and $B'_E = \{e'_1, \dots, e'_n, m_1, \dots, m_l\}$ where $\{m_1, \dots, m_l\}$ is a basis of M and $\phi(e_i) = \phi'(e'_i)$. Let $B_D = \{(e_i, e'_i) \mid i = 1, \dots, n\} \cup \{(m_i, 0) \mid i = 1, \dots, l\} \cup \{(0, m_j) \mid j = 1, \dots, l\}$. It is easy to verify that B_D is a basis of D. Define map φ_D on B_D by setting $\varphi_D((e_i, e'_i)) := (\varphi_E(e_i), \varphi_{E'}(e'_i))$, $\varphi_D((m_i, 0)) := (m_i^{[p]}, 0)$, and $\varphi_D((0, m_j)) := (0, m_j^{[p]})$. It is easily seen that (D, B_D, φ_D) is a generalized restricted Lie algebra. Example 2.1 shows that $(D/J, B_{D/J}, \varphi_{D/J}, \psi)$ constitutes a generalized restricted extension of M by L. Hence $\mathrm{ext}_s(M, L)$ is a subgroup of $\mathrm{ext}(M, L)$.

The scalar multiple of the similarity class of (E, ϕ) by an element $\alpha \in F$ is the similarity class of the extension defined as follows. First, construct a Lie algebra (E, M) whose space is the direct sum of E and M, and where the commutation is defined by the formula

$$\big[(e, m), \big(e', m'\big)\big] = \big(\big[e, e'\big], e \cdot m' - e' \cdot m\big).$$

Let J be the ideal of (E, M) which consists of the elements of the form $(m, -\alpha m)$, with $m \in M$. In $(E, M)/J$, identify $(M, M)/J$ with M by the homomorphism $(m, m') \to \alpha m + m'$ whose kernel is J. Let ψ be the homomorphism $(e, m) \to \phi(e)$. Then the similarity class of $((E, M)/J, \psi)$ is the desired α-multiple of the similarity class of (E, ϕ).

According to Lemma 4.5, every element of $\mathrm{ext}_s(M, L)$ is represented by a generalized extension $(E, B_E, \varphi_E, \phi)$ in which M is strongly abelian. According to the proof of Lemma 4.5, we can assume $B_E = \{e_1, \dots, e_n, m_1, \dots, m_l\}$. Let $B_{(E,M)} = \{(e_1, 0), \dots, (e_n, 0), (m_1, 0), \dots, (m_l, 0)\} \cup \{(0, m_j) \mid j = 1, \dots, l\}$. It is easily seen that $B_{(E,M)}$ is a basis of (E, M). Define a map $\varphi_{(E,M)}$ on $B_{(E,M)}$ by setting $\varphi_{(E,M)}((e_i, 0)) := (\varphi_E(e_i), 0)$, $\varphi_{(E,M)}((m_i, 0)) := (0, 0)$, and $\varphi_{(E,M)}((0, m_j)) := (0, 0)$. It is easy to verify that $(\mathrm{ad}(e_i, 0))^{p^r} = \mathrm{ad}(\varphi_E(e_i), 0)$, $(\mathrm{ad}(m_i, 0))^p = \mathrm{ad}\,\varphi_{(E,M)}((m_i, 0))$, and $(\mathrm{ad}(0, m_j))^p = \mathrm{ad}\,\varphi_{(E,M)}((0, m_j))$. Hence $((E, M), B_{(E,M)}, \varphi_{(E,M)})$ is a generalized restricted Lie algebra. Example 2.1 ensures that $((E, M)/J, B_{(E,M)/J}, \varphi_{(E,M)/J}, \psi)$ is a generalized restricted extension of M by L. Hence we have proved the following proposition:

Proposition 4.6. $\mathrm{ext}_s(M, L)$ *is a subspace of* $\mathrm{ext}(M, L)$.

Let $\mathrm{ext}_{\varphi_L}(M, L)$ be the equivalence classes of those generalized restricted extensions of M by L. If we assign to each such equivalence class the similarity class to which it belongs, we obtain what we shall call the canonical map of $\mathrm{ext}_{\varphi_L}(M, L)$ into $\mathrm{ext}_s(M, L)$. It is clear that, no matter what the given p-mapping in M actually is, the canonical map is a group homomorphism of $\mathrm{ext}_{\varphi_L}(M, L)$ onto $\mathrm{ext}_s(M, L)$. Furthermore, if M is strongly abelian, then the canonical map is an F-linear map.

5. Cohomology of generalized restricted Lie algebras

As we know, there is a canonical isomorphism of $\mathrm{ext}(M, L)$ onto $H^2(L, M)$. Let f be any Lie 2-cocycle for L in M. The extension of M by L, which corresponds to the cohomology class c of f by the canonical isomorphism, is the following. The underlying space of the extended algebra is the direct sum of L and M, and $[(x, m), (x_1, m_1)] := ([x, x_1], x \cdot m_1 - x_1 \cdot m + f(x, x_1))$ for $x, x_1 \in L$, $m_1, m \in M$. The homomorphism onto L is the map $(x, m) \to x$.

Set $k_x(x_1) := \sum_{i=0}^{p^r-1} x^i \cdot f(x, \mathrm{ad}\, x^{p^r-1-i}(x_1))$ for $x \in B_L$ and write $f_y(x) = f(x, y)$. By directly computing, we have

$$\big(\mathrm{ad}(x, 0)\big)^{p^r}(x_1, 0) = \big(\big[\varphi_L(x), x_1\big], k_x(x_1)\big). \tag{12}$$

Applying $(\mathrm{ad}(x, 0)^{p^r})$ to $[(x_1, 0), (x_2, 0)]$ and considering the fact that $(\mathrm{ad}(x, 0))^{p^r}$ is a derivation and f is a Lie 2-cocycle for L in M, we can obtain that $k_x + f_{\varphi_L(x)}$ is a Lie 1-cocycle for L in M for each fixed $x \in B_L$.

If f is a coboundary, ∂g, then

$$k_x(x_1) = \varphi_L(x) \cdot g(x_1) - g\big(\big[\varphi_L(x), x_1\big]\big) - \sum_{i=0}^{p^r-1} x^i\, \mathrm{ad}\, x^{p^r-1-i}(x_1) \cdot g(x).$$

Applying formula (2), we obtain

$$k_x(x_1) = \varphi_L(x) \cdot g(x_1) - g\big(\big[\varphi_L(x), x_1\big]\big) - x_1 x^{p^r-1} \cdot g(x)$$

whence

$$k_x(x_1) + f\big(x_1, \varphi_L(x)\big) = x_1 \cdot \big(g\big(\varphi_L(x)\big) - x^{p^r-1} \cdot g(x)\big).$$

It follows that the cohomology class of $k_x + f_{\varphi_L(x)}$ depends only on the cohomology class c of f. We shall denote it by $c(x)$. Let $H_s^2(L, M)$ be the image of $\mathrm{ext}_s(M, L)$ in $H^2(L, M)$ by the canonical isomorphism of $\mathrm{ext}(M, L)$ onto $H^2(L, M)$. Then

Lemma 5.1. *$c \in H_s^2(L, M)$ if and only if $c(x) = 0$ for all $x \in B_L$.*

Proof. Let $c \in H_s^2(L, M)$ and let $(E, B_E, \varphi_E, \phi)$ be a generalized restricted extension whose similarity class corresponds to c. It can be easily seen from the proof of Lemma 4.5 that we can assume $B_E = \{(x, 0) \mid x \in B_L\} \cup \{(0, m_1), \ldots, (0, m_l)\}$ (where $\{m_1, \ldots, m_l\}$ is a basis of M). Lemma 4.2 means that we can write $\varphi_E((x, 0)) = (\varphi_L(x), \rho(x))$ for $x \in B_L$. Moreover, $(\operatorname{ad}(x, 0))^{p^r} \cdot (x_1, 0) = \operatorname{ad}\varphi_E(x, 0) \cdot (x_1, 0)$ means $k_x(x_1) = f(\varphi_L(x), x_1) - x_1 \cdot \rho(x)$, i.e., $k_x + f_{\varphi_L(x)} = \partial(-\rho(x))$, so that $c(x) = 0$.

Conversely, let g be an associative 2-cocycle for $U(L)^+$ in M which represents the given cohomology class $c \in H^2(L, M)$. If $f(x, y) = g(x, y) - g(y, x)$. Then f is a Lie 2-cocycle belonging to c. If we express $k_x(x_1)$ in terms of values g, we have

$$k_x(x_1) = g\big(x^{p^r}, x_1\big) - \sum_{i=0}^{p^r-1} g\big(x^i \operatorname{ad} x^{p^r-1-i} \cdot x_1, x\big).$$

By formula (2), this becomes

$$\begin{aligned} k_x(x_1) &= g\big(x^{p^r}, x_1\big) - g\big(x_1 x^{p^r-1}, x\big) \\ &= g\big(x^{p^r}, x_1\big) - g\big(x_1, x^{p^r}\big) - x_1 \cdot g\big(x^{p^r-1}, x\big). \end{aligned} \tag{13}$$

Hence $c(x)$ has the representative 1-cocycle g'_x, where

$$g'_x(x_1) = g\big(x^{p^r} - \varphi_L(x), x_1\big) - g\big(x_1, x^{p^r} - \varphi_L(x)\big).$$

If $c(x) = 0$ for all $x \in B_L$, we can find a map σ of B_L into M such that $g'_x(x_1) = x_1 \cdot \sigma(x)$. Let $B_E = \{(x, 0) \mid x \in B_L\} \cup \{(0, m_1), \ldots, (0, m_l)\}$ and define $\varphi_E((x, 0)) = (\varphi_L(x), g(x^{p^r-1}, x) - \sigma(x))$ for all $x \in B_L$, $\varphi_E((0, m_i)) = (0, 0)$. It is clear that $(\operatorname{ad}(0, m_i))^p = \operatorname{ad}\varphi_E((0, m_i))$. Now we verify the identity $(\operatorname{ad}(x, 0))^{p^r} = \operatorname{ad}\varphi_E((x, 0))$:

$$\begin{aligned} \big(\operatorname{ad}(x, 0)\big)^{p^r}(x_1, m_1) &= \big(\big[\varphi_L(x), x_1\big], \varphi_L(x) \cdot m_1 + k_x(x_1)\big) \\ &= \big(\big[\varphi_L(x), x_1\big], \varphi_L(x) \cdot m_1 - x_1 \cdot g\big(x^{p^r-1}, x\big) \\ &\quad + g\big(x^{p^r}, x_1\big) - g\big(x_1, x^{p^r}\big)\big) \end{aligned}$$

while

$$\begin{aligned} \operatorname{ad}\varphi_E\big((x, 0)\big)(x_1, m_1) &= \big(\big[\varphi_L(x), x_1\big], \varphi_L(x) \cdot m_1 - x_1 \cdot g\big(x^{p^r-1}, x\big) \\ &\quad + x_1\sigma(x) + g\big(\varphi_L(x), x_1\big) - g\big(x_1, \varphi_L(x)\big)\big) \\ &= \big(\big[\varphi_L(x), x_1\big], \varphi_L(x) \cdot m_1 - x_1 \cdot g\big(x^{p^r-1}, x\big) \\ &\quad + g'_x(x_1) + g\big(\varphi_L(x), x_1\big) - g\big(x_1, \varphi_L(x)\big)\big) \\ &= \big(\big[\varphi_L(x), x_1\big], \varphi_L(x) \cdot m_1 - x_1 \cdot g\big(x^{p^r-1}, x\big) \\ &\quad + g\big(x^{p^r}, x_1\big) - g\big(x_1, x^{p^r}\big)\big). \end{aligned}$$

Thus $(\mathrm{ad}(x,0))^{p^r} = \mathrm{ad}\,\varphi_E((x,0))$. Hence $(E, B_E, \varphi_E, \phi)$ is a generalized restricted extension of M by L, i.e., $c \in H_s^2(L, M)$. □

Let $\mathrm{Map}(B_L, H^1(L, M))$ be the space of all maps of B_L into $H^1(L, M)$. Then we have proved

Proposition 5.2. *Let L be a generalized restricted Lie algebra and M be a generalized restricted L-module. For an associative 2-cocycle g, define $g'_x(x_1) := g(x^{p^r} - \varphi_L(x), x_1) - g(x_1, x^{p^r} - \varphi_L(x))$. Then map $g \to g'$ induces an F-linear map of $H^2(L, M)$ into $\mathrm{Map}(B_L, H^1(L, M))$ whose kernel is precisely the subspace $H_s^2(L, M)$.*

Let γ be a similarity isomorphism of a generalized restricted extension $(E, B_E, \varphi_E, \phi)$ onto a generalized restricted extension $(E', B_{E'}, \varphi_{E'}, \phi')$. Let $e \in B_E$. Then there exist $e' \in B_{E'}$ such that $\gamma(e) = e' + m$ $(m \in M)$. According to formula (10), we have

$$\varphi_{E'}\big(\gamma(e)\big) = \varphi_{E'}\big(e'\big) + \sum_{i=0}^{r}\big(\big(\mathrm{ad}\,e'\big)^{p^i-1}m\big)^{[p]^{r-i}}.$$

Define $g(x) := \gamma(\varphi_E(e)) - \varphi_{E'}(\gamma(e))$ (where $\phi(e) = x \in B_L$). Then

$$\begin{aligned}\phi'\big(g(x)\big) &= \phi'\big(\gamma\big(\varphi_E(e)\big) - \varphi_{E'}\big(\gamma(e)\big)\big) = \phi\big(\varphi_E(e)\big) - \phi'\varphi_{E'}\big(e'\big)\\ &= \varphi_L(x) - \varphi_L(x) = 0,\end{aligned}$$

hence $g(x) \in M$. For $y \in L$, there exists $z \in E$ such that $y = \phi'\gamma(z)$, thus

$$\begin{aligned}y \cdot g\big(\phi(e)\big) &= \big[\gamma(z), \big(\gamma\big(\varphi_E(e)\big) - \varphi_{E'}\big(\gamma(e)\big)\big)\big]\\ &= \gamma\big(\big[z, \varphi_E(e)\big]\big) + \mathrm{ad}\,\gamma(e)^{p^r}\big(\gamma(z)\big)\\ &= \gamma\big(\big[z, \varphi_E(e)\big]\big) + \gamma\big(\mathrm{ad}\,e^{p^r}(z)\big) = 0,\end{aligned}$$

i.e., $g \in \mathrm{Map}(B_L, M^L)$ (the set of all maps of B_L into M^L). Define $\varphi_{E_g}(e) := \varphi_E(e) - g(\phi(e))$ for $e \in B_E$. It is obvious that $(E, B_E, \varphi_{E_g}, \phi)$ is a generalized restricted extension of M by L. Lemma 4.4 ensures that γ is an equivalence isomorphism of $(E, B_E, \varphi_{E_g}, \phi)$ onto $(E', B_{E'}, \varphi_{E'}, \phi')$.

Conversely, if $g \in \mathrm{Map}(B_L, M^L)$, then $(E, B_E, \varphi_{E_g}, \phi)$ is obviously a generalized restricted extension of M by L which is similar to $(E, B_E, \varphi_E, \phi)$.

With every $g \in \mathrm{Map}(B, M^L)$, we associate the transformation g^* on $\mathrm{ext}_{\varphi_L}(M, L)$ which is induced by the map $(E, B_E, \varphi_E, \phi) \to (E, B_E, \varphi_{E_g}, \phi)$. Let π be the canonical map of $\mathrm{ext}_{\varphi_L}(M, L)$ onto $\mathrm{ext}_s(M, L)$ and $S_c = \pi^{-1}(c)$ where $c \in \mathrm{ext}_s(M, L)$.

Lemma 5.3. *The following statements hold*:

(1) $g^*(S_c) \subset S_c$, *and* $\mathrm{Map}(B_L, M^L)$ *operates transitively on each* S_c.

(2) *Let* s_0 *denote the* 0*-element of* $\operatorname{ext}_{\varphi_L}(M, L)$ *and* $s \in \operatorname{ext}_{\varphi_L}(M, L)$*. Then* $g^*(s) = g^*(s_0) + s$*.*

(3) *The maps* $g \to g^*$ *and* $g \to g^*(s_0)$ *are group homomorphisms.*

(4) *The kernel of the map* $g \to g^*$ *is* $\{g \mid g(x) = \sum_{i=0}^{r}(x^{p^i-1}f(x))^{[p]^{r-i}} - f(\varphi_L(x)), x \in B_L, f \in Z^1(L, M)\}$*, where* $Z^1(L, M) = \ker(\partial^1 : C^1(L, M) \to C^2(L, M))$*.*

Proof. (1) has been shown by the above discussion.

(2) Let $(E_0, B_{E_0}, \varphi_{E_0}, \phi')$ be a representation of s_0 and $(E, B_E, \varphi_E, \phi)$ a representation of s. According the proof of Lemma 4.5, we can assume that $B_{E_0} = \{m_1, \ldots, m_l, e'_1, \ldots, e'_n\}$ and $B_E = \{m_1, \ldots, m_l, e_1, \ldots, e_n\}$ (where $\phi(e_i) = \phi(e'_i) = x_i \in B_L$). Let $D = \{(e, e')\colon \phi(e) = \phi'(e')\}$, $J = \{(m, -m)\colon m \in M\}$, $B_D = \{(e_i, e'_i) \mid i = 1, \ldots, n\} \cup \{(m_i, 0) \mid i = 1, \ldots, l\} \cup \{(0, m_j) \mid j = 1, \ldots, l\}$, and $\psi(e, e') = \phi(e)$.

Then $g^*(s) = g^*(s) + s_0$ has a representation $(D/J, B_D/J, \varphi_{D/J}, \bar{\psi})$ and $\varphi_{D/J}$ is induced by $\varphi_D : B_D \to D$, where

$$\varphi_D\big((e_i, e'_i)\big) := \big(\varphi_{E_g}(e_i), \varphi_{E_0}(e'_i)\big) = \big(\varphi_E(e_i) - g(x_i), \varphi_{E_0}(e'_i)\big),$$

$$\varphi_D\big((m_i, 0)\big) := \big(m_i^{[p]}, 0\big), \quad \text{and} \quad \varphi_D\big((0, m_j)\big) := \big(0, m_j^{[p]}\big).$$

Similarly, $s + g^*(s_0)$ has a representation $(D/J, B_{D/J}, \varphi'_{D/J}, \psi)$ and $\varphi_{D/J}$ is induced by $\varphi_D : B_D \to D$, where

$$\varphi_D\big((e_i, e'_i)\big) := \big(\varphi_E(e_i), \varphi_{(E_0)_g}(e'_i)\big) = \big(\varphi_E(e_i), \varphi_{E_0}(e'_i) - g(x_i)\big),$$

$$\varphi_D\big((m_i, 0)\big) := \big(m_i^{[p]}, 0\big), \quad \text{and} \quad \varphi_D\big((0, m_j)\big) := \big(0, m_j^{[p]}\big).$$

Thus we have $g^*(s) = s + g^*(s_0)$.

(3) $g_1^*(g_2^*((E, B_E, \varphi_E, \phi))) = g_1^*((E, B_E, \varphi_{E_{g_2}}, \phi)) = (E, B_E, \varphi_{(E_{g_2})_{g_1}}, \phi)$, by the definition of φ_{E_g} to obtain $\varphi_{(E_{g_2})_{g_1}} = \varphi_{E_{(g_1+g_2)}}$, hence $(g_1 + g_2)^* = g_1^* g_2^*$.

(4) Suppose that ζ is an equivalence isomorphism of $(E, B_E, \varphi_{E_{g_1}}, \phi)$ onto $(E, B_E, \varphi_{E_{g_2}}, \phi)$. Lemma 4.1 ensures $\zeta(e) = e + h(\phi(e))$ where h is a Lie 1-cocycle for L in M. By Lemma 4.4, we have $\varphi_{E_{g_2}}(\zeta(e)) = \zeta(\varphi_{E_{g_1}}(e))$, $e \in B_E$. Writing out the relation $\zeta(\varphi_{E_{g_1}}(e)) = \varphi_{E_{g_2}}(\zeta(e))$, we have

$$g_2(x) - g_1(x) = x^{p^r-1} \cdot h(x) - h\big(\varphi_L(x)\big) + \sum_{i=0}^{r-1}\big(x^{p^i-1} \cdot h(x)\big)^{[p]^{r-i}}, \quad x \in B_L.$$

Conversely, let f be any Lie 1-cocycle for L in M. Then f is the restriction to L of an associative cocycle $\tilde{f}$ defined on $U(L)^+$. For $x \in B_L$, set

$$g(x) := x^{p^r-1} f(x) - f\big(\varphi_L(x)\big) + \sum_{i=0}^{r-1}\big(x^{p^i-1} f(x)\big)^{[p]^{r-i}}$$

$$= \tilde{f}\big(x^{p^r}\big) - f\big(\varphi_L(x)\big) + \sum_{i=0}^{r-1}\big(x^{p^i-1}f(x)\big)^{[p]^{r-i}}$$

$$= \tilde{f}\big(x^{p^r} - \varphi_L(x)\big) + \sum_{i=0}^{r-1}\big(x^{p^i-1}f(x)\big)^{[p]^{r-i}}.$$

For $y \in L$, we have

$$yg(x) = y\tilde{f}\big(x^{p^r} - \varphi_L(x)\big) = \tilde{f}\big(y\big(x^{p^r} - \varphi_L(x)\big)\big)$$
$$= \tilde{f}\big(\big(x^{p^r} - \varphi_L(x)\big)y\big) = \big(x^{p^r} - \varphi_L(x)\big)f(y) = 0,$$

whence g is a map from B_L into M^L. Now we verify that the map $e \to \zeta(e) = e + f(\phi(e))$ is an equivalence isomorphism of $(E, B_E, \varphi_E, \phi)$ onto $(E, B_E, \varphi_{E_g}, \phi)$. Lemma 4.1 ensures that ζ is a similarity isomorphism,

$$\varphi_{E_g}\big(\zeta(e)\big) = \varphi_{E_g}(e) + \sum_{i=0}^{r}\big((\operatorname{ad} e)^{p^i-1}f\big(\phi(e)\big)\big)^{[p]^{r-i}}$$

$$= \varphi_E(e) - g\big(\phi(e)\big) + \sum_{i=0}^{r}\big((\operatorname{ad} e)^{p^i-1}f\big(\phi(e)\big)\big)^{[p]^{r-i}}$$

$$= \varphi_E(e) + f\big(\phi\big(\varphi_E(e)\big)\big) = \zeta\big(\varphi_E(e)\big).$$

Lemma 4.4 ensures that ζ is an equivalence isomorphism. □

For $f \in Z^1(L, M)$, define $\hat{f}(x) := \sum_{i=0}^{r}(x^{p^i-1}f(x))^{[p]^{r-i}} - f(\varphi_L(x))$ for $x \in B_L$. Then we have

Proposition 5.4. *Let M be an abelian restricted Lie algebra on which the generalized restricted Lie algebra L operates. Then the map $f \to \hat{f}$ of $Z^1(L, M)$ into $\operatorname{Map}(B_L, M^L)$, the homomorphism $g \to g^*(s_0)$ of $\operatorname{Map}(B_L, M^L)$ into $\operatorname{ext}_{\varphi_L}(M, L)$, and the canonical homomorphism of $\operatorname{ext}_{\varphi_L}(M, L)$ onto $\operatorname{ext}_s(M, L)$ constitute an exact sequence*

$$Z^1(L, M) \to \operatorname{Map}\big(B_L, M^L\big) \to \operatorname{ext}_{\varphi_L}(M, L) \to \operatorname{ext}_s(M, L) \to 0.$$

Proof. Part (2) of Lemma 5.3 implies that $\ker(g \to g^*(s_0)) = \ker(g \to g^*)$, and (4) of Lemma 5.3 implies that $\ker(g \to g^*) = \operatorname{im}(f \to \hat{f})$, i.e., $\ker(g \to g^*(s_0)) = \operatorname{im}(f \to \hat{f})$. Part (1) of Lemma 5.3 indicates that $\operatorname{im}(g \to g^*(s_0))$ is the kernel of the canonical homomorphism of $\operatorname{ext}_{\varphi_L}(M, L)$ onto $\operatorname{ext}_s(M, L)$. □

Suppose that M is strongly abelian and an associative 2-cocycle g for $u_{\varphi_L}(L)^+$ in M is given. We can construct a corresponding generalized restricted extension of M by L as follows. The underlying vector space of the extended Lie algebra, E_g, is the direct sum of L and M. The commutation is defined by the formula

$$\big[(x_1,m_1),(x_2,m_2)\big]=\big([x_1,x_2],x_1m_2-x_2m_1+g(x_1,x_2)-g(x_2,x_1)\big).$$

The homomorphism ϕ_g of E_g onto L is defined by $\phi_g((x,m))=x$. Define $\varphi_g(x,0):=(\varphi_L(x),g(x^{p^r-1},x))$ for $x\in B_L$ and $\varphi_g(0,m):=(0,0)$ for $m\in M$. Then

$$\begin{aligned}\operatorname{ad}\varphi_g(x,0)\cdot(x_1,m)&=\big[\big(\varphi_L(x),g\big(x^{p^r-1},x\big)\big),(x_1,m)\big]\\&=\big(\big[\varphi_L(x),x_1\big],\varphi_L(x)m-x_1g\big(x^{p^r-1},x\big)\\&\quad+g\big(\varphi_L(x),x_1\big)-g\big(x_1,\varphi_L(x)\big)\big).\end{aligned}$$

According to formulas (12) and (13), we have

$$\begin{aligned}\big(\operatorname{ad}(x,0)\big)^{p^r}\cdot(x_1,m)&=\big(\big[\varphi_L(x),x_1\big],\varphi_L(x)m-x_1g\big(x^{p^r-1},x\big)\\&\quad+g\big(x^{p^r},x_1\big)-g\big(x_1,x^{p^r}\big)\big).\end{aligned}$$

The fact that g is an associative 2-cocycle for $u_{\varphi_L}(L)^+$ in M implies that $g(\varphi_L(x),x_1)=g(x^{p^r},x_1)$ and $g(x_1,\varphi_L(x))=g(x_1,x^{p^r})$, hence $\operatorname{ad}\varphi_g(x,0)=\operatorname{ad}(x,0)^{p^r}$. This means that $(E_g,B_E,\varphi_g,\phi_g)$ is a generalized restricted extension of M by L.

Lemma 5.5. *Let h be any 1-cochain for $u_{\varphi_L}(L)^+$ in M. For $(x,m)\in E_{g+\partial h}$, set $\alpha(x,m):=(x,m+h(x))\in E_g$. Then α is an equivalence isomorphism of $(E_{g+\partial h},B_E,\varphi_{g+\partial h},\phi_{g+\partial h})$ onto $(E_g,B_E,\varphi_g,\phi_g)$.*

Proof. Let $(x_1,m_1),(x_2,m_2)\in E_{g+\partial h}$. Then

$$\begin{aligned}\big[(x_1,m_1),(x_2,m_2)\big]&=\big([x_1,x_2],x_1m_2-x_2m_1+g(x_1,x_2)-g(x_2,x_1)\\&\quad+\partial h(x_1,x_2)-\partial h(x_2,x_1)\big)\\&=\big([x_1,x_2],x_1m_2-x_2m_1+g(x_1,x_2)-g(x_2,x_1)\\&\quad+x_1h(x_2)-x_2h(x_1)-h\big([x_1,x_2]\big)\big),\end{aligned}$$

whence

$$\begin{aligned}\alpha\big(\big[(x_1,m_1),(x_2,m_2)\big]\big)&=\big([x_1,x_2],x_1m_2-x_2m_1+g(x_1,x_2)-g(x_2,x_1)\\&\quad+x_1h(x_2)-x_2h(x_1)\big).\end{aligned}$$

While for $(x_1,m_1+h(x_1)),(x_2,m_2+h(x_2))\in E_g$, we have

$$\begin{aligned}\big[\big(x_1,m_1+h(x_1)\big),\big(x_2,m_2+h(x_2)\big)\big]&=\big([x_1,x_2],x_1\big(m_2+h(x_2)\big)-x_2\big(m_1+h(x_1)\big)\\&\quad+g(x_1,x_2)-g(x_2,x_1)\big),\end{aligned}$$

i.e., α is Lie algebra homomorphism of $E_{g+\partial h}$ to E_g. It is easy to verify that α is an isomorphism and $\phi_{g+\partial h}\alpha = \phi_g$ and $\alpha(0,m) = (0,m)$. Hence α is a similarity isomorphism of $(E_{g+\partial h}, B_E, \varphi_{g+\partial h}, \phi_{g+\partial h})$ onto $(E_g, B_E, \varphi_g, \phi_g)$.

For $x \in B_L$, we have

$$\begin{aligned}\alpha\big(\varphi_{g+\partial h}(x,0)\big) &= \alpha\big(\varphi_L(x), g\big(x^{p^r-1}, x\big) + \partial h\big(x^{p^r-1}, x\big)\big)\\ &= \big(\varphi_L(x), g\big(x^{p^r-1}, x\big) + x^{p^r-1}h(x) - h\big(x^{p^r}\big) + h\big(\varphi_L(x)\big)\big)\\ &= \big(\varphi_L(x), g\big(x^{p^r-1}, x\big) + x^{p^r-1}h(x)\big),\end{aligned}$$

while

$$\varphi_g\big(\alpha(x,0)\big) = \varphi_g\big(x, h(x)\big) = \big(\varphi_L(x), g\big(x^{p^r-1}, x\big) + x^{p^r-1}h(x)\big)$$

whence $\alpha(\varphi_{g+\partial h}(x,0)) = \varphi_g(\alpha(x,0))$. Lemma 4.4 ensures that α is an equivalence isomorphism. □

Lemma 5.5 ensures that the correspondence $g \to (E_g, B_E, \varphi_g, \phi_g)$ induces a map $H^2_{\varphi_L}(L,M)$ into $\mathrm{ext}_{\varphi_L}(M,L)$. We denote it by ρ.

Lemma 5.6. *The map ρ is a F-linear homomorphism of $H^2_{\varphi_L}(L,M)$ into $\mathrm{ext}_{\varphi_L}(M,L)$.*

Proof. According to the discussion before Proposition 4.6, the difference of the equivalence class of $(E_{g_1}, B_E, \varphi_{g_1}, \phi_{g_1})$ and the equivalence class of $(E_{g_2}, B_E, \varphi_{g_2}, \phi_{g_2})$ has a representation $(D/J, B_{D/J}, \varphi_{D/J}, \bar{\phi})$, where $D = \{((x,m),(x,m')) \mid x \in L, m, m' \in M\}$, $J = \{((0,m),(0,m)) \mid m \in M\}$, $B_D = \{((x_i,0),(x_i,0)) \mid x_i \in B_L,\ i = 1,\dots,n\} \cup \{((m_i,0),(0,0)) \mid i = 1,\dots,l\} \cup \{((0,0),(0,m_j)) \mid j = 1,\dots,l\}$, $\bar{\phi}$ is induced by $\phi((x,m),\,(x,m')) = x$, and $\varphi_{D/J}$ is induced by

$$\begin{aligned}\varphi_D\big((x_i,m),\big(x_i,m'\big)\big) = \big(&\big(\varphi_L(x_i), x_i^{p^r-1}m + g_1\big(x_i^{p^r-1}, x_i\big)\big),\\ &\big(\varphi_L(x_i), x_i^{p^r-1}m' + g_2\big(x_i^{p^r-1}, x_i\big)\big)\big).\end{aligned}$$

Define $\gamma : D \to E_{g_1-g_2}$, via $\gamma((x,m),\,(x,m')) := (x, m-m')$. For $x, y \in L$ and m_1, m_1', $m_2, m_2' \in M$, we have

$$\begin{aligned}&\gamma\big(\big[\big((x,m_1),\big(x,m_1'\big)\big),\big((y,m_2),\big(y,m_2'\big)\big)\big]\big)\\ &\quad= \gamma\big(\big[(x,m_1),(y,m_2)\big],\big[\big(x,m_1'\big),\big(y,m_2'\big)\big]\big)\\ &\quad= \big(\gamma\big([x,y], xm_2 - ym_1 + g_1(x,y) - g_1(x,y)\big),\\ &\qquad\big([x,y], xm_2' - ym_1' + g_2(x,y) - g_2(y,x)\big)\big)\\ &\quad= \big([x,y], x\big(m_2 - m_2'\big) - y\big(m_1 - m_1'\big)\\ &\qquad + g_1(x,y) - g_2(x,y) - g_1(y,x) + g_2(y,x)\big)\\ &\quad= \big[\gamma\big((x,m_1),\big(x,m_1'\big)\big), \gamma\big((y,m_2),\big(y,m_2'\big)\big)\big].\end{aligned}$$

Hence γ is a Lie algebra homomorphism of D onto $E_{g_1-g_2}$. It is clear that $\ker(\gamma)=J$, thus γ induces an isomorphism $\bar{\gamma}: D/J \to E_{g_1-g_2}$. For $x_i \in B_L$ and $m, m' \in M$, we have

$$\begin{aligned}
&\gamma\big(\varphi_D\big((x_i,m),\big(x_i,m'\big)\big)\big)\\
&\quad=\gamma\big(\big(\varphi_L(x_i),x_i^{p^r-1}m+g_1\big(x_i^{p^r-1},x_i\big)\big),\big(\varphi_L(x_i),x_i^{p^r-1}m'+g_2\big(x_i^{p^r-1},x_i\big)\big)\big)\\
&\quad=\big(\varphi_L(x_i),x_i^{p^r-1}m-x_i^{p^r-1}m'+g_1\big(x_i^{p^r-1},x_i\big)-g_2\big(x_i^{p^r-1},x_i\big)\big)\\
&\quad=\varphi_{g_1-g_2}\gamma\big((x_i,m),\big(x_i,m'\big)\big).
\end{aligned}$$

This means that $\bar{\gamma}$ is an equivalence isomorphism of $(D/J, B_{D/J}, \varphi_{D/J}, \bar{\phi})$ onto $(E_{g_1-g_2}, B_E, \varphi_{g_1-g_2}, \phi_{g_1-g_2})$.

According to the discussion before Proposition 4.6, the scalar multiple by an element $\alpha \in F$ of the equivalence class of $(E_g, B_E, \varphi_g, \phi_g)$ has a representation $(D/J, B_{D/J}, \varphi_{D/J}, \bar{\phi})$ where $D=\{((x,m),m') \mid x\in L,\ m,m'\in M\}$, $J=\{((0,-\alpha m),m) \mid m\in M\}$, $B_D=\{((x_i,0),0) \mid x_i\in B_L,\ j=1,\dots,l\}\cup\{((0,m_i),0) \mid i=1,\dots,l\}\cup\{((0,0),m_j) \mid j=1,\dots,l\}$, $\varphi_{D/J}$ is induced by

$$\varphi_D\big((x_i,m),m'\big)=\big(\big(\varphi_L(x_i),x_i^{p^r-1}m+g\big(x_i^{p^r-1},x_i\big)\big),x_i^{p^r-1}m'\big),$$

and $\bar{\phi}$ is induced by $\phi((x,m),m')=x$. Define $\gamma: D\to E_{\alpha g}$ by $\gamma((x,m_1),m_2)=(x,\alpha m_1+m_2)$. For $((x,m_1),m_1'), ((y,m_2), m_2')\in D$, we have

$$\begin{aligned}
&\gamma\big(\big[\big((x,m_1),m_1'\big),\big((y,m_2),m_2'\big)\big]\big)\\
&\quad=\gamma\big([x,y],xm_2-ym_1+g(x,y)\big),\big(xm_2'-ym_1'\big)\\
&\quad=\big([x,y],\alpha xm_2-\alpha ym_1+\alpha g(x,y)+xm_2'-ym_1'\big),
\end{aligned}$$

while

$$\begin{aligned}
&\big[\gamma\big((x,m_1),m_1'\big),\gamma\big((y,m_2),m_2'\big)\big]\\
&\quad=\big[\big(x,\alpha m_1+m_1'\big),\big(y,\alpha m_2+m_2'\big)\big]\\
&\quad=\big([x,y],\alpha xm_2+xm_2'-\alpha ym_1-ym_1'+\alpha g(x,y)\big).
\end{aligned}$$

Hence γ is a Lie algebra homomorphism of D onto $E_{\alpha g}$. It is clear that $\ker(\gamma)=J$; thus, γ induces an isomorphism $\bar{\gamma}: D/J\to E_{\alpha g}$. For $x_i\in B_L$ and $m,m'\in M$, we have

$$\begin{aligned}
\gamma\varphi_D\big((x_i,m),m'\big)&=\gamma\big(\big(\varphi_L(x_i),x_i^{p^r-1}m+g\big(x_i^{p^r-1},x_i\big)\big),x_i^{p^r-1}m'\big)\\
&=\big(\varphi_L(x_i),\alpha g\big(x_i^{p^r-1},x_i\big)+\alpha x_i^{p^r-1}m+x_i^{p^r-1}m'\big),
\end{aligned}$$

while

$$\begin{aligned}\varphi_{\alpha g}\gamma\big((x_i, m), m'\big) &= \varphi_{\alpha g}\big(x_i, \alpha m + m'\big)\\ &= \big(\varphi_L(x_i), x_i^{p^r-1}\big(\alpha m + m'\big) + \alpha g\big(x_i^{p^r-1}, x_i\big)\big).\end{aligned}$$

Hence $\gamma\psi((x_i, m), m') = \varphi_{\alpha g}\gamma((x_i, m), m')$. This means that $\bar{\gamma}$ is an equivalence isomorphism of $(D/J, B_{D/J}, \varphi_{D/J}, \bar{\phi})$ onto $(E_{\alpha g}, B_E, \varphi_{\alpha g}, \phi_{\alpha g})$. Hence ρ is an F-linear homomorphism. □

Lemma 5.7. *ρ is an isomorphism.*

Proof. Let $(E, B_E, \varphi_E, \phi)$ be a generalized restricted extension of M by L; $\tilde{\phi}$ the homomorphism of $u_{\varphi_E}(E)$ onto $u_{\varphi_L}(L)$, which is extended by ϕ; and $\{m_1, \ldots, m_l\}$ a basis of M. Define a map γ of $u_{\varphi_E}(E)M$ onto M by $\gamma(\sum_i u_i m_i) = \sum_i \tilde{\phi}(u_i)m_i$. Let ψ be linear map of L into E which is inverse to ϕ. Extend ψ to a linear map $\tilde{\psi}$ of $u_{\varphi_L}(L)$ into $u_{\varphi_E}(E)$ such that $\tilde{\phi}\tilde{\psi}$ is the identity map on $u_{\varphi_L}(L)$. Define $g(x, y) := \gamma(\tilde{\psi}(x)\tilde{\psi}(y) - \tilde{\psi}(xy))$ and a map $\alpha : E_g \to E$ by setting $\alpha(x, m) := \psi(x) + m$. As done in [3, p. 573], we can verify the following facts:

(1) γ is a $u_{\varphi_E}(E)$-homomorphism.
(2) $\gamma(u_{\varphi_E}(E)Mu_{\varphi_E}(E)^+) = 0$.
(3) g is an associative 2-cocycle for $u_{\varphi_L}(L)^+$ in M.
(4) α is a similarity isomorphism.

Now we prove that α is generalized restricted equivalence isomorphism. For $x \in B_L$, $\phi\psi(x) = x$ implies that there is $e \in B_E$ and $m' \in M$ such that $\psi(x) = e + m'$. According to formula (10), $\varphi_E(\psi(x))$ is well defined and we have

$$\varphi_E\big(\alpha(x, m)\big) = \varphi_E\big(\psi(x)\big) + x^{p^r-1}m,$$

while

$$\begin{aligned}\alpha\varphi_g(x, m) &= \alpha\big(\varphi_L(x), g\big(x^{p^r-1}, x\big) + x^{p^r-1}m\big)\\ &= \psi\big(\varphi_L(x)\big) + g\big(x^{p^r-1}, x\big) + x^{p^r-1}m.\end{aligned}$$

Since $(\tilde{\psi}(x^{p^r-1}) - \psi(x)^{p^r-1})\psi(x) \in u_{\varphi_E}(E)Mu_{\varphi_E}(E)^+$ and $\varphi_E(\psi(x)) - \psi(\varphi_L(x)) \in M$, we have

$$\begin{aligned}g\big(x^{p^r-1}, x\big) &= \gamma\big(\tilde{\psi}\big(x^{p^r-1}\big)\psi(x) - \tilde{\psi}\big(x^{p^r}\big)\big)\\ &= \gamma\big(\big(\psi\big(x^{p^r-1}\big) - \psi(x)^{p^r-1}\big)\psi(x)\big) + \gamma\big(\psi(x)^{p^r} - \tilde{\psi}\big(x^{p^r}\big)\big)\\ &= \gamma\big(\varphi_E\big(\psi(x)\big) - \psi\big(\varphi_L(x)\big)\big) = \varphi_E\big(\psi(x)\big) - \psi\big(\varphi_L(x)\big),\end{aligned}$$

whence $\varphi_E\alpha(x, m) = \alpha\varphi_g(x, m)$. According to Lemma 4.4, α is an equivalence isomorphism of $(E, B_E, \varphi_E, \phi)$ to $(E_g, B_E, \varphi_g, \phi_g)$, i.e., ρ is a surjective map. Similarly as done in [3, p. 573], we can show that ρ is an injective map. □

Hence we have

Theorem 5.8. *Let M be a strongly abelian restricted Lie algebra on which the generalized restricted Lie algebra L operates. Then there is a canonical isomorphism of* $\mathrm{ext}_{\varphi_L}(M, L)$ *onto* $H^2_{\varphi_L}(L, M)$ *such that the following diagram of canonical F-linear maps is commutative*:

$$\begin{array}{ccc} \mathrm{ext}_{\varphi_L}(M, L) & \longrightarrow & H^2_{\varphi_L}(L, M) \\ \downarrow & & \downarrow \\ \mathrm{ext}_s(M, L) & \longrightarrow & H^2_s(L, M). \end{array}$$

In the case that M is strongly abelian, we can replace $Z^1(L, M)$ by $H^1(L, M)$ in the exact sequence of Proposition 5.4. Taking account of Propositions 5.2 and 3.1, we obtain that the sequence

$$\begin{aligned} 0 &\to H^1_{\varphi_L}(L, M) \to H^1(L, M) \to \mathrm{Map}\big(B_L, M^L\big) \\ &\to H^2_{\varphi_L}(L, M) \to H^2(L, M) \to \mathrm{Map}\big(B_L, H^1(L, M)\big) \end{aligned}$$

is exact.

Since $\mathrm{Map}(B_L, M^L) \cong \hom_F(L, M^L)$, $\mathrm{Map}(B_L, H^1(L, M)) \cong \hom_F(L, H^1(L, M))$, we have

Theorem 5.9. *For every generalized restricted L-module M, the following sequence is exact*:

$$\begin{aligned} 0 &\to H^1_{\varphi_L}(L, M) \to H^1(L, M) \to \hom_F\big(L, M^L\big) \\ &\to H^2_{\varphi_L}(L, M) \to H^2(L, M) \to \hom_F\big(L, H^1(L, M)\big). \end{aligned} \tag{14}$$

In [2, p. 2876], Professor Jörg Feldvoss has proved

Proposition 5.10. *Let L be a restricted Lie algebra and M a restricted L-module. Then*

$$\widehat{H}^n_p(L, M) \cong \widehat{H}^{n-r}_p\big(L, M^{(r)}\big) \quad \text{and} \quad \widehat{H}^0_p(L, M) \cong M^L/(s \cdot M),$$

where s is a left integral of $u_p(L)$.

In [4], the author has shown that the result of Proposition 5.10 holds if L is a generalized restricted Lie algebra and M is a generalized restricted L-module.

Let $L = X(n, \mathbf{m})$ $(X = W, S, H, K)$ and M be an irreducible L-module. Then L is a generalized restricted Lie algebra. If M is not a generalized restricted L-module, then according to [1, Theorem 1, p. 128], we can easily obtain that $H^1(L, M) = 0$. If M

is a trivial irreducible L-module, then $[L,L]=L$ implies $H^1(L,M)=0$. If M is a generalized restricted L-module, then sequence (14) implies $H^1(L,M)\cong H^1_{\varphi_L}(L,M)$. Proposition 5.10 provides that

$$H^1_{\varphi_L}(L,M)\cong \widehat{H}^0_p\big(L,M^{(1)}\big)\cong \big(u_{\varphi_L}(L)_+\otimes M\big)^L/s_L\cdot\big((u_{\varphi_L}(L)_+\otimes M)\big).$$

So we have $H^1(L,M)\cong (u_{\varphi_L}(L)_+\otimes M)^L/s_L\cdot((u_{\varphi_L}(L)_+\otimes M))$.

If $M=F$, then sequence (14) and $H^1(L,F)=0$ ensure that the sequence

$$0\to \hom_F(L,F)\to H^2_{\varphi_L}(L,F)\to H^2(L,F)\to 0 \tag{15}$$

is exact. Proposition 5.10 provides that

$$H^2_{\varphi_L}(L,F)\cong \hat{H}^0_{\varphi_L}\big(L,F^{(2)}\big)\cong \big(u_{\varphi_L}(L)_+\otimes u_{\varphi_L}(L)_+\big)^L/s_L\cdot\big(\big(u_{\varphi_L}(L)_+\otimes u_{\varphi_L}(L)_+\big)\big).$$

So the sequence (15) gives us a way to study the center extensions of the graded Lie algebras of Cartan type.

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