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# Lie symmetry analysis, conservation laws and exact solutions of the seventh-order time fractional Sawada–Kotera–Ito equation

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ABSTRACT

In this paper Lie symmetry analysis of the seventh-order time fractional Sawada–Kotera–Ito (FSKI) equation with Riemann–Liouville derivative is performed. Using the Lie point symmetries of FSKI equation, it is shown that it can be transformed into a nonlinear ordinary differential equation of fractional order with a new dependent variable. In the reduced equation the derivative is in Erdelyi–Kober sense. Furthermore, adapting the Ibragimov's nonlocal conservation method to time fractional partial differential equations, we obtain conservation laws of the underlying equation. In addition, we construct some exact travelling wave solutions for the FSKI equation using the sub-equation method.



## 1. Introduction

Fractional partial differential equations (FPDEs) appear in various research and engineering applications such as physics, biology, rheology, viscoelasticity, control theory, signal processing, systems identification and electrochemistry [1–8]. Recently, they have attracted considerable interest and there has been a significant theoretical development in this area.

It is well-known that in order to describe nonlinear physical phenomena finding exact solutions of nonlinear fractional partial differential equations (NLFPDEs) is the key instrument. A physical phenomenon may depend not only on the time instant but also on the time history, which can be successfully modelled using the theory of derivatives and integrals of fractional order [1–8].

Very recently, several powerful methods have been developed in the literature for finding exact solutions of NLFPDEs. Some of the most important methods found in the literature include the exp function method, the fractional sub-equation method, the first integral method, the $(G'/G)$-expansion method, and the Lie symmetry method [9–28].

Lie symmetry analysis is one of the most general and effective methods for obtaining exact solutions of nonlinear partial differential equations (PDEs). In the last few decades, Lie's method has been described in a number of excellent textbooks and has been applied to a number of physical and engineering models. See for example [29–33] and references therein.

However, application of the Lie symmetry analysis to FPDEs is quite new. We observe only a few studies in the literature. For instance, the authors of [15] considered the time fractional linear wave-diffusion equation and obtained a group of dilations. Using dilation symmetries invariant solutions were constructed. In [16], an attempt has been made to extend the Lie symmetry analysis to FPDEs (see also [17]). In addition, in [18], this method has been applied to time fractional generalized Burgers and Korteweg–de Vries equations. In [19], group-analysis of time fractional Harry–Dym equation with Riemann–Liouville derivative was performed and symmetry reductions and group-invariant solutions were obtained. The authors of [20] investigated the invariance properties of fractional Sharma–Tasso–Olver equation using the Lie group analysis method. In [21] an algorithm for the symbolic computation of Lie point symmetries for fractional differential equations (FDEs) was presented.

On the other hand, as stated in [30] conservation laws are very important tools in the study of differential equations from mathematical as well as physical point of view. If the underlying system has conservation laws then its integrability is quite possible [29,32]. Noether theorem [34] provides us with a systematic method for finding conservation laws of PDEs provided a Noether symmetry associated with a Lagrangian is known for Euler–Lagrange equations. However, there exist some approaches in the

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literature for obtaining the conservation laws of the PDEs, which do not have a Lagrangian [35–39].

The family of seventh-order Korteweg–de Vries (KdV) equations are given by [40]

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0, \quad (1)$$

where $a, b, c, d, e, f$ and $g$ are non-zero constants. In fact, the seventh-order KdV was introduced by Pomeau et al. [41] and its structural stability was discussed under a singular perturbation. In this paper, we study the seventh-order time fractional Sawada–Kotera–Ito (FSKI) equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0, \quad (2)$$

where $\alpha (0 < \alpha \leqslant 1)$ is a parameter describing the order of the fractional time derivative. When $\alpha = 1$, Eq. (2) becomes

$$\frac{\partial u}{\partial t} + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0, \quad (3)$$

which has been widely studied in the literature. For instance, the $N$-soliton solutions, bilinear form and Lax pair for Eq. (3) have been investigated in [42]. The bilinear and tanh–coth methods have been applied to Eq. (3) and soliton solutions were obtained in [43]. A decomposition was implemented for approximating its solutions [44]. In [45], using the Bell polynomial approach, Lax pair and infinite conservation laws were deduced for Eq. (3).

Our aim in the present work is to study symmetry reductions and conservation laws of the time FSKI equation (2) with the help of Lie symmetry analysis and Ibragimov's nonlocal conservation method [38], respectively. In addition, we intend to obtain exact travelling wave solutions of the FSKI equation by the sub-equation method.

The paper is organized as follows. In Section 2, firstly some basic properties of Riemann–Liouville derivative are recalled. Then, Lie group method for FPDEs are presented. In Section 3, we apply the Lie group analysis method to the time FSKI equation (2) and obtain symmetry reductions. Then in Section 4, conservation laws of the time FSKI equation(2) are derived using the Ibragimov's nonlocal conservation theorem. In Section 5, we construct exact travelling wave solutions of the time FSKI equation (2) via the sub-equation method. Lastly, concluding remarks are given in Section 6.

## 2. Preliminaries

We recall that the Riemann–Liouville derivative [4,5] of order $\alpha$ is defined by the following expression:

$$\partial_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(\tau,x)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n, \ n \epsilon N, \\ \frac{\partial^n f}{\partial t^n}, & \alpha = n \epsilon N. \end{cases} \quad (4)$$

### 2.1. Description of Lie symmetry method for the time FPDEs

In this section, we present some notations and definitions that will be used in the sequel. For details see for example [16–20].

Consider a scalar time FPDE having the form [18,20]

$$E \equiv \frac{\partial^\alpha u}{\partial t^\alpha} - F(x, t, u, u_x, u_{xx}, u_{xxx}, u_{4x}, u_{5x}, u_{6x}, u_{7x}) = 0, \quad (5)$$

where $\alpha (0 < \alpha \leqslant 1)$ is a parameter. Consider a one-parameter Lie group of infinitesimal transformations given by

$$\begin{aligned}
\bar{t} &= t + \epsilon\tau(x,t,u) + O(\epsilon^2),\\
\bar{x} &= x + \epsilon\xi(x,t,u) + O(\epsilon^2),\\
\bar{u} &= u + \epsilon\eta(x,t,u) + O(\epsilon^2),\\
\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_\alpha^0(x,t,u) + O(\epsilon^2),\\
\frac{\partial \bar{u}}{\partial \bar{x}} &= \frac{\partial u}{\partial x} + \epsilon\eta^x(x,t,u) + O(\epsilon^2),\\
\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(x,t,u) + O(\epsilon^2),\\
\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \epsilon\eta^{xxx}(x,t,u) + O(\epsilon^2),\\
\frac{\partial^4 \bar{u}}{\partial \bar{x}^4} &= \frac{\partial^4 u}{\partial x^4} + \epsilon\eta^{xxxx}(x,t,u) + O(\epsilon^2),\\
\frac{\partial^5 \bar{u}}{\partial \bar{x}^5} &= \frac{\partial^5 u}{\partial x^5} + \epsilon\eta^{xxxxx}(x,t,u) + O(\epsilon^2),\\
\frac{\partial^6 \bar{u}}{\partial \bar{x}^6} &= \frac{\partial^6 u}{\partial x^6} + \epsilon\eta^{xxxxxx}(x,t,u) + O(\epsilon^2),\\
\frac{\partial^7 \bar{u}}{\partial \bar{x}^7} &= \frac{\partial^7 u}{\partial x^7} + \epsilon\eta^{xxxxxxx}(x,t,u) + O(\epsilon^2),
\end{aligned} \quad (6)$$

where

$$\begin{aligned}
\eta^x &= D_x(\eta) - u_xD_x(\xi) - u_tD_x(\tau),\\
\eta^{xx} &= D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\xi),\\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt}D_x(\tau) - u_{xxxx}D_x(\xi),\\
\eta^{xxxxx} &= D_x(\eta^{xxxx}) - u_{xxxxt}D_x(\tau) - u_{xxxxx}D_x(\xi),\\
\eta^{xxxxxx} &= D_x(\eta^{xxxxx}) - u_{xxxxxt}D_x(\tau) - u_{xxxxxx}D_x(\xi),\\
\eta^{xxxxxxx} &= D_x(\eta^{xxxxxx}) - u_{xxxxxxt}D_x(\tau) - u_{xxxxxxx}D_x(\xi)
\end{aligned}$$

and $D_x$ denotes the total differentiation operator defined by

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots.$$

Then the associated Lie algebra of symmetries is the set of vector fields of the form

$$X = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \quad (7)$$

The vector field (7) is a Lie point symmetry of (5) provided

$$\mathrm{pr}^{(7)}X(E)|_{E=0} = 0. \quad (8)$$

Also, the invariance condition yields [20]

$$\tau(x,t,u)|_{t=0} = 0 \quad (9)$$

and the $\alpha$th extended infinitesimal related to Riemann–Liouville fractional time derivative with (9) is given by [16,17]

$$\eta_\alpha^0 = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)\,D_t^{\alpha-n}(u_x) + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u), \quad (10)$$

where

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^{r}\frac{\partial^{m}}{\partial t^{m}}\times\left[u^{k-r}\right]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^{k}}. \tag{11}$$

It should be noted that we have $\mu=0$ when the infinitesimal $\eta$ is linear in $u$, because of the existence of the derivatives $\frac{\partial^k\eta}{\partial u^k}, k\geqslant 2$ in the above expression.

**Definition 1.** The function $u=\theta(x,t)$ is an invariant solution of Eq. (5) corresponding to the infinitesimal generator (7) if and only if

(1) $u=\theta(x,t)$ satisfies Eq. (5).
(2) $u=\theta(x,t)$ is an invariant surface of (6), namely, it fulfils the invariant surface condition

$$\tau(x,t,\theta)\theta_t+\xi(x,t,\theta)\theta_x=\eta(x,t,\theta).$$

## 3. Lie symmetries and reductions for the time FSKI equation

Let us assume that the time FSKI equation (2) is invariant under the one-parameter group of transformations (6) and so we have

$$\frac{\partial^\alpha\bar{u}}{\partial\bar{t}^\alpha}+252\bar{u}^3\bar{u}_x+63\bar{u}_x^3+378\bar{u}\bar{u}_x\bar{u}_{xx}+126\bar{u}^2\bar{u}_{xxx}+63\bar{u}_{2x}\bar{u}_{3x}+42\bar{u}_x\bar{u}_{4x}+21\bar{u}\bar{u}_{5x}+\bar{u}_{7x}=0 \tag{12}$$

provided $u=u(x,t)$ satisfies (2). Using the point transformations (6) in (12) we obtain the invariant equation

$$\eta_\alpha^0+(756u^2u_x+378u_xu_{xx}+252uu_{xxx}+21u_{5x})\eta+(252u^3+189u_x^2+378uu_{xx}+42u_{4x})\eta^x+(378uu_x+63u_{3x})\eta^{xx}+(126u^2+63u_{2x})\eta^{xxx}+42u_x\eta^{xxxx}+21u\eta^{xxxxx}+\eta^{xxxxxxx}=0. \tag{13}$$

Substituting the values of $\eta_\alpha^0,\eta^x,\eta^{xx},\eta^{xxx},\eta^{xxxx},\eta^{xxxxx}$ and $\eta^{xxxxxxx}$ from (6) and (10) into (13) and equating various powers of the derivatives of $u$ to zero, we obtain an overdetermined system of linear equations. These are (with the aid of [21])

$$\tau_x=\tau_u=\xi_t=\xi_u=\eta_{uu}=0,$$
$$\binom{\alpha}{n}\partial_t^n(\eta_u)-\binom{\alpha}{n+1}D_t^{n+1}(\tau)=0,\quad n=1,2,3,\ldots,$$
$$7\xi'(x)-\alpha\tau'(t)=0,$$
$$\partial_t^\alpha(\eta)-u\partial_t^\alpha(\eta_u)+252u^3\eta_x+126u^2\eta_{xxx}+21u\eta_{xxxxx}+\eta_{xxxxxxx}=0.$$

Now solving the above equations, we obtain

$$\xi=\alpha xc_1+c_2,\quad \tau=7tc_1,\quad \eta=-2\alpha uc_1,$$

where $c_1$ and $c_2$ are arbitrary constants. Hence the infinitesimal symmetry group of the time FSKI equation (2) is spanned by the two vector fields

$$X_1=\frac{\partial}{\partial x},\quad X_2=7t\frac{\partial}{\partial t}+\alpha x\frac{\partial}{\partial x}-2\alpha u\frac{\partial}{\partial u}. \tag{14}$$

In what follows, we perform similarity reductions, present the reduced nonlinear fractional ordinary differential equations (ODEs) and classify the corresponding group-invariant solutions of the time FSKI equation (2) for the two Lie point symmetries (14).

**Case 1.** $X_1=\partial/\partial x$. Integration of the invariant surface condition

$$\frac{dt}{0}=\frac{dx}{1}=\frac{du}{0}$$

gives the similarity variables $t$ and $u$. Thus, we have the ansatz $u=f(t)$. Inserting this value of $u$ into Eq. (2), we obtain the reduced fractional ODE

$$\partial_t^\alpha f(t)=0.$$

Solving the above equation yields the group-invariant solution

$$u=a_1t^{\alpha-1},$$

where $a_1$ is an arbitrary constant of integration.

**Case 2.** $X_2=7t\,\partial/\partial t+\alpha x\,\partial/\partial x-2\alpha u\,\partial/\partial u$.

The similarity variables corresponding to the infinitesimal generator $X_2$ can be obtained by solving the associated characteristic equations given by

$$\frac{dt}{7t}=\frac{dx}{\alpha x}=\frac{du}{-2\alpha u}.$$

Solving the above equations, we obtain the two invariants

$$I_1=ut^{2\alpha/7},\quad I_2=xt^{-\alpha/7}. \tag{15}$$

Thus, the symmetry $X_2$ gives the group-invariant solution

$$u=t^{-2\alpha/7}g(\xi),\quad \xi=xt^{-\alpha/7}, \tag{16}$$

where $g$ is an arbitrary function of $\xi$. Using these invariants, Eq. (2) tranforms to a special nonlinear ODE of fractional order. Thus, we have the following theorem corresponding to this case.

**Theorem 1.** *The similarity transformation (16) reduces (2) to the following nonlinear ODE of fractional order:*

$$\left(P_{\frac{7}{\alpha}}^{1-\frac{9\alpha}{7},\alpha}g\right)(\xi)+252g^3g_\xi+63g_\xi^3+378gg_\xi g_{\xi\xi}+126g^2g_{\xi\xi\xi}+63g_{2\xi}g_{3\xi}+42g_\xi g_{4\xi}+21gg_{5\xi}+g_{7\xi}=0 \tag{17}$$

*with the Erdelyi–Kober fractional differential operator [4]*

$$\left(P_\beta^{\tau,\alpha}g\right):=\prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\beta}\xi\frac{d}{d\xi}\right)\left(K_\beta^{\tau+\alpha,n-\alpha}g\right)(\xi), \tag{18}$$

$$n=\begin{cases}[\alpha]+1, & \alpha\notin N,\\ \alpha, & \alpha\in N,\end{cases} \tag{19}$$

*where*

$$\left(K_\beta^{\tau,\alpha}g\right)(\xi):=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^\infty(u-1)^{\alpha-1}u^{-(\tau+\alpha)}g(\xi u^{\frac{1}{\beta}})du, & \alpha>0,\\ g(\xi), & \alpha=0,\end{cases} \tag{20}$$

*is the Erdelyi–Kober fractional integral operator.*

(See also [18,20]). Let $n-1<\alpha<n,\ n=1,2,3,\ldots$. Based on the Riemann–Liouville fractional derivative for the similarity transformation (16), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}s^{\frac{-2\alpha}{7}}g(xs^{\frac{-\alpha}{7}})ds\right]. \tag{21}$$

Letting $v=t/s$, one can get $ds=-(t/v^2)dv$. Then Eq. (21) can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{9\alpha}{7}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty(v-1)^{n-\alpha-1}v^{-(n+1-\frac{9\alpha}{7})}g(\xi v^{\frac{\alpha}{7}})dv\right]. \tag{22}$$

If one uses the definition of Erdelyi–Kober fractional integral operator (20), then Eq. (22) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{9\alpha}{7}}\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right]. \tag{23}$$

We now intend to simplify the right hand side of (23). Taking into consideration $\xi = xt^{\frac{-\alpha}{7}}, \phi \in C^1(0,\infty)$, we can obtain

$$t\frac{\partial}{\partial t}\phi(\xi) = tx\left(-\frac{\alpha}{7}\right)t^{-\frac{\alpha}{7}-1}\phi'(\xi) = -\frac{\alpha}{7}\xi\frac{\partial}{\partial \xi}\phi(\xi). \tag{24}$$

Thus, we have

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{9\alpha}{7}}\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right] = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{9\alpha}{7}}\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right)\right] = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{9\alpha}{7}-1}\left(n-\frac{9\alpha}{7}-\frac{\alpha}{7}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right]. \tag{25}$$

Repeating the same procedure $n-1$ times, one can obtain

$$\begin{aligned}\frac{\partial^n}{\partial t^n}&\left[t^{n-\frac{9\alpha}{7}}\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{9\alpha}{7}}\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{9\alpha}{7}-1}\left(n-\frac{9\alpha}{7}-\frac{\alpha}{7}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right]\\ &\;\;\vdots\\ &= t^{-\frac{9\alpha}{7}}\prod_{j=0}^{n-1}\left(1-\frac{9\alpha}{7}+j-\frac{\alpha}{7}\xi\frac{d}{d\xi}\right)\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi). \end{aligned}\tag{26}$$

Using the definition of the Erdélyi–Kober fractional differential operator (18) in the above, we get

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{9\alpha}{7}}\left(K_{\frac{7}{\alpha}}^{1-\frac{2\alpha}{7},n-\alpha}g\right)(\xi)\right] = t^{-\frac{9\alpha}{7}}\left(P_{\frac{7}{\alpha}}^{1-\frac{9\alpha}{7},\alpha}g\right)(\xi). \tag{27}$$

Now substituting (27) into (23), we obtain an expression for the time fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{9\alpha}{7}}\left(P_{\frac{7}{\alpha}}^{1-\frac{9\alpha}{7},\alpha}g\right)(\xi).$$

Thus, the time FSKI equation (2) can be reduced into a fractional order ODE

$$\left(P_{\frac{7}{\alpha}}^{1-\frac{9\alpha}{7},\alpha}g\right)(\xi) + 252g^3g_\xi + 63g_\xi^3 + 378gg_\xi g_{\xi\xi} + 126g^2g_{\xi\xi\xi} + 63g_{2\xi}g_{3\xi} + 42g_\xi g_{4\xi} + 21gg_{5\xi} + g_{7\xi} = 0. \tag{28}$$

This completes the proof of the theorem.

## 4. Conservation laws

We now construct the conservation laws of the FSKI equation (2). However, we first recall some basic definitions including the definitions of derivative and integral operators that we use in our work. The Riemann–Liouville left-sided time-fractional derivative

$${}_0D_t^\alpha u = D_t^n\left({}_0I_t^{n-\alpha}u\right)$$

and the left-sided time-fractional integral of order $n-\alpha$, namely, ${}_0I_t^{n-\alpha}$ defined by

$$\left({}_0I_t^{n-\alpha}u\right)(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(\theta,x)}{(t-\theta)^{1-n+\alpha}}d\theta,$$

will be employed. Here $\Gamma(z)$ denotes the Gamma function, $D_t$ is the operator of differentiation with respect to $t$ and $n=[\alpha]+1$ [27].

A conservation law for Eq. (2) is written as

$$D_t(C^t) + D_x(C^x) = 0,$$

which holds for all solutions $u(x,t)$ of the Eq. (2).

We now use Ibragimov method [38] for constructing the conservation laws of Eq. (2). It can easily be seen that the FSKI equation (2) has the formal Lagrangian

$$L = v(t,x)\left[\frac{\partial^\alpha u}{\partial t^\alpha} + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x}\right],$$

where $v(t,x)$ is a new dependent variable. The Euler–Lagrange operator is [26,27]

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \left(D_t^\alpha\right)^*\frac{\partial}{\partial D_t^\alpha u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}} + D_x^4\frac{\partial}{\partial u_{4x}} - D_x^5\frac{\partial}{\partial u_{5x}} - D_x^7\frac{\partial}{\partial u_{7x}},$$

where $\left(D_t^\alpha\right)^*$ is the adjoint operator of $\left(D_t^\alpha\right)$.

The adjoint equation to Eq. (2) is given by [38]

$$\frac{\delta L}{\delta u} = 0.$$

Also, we have [38]

$$\overline{X} + D_t(\tau)I + D_t(\xi)I = W\frac{\delta}{\delta u} + D_tN^t + D_xN^x,$$

where $I$ is the identity operator, $N^t andN^x$ are the Noether operators, and the prolonged vector field $\overline{X}$ is defined by

$$\overline{X} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta_\alpha^0\frac{\partial}{\partial D_t^\alpha u} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xxxx}\frac{\partial}{\partial u_{xxxx}} + \eta^{xxxxx}\frac{\partial}{\partial u_{xxxxx}} + \eta^{xxxxxx}\frac{\partial}{\partial u_{xxxxxx}} + \eta^{xxxxxxx}\frac{\partial}{\partial u_{xxxxxxx}}$$

and the Lie characteristic function $W$ is given by

$$W = \eta - \tau u_t - \xi u_x.$$

In the case when Riemann–Liouville time-fractional derivative is used in Eq. (2), the operator $N^t$ is given by [26–28]

$$N^t = \tau l + \sum_{k=0}^{n-1}(-1)^k{}_0D_t^{\alpha-1-k}(W)D_t^k\frac{\partial}{\partial_0 D_t^\alpha u} - (-1)^nJ\left(W, D_t^n\frac{\partial}{\partial_0 D_t^\alpha u}\right) \tag{29}$$

with $J$ defined by

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}}d\mu dt. \tag{30}$$

For the seventh-order space derivatives, the operator $N^x$ is given by

$$\begin{aligned}N^x = \xi l &+ W\left(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}} + D_x^2\frac{\partial}{\partial u_{xxx}} - D_x^3\frac{\partial}{\partial u_{4x}} + D_x^4\frac{\partial}{\partial u_{5x}} - D_x^5\frac{\partial}{\partial u_{6x}} + D_x^6\frac{\partial}{\partial u_{7x}}\right)\\ &+ D_x(W)\left(\frac{\partial}{\partial u_{xx}} - D_x\frac{\partial}{\partial u_{xxx}} + D_x^2\frac{\partial}{\partial u_{4x}} - D_x^3\frac{\partial}{\partial u_{5x}} + D_x^4\frac{\partial}{\partial u_{6x}} - D_x^5\frac{\partial}{\partial u_{7x}}\right)\\ &+ D_x^2(W)\left(\frac{\partial}{\partial u_{xxx}} - D_x\frac{\partial}{\partial u_{4x}} + D_x^2\frac{\partial}{\partial u_{5x}} - D_x^3\frac{\partial}{\partial u_{6x}} + D_x^4\frac{\partial}{\partial u_{7x}}\right)\\ &+ D_x^3(W)\left(\frac{\partial}{\partial u_{4x}} - D_x\frac{\partial}{\partial u_{5x}} + D_x^2\frac{\partial}{\partial u_{6x}} - D_x^3\frac{\partial}{\partial u_{7x}}\right)\\ &+ D_x^4(W)\left(\frac{\partial}{\partial u_{5x}} - D_x\frac{\partial}{\partial u_{6x}} + D_x^2\frac{\partial}{\partial u_{7x}}\right) + D_x^5(W)\left(\frac{\partial}{\partial u_{6x}} - D_x\frac{\partial}{\partial u_{7x}}\right)\\ &+ D_x^6(W)\frac{\partial}{\partial u_{7x}}.\end{aligned}\tag{31}$$

The invariance condition for any given generator $X$ of (2) and its solutions reads

$$\left(\overline{X}L + D_t(\tau)L + D_x(\xi)L\right)|_{(2)} = 0 \tag{32}$$

and consequently the conservation law of Eq. (2) can be written as

$$D_t(N^tL) + D_x(N^xL) = 0. \tag{33}$$

Now, we present the conservation laws of Eq. (2) using the above formalism. We consider two subcases corresponding to the order of $\alpha$.

**Case 1.** When $\alpha \in (0,1)$, with the help of (29) and (30), the components of the conserved vectors are

$$C_i^t = \tau L + (-1)^0 {}_0D_t^{\alpha-1}(W_i) D_t^0 \frac{\partial L}{\partial\{{}_0D_t^\alpha u\}} - (-1)^1 J\left(W_i, D_t^1 \frac{\partial L}{\partial\{{}_0D_t^\alpha u\}}\right)$$
$$= {}_0D_t^{\alpha-1}(W_i)v + J(W_i, v_t),$$

$$C_i^x = \xi L + W_i\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} + D_x^2\frac{\partial L}{\partial u_{xxx}} - D_x^3\frac{\partial L}{\partial u_{4x}} + D_x^4\frac{\partial L}{\partial u_{5x}} - D_x^5\frac{\partial L}{\partial u_{6x}} + D_x^6\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x(W_i)\left(\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}} + D_x^2\frac{\partial L}{\partial u_{4x}} - D_x^3\frac{\partial L}{\partial u_{5x}} + D_x^4\frac{\partial L}{\partial u_{6x}} - D_x^5\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^2(W_i)\left(\frac{\partial L}{\partial u_{xxx}} - D_x\frac{\partial L}{\partial u_{4x}} + D_x^2\frac{\partial L}{\partial u_{5x}} - D_x^3\frac{\partial L}{\partial u_{6x}} + D_x^4\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^3(W_i)\left(\frac{\partial L}{\partial u_{4x}} - D_x\frac{\partial L}{\partial u_{5x}} + D_x^2\frac{\partial L}{\partial u_{6x}} - D_x^3\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^4(W_i)\left(\frac{\partial L}{\partial u_{5x}} - D_x\frac{\partial L}{\partial u_{6x}} + D_x^2\frac{\partial L}{\partial u_{7x}}\right) + D_x^5(W_i)\left(\frac{\partial L}{\partial u_{6x}} - D_x\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^6(W_i)\frac{\partial L}{\partial u_{7x}}$$

$$= W_i\{v(252u^3 + 189u_x^2 + 378uu_{xx} + 42u_{4x})$$
$$- D_x(v(378uu_x + 63u_{3x})) + D_x^2(v(126u^2 + 63u_{2x})) - D_x^3(42vu_x)$$
$$+ D_x^4(21vu) + D_x^6 v\}$$
$$+ D_x(W_i)\{v(378uu_x + 63u_{3x}) - D_x(v(126u^2 + 63u_{2x}))$$
$$+ D_x^2(42vu_x) - D_x^3(21vu) - D_x^5 v\}$$
$$+ D_x^2(W_i)\{v(126u^2 + 63u_{2x}) - D_x(42vu_x) + D_x^2(21vu) + D_x^4 v\}$$
$$+ D_x^3(W_i)\{42vu_x - D_x(21vu) - D_x^3 v\} + D_x^4(W_i)\{21vu + D_x^2 v\}$$
$$- v_x D_x^5(W_i) + vD_x^6(W_i),$$

where $i = 1,2$ and the Lie characteristics functions $W_i$ are given by

$$W_1 = -u_x, \quad W_2 = -2\alpha u - 7tu_t - \alpha x u_x.$$

**Case 2.** When $\alpha \in (1,2)$, likewise as before the components of conserved vectors are given by

$$C_i^t = \tau L + (-1)^0 {}_0D_t^{\alpha-1}(W_i) D_t^0 \frac{\partial L}{\partial\{{}_0D_t^\alpha u\}} - (-1)^1 J\left(W_i, D_t^1 \frac{\partial L}{\partial\{{}_0D_t^\alpha u\}}\right)$$
$$+ (-1)^1 {}_0D_t^{\alpha-2}(W_i) D_t^1 \frac{\partial L}{\partial\{{}_0D_t^\alpha u\}} - (-1)^2 J\left(W_i, D_t^2 \frac{\partial L}{\partial\{{}_0D_t^\alpha u\}}\right)$$
$$= v\ {}_0D_t^{\alpha-1}(W_i) + J(W_i, v_t) - v_t\ {}_0D_t^{\alpha-2}(W_i) - J(W_i, v_{tt}),$$

$$C_i^x = \xi L + W_i\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} + D_x^2\frac{\partial L}{\partial u_{xxx}} - D_x^3\frac{\partial L}{\partial u_{4x}} + D_x^4\frac{\partial L}{\partial u_{5x}} - D_x^5\frac{\partial L}{\partial u_{6x}} + D_x^6\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x(W_i)\left(\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}} + D_x^2\frac{\partial L}{\partial u_{4x}} - D_x^3\frac{\partial L}{\partial u_{5x}} + D_x^4\frac{\partial L}{\partial u_{6x}} - D_x^5\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^2(W_i)\left(\frac{\partial L}{\partial u_{xxx}} - D_x\frac{\partial L}{\partial u_{4x}} + D_x^2\frac{\partial L}{\partial u_{5x}} - D_x^3\frac{\partial L}{\partial u_{6x}} + D_x^4\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^3(W_i)\left(\frac{\partial L}{\partial u_{4x}} - D_x\frac{\partial L}{\partial u_{5x}} + D_x^2\frac{\partial L}{\partial u_{6x}} - D_x^3\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^4(W_i)\left(\frac{\partial L}{\partial u_{5x}} - D_x\frac{\partial L}{\partial u_{6x}} + D_x^2\frac{\partial L}{\partial u_{7x}}\right) + D_x^5(W_i)\left(\frac{\partial L}{\partial u_{6x}} - D_x\frac{\partial L}{\partial u_{7x}}\right)$$
$$+ D_x^6(W_i)\frac{\partial L}{\partial u_{7x}}$$

$$= W_i\{v(252u^3 + 189u_x^2 + 378uu_{xx} + 42u_{4x})$$
$$- D_x(v(378uu_x + 63u_{3x}))$$
$$+ D_x^2(v(126u^2 + 63u_{2x})) - D_x^3(42vu_x) + D_x^4(21vu) + D_x^6 v\}$$
$$+ D_x(W_i)\{v(378uu_x + 63u_{3x}) - D_x(v(126u^2 + 63u_{2x})$$
$$+ D_x^2(42vu_x) - D_x^3(21vu) - D_x^5 v\}$$
$$+ D_x^2(W_i)\{v(126u^2 + 63u_{2x}) - D_x(42vu_x) + D_x^2(21vu) + D_x^4 v\}$$
$$+ D_x^3(W_i)\{42vu_x - D_x(21vu) - D_x^3 v\} + D_x^4(W_i)\{21vu + D_x^2 v\}$$
$$- v_x D_x^5(W_i) + vD_x^6(W_i),$$

where $i = 1,2$ and functions $W_i$ are given by

$$W_1 = -u_x, \quad W_2 = -2\alpha u - 7tu_t - \alpha x u_x.$$

## 5. Exact travelling wave solutions of the time FSKI equation

In this section we construct exact travelling wave solutions of the time FSKI equation (2). For this aim, we consider the fractional derivative appears in the Eq. (2) in the sense of modified Riemann–Liouville derivative. The Jumarie's modified Riemann–Liouville derivative [46] of order $\alpha$ is defined by the following expression

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi) - f(0))d\xi, & 0 < \alpha < 1, \\ \left[f^{(n)}(t)\right]^{(\alpha-n)}, & n \leqslant \alpha < n+1,\ n \geqslant 1. \end{cases} \tag{34}$$

where $f : R \to R, t \to f(t)$ denotes a continuous (but not necessarily first-order-differentiable) function and $\Gamma(.)$ is the Gamma function defined by:

$$\Gamma(\alpha) = lim_{n\to\infty}\frac{n!n^\alpha}{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n)} \tag{35}$$

Modified Riemann–Liouville derivative has the following important property:

$$D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}, \quad \gamma > 0. \tag{36}$$

### 5.1. Description of the sub-equation method

In this subsection, we employ the sub-equation method, which was developed by Zhang and Zhang [12]. We recall this method here.

We consider the NLFPDEs of the type [12,20]

$$P(u, u_t, u_x, D_t^\alpha u, D_x^\alpha u, \ldots) = 0, \quad 0 < \alpha \leqslant 1, \tag{37}$$

where $u$ is an unknown function, $F$ is a polynomial of $u$ and its partial fractional derivatives, $D_x^\alpha u$ and $D_t^\alpha u$ are the modified Riemann–Liouville derivatives of $u$ with respect to $t$ and $x$, respectively. We present the main steps of the sub-equation method.

**Step 1:** By making use of the travelling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x + ct, \tag{38}$$

where $c$ is a nonzero constant to be determined later, we can rewrite (37) in the following nonlinear fractional ordinary differential equation (NFODE):

$$P(u, cu', u', c^\alpha D_\xi^\alpha u, D_\xi^\alpha u, \ldots) = 0, \quad 0 < \alpha \leqslant 1. \tag{39}$$

**Step 2:** According to sub-equation method, we assume that the travelling wave solution can be expressed in the form

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i(\phi(\xi))^i, \tag{40}$$

where $a_i$ $(i = 1, \ldots, n)$ are constants to be determined later, $n$ is a positive integer which is determined by balancing the highest order derivatives and nonlinear terms in (37) and the function $\phi(\xi)$ satisfies the fractional Riccati equation

$$D_\xi^\alpha \phi(\xi) = \sigma + \phi^2(\xi), \tag{41}$$

where $\sigma$ is a constant. Some special solutions of the fractional Riccati equation (41) are

$$\phi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0, \\ -\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0, \\ \sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi), & \sigma > 0, \\ -\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi), & \sigma > 0, \\ -\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}, & \omega = \text{const.},\ \sigma = 0. \end{cases} \tag{42}$$

**Step 3**: Substituting (40) into (39) and setting the coefficients of $\phi^i$ to be zero, one obtains an over-determined nonlinear algebraic system in $a_i$ $(i = 1, \ldots, n)$ and $c$.

**Step 4:** Then substituting these constants and the solutions of (41) into (40), we get the exact travelling solutions of the Eq. (37).

*5.2. Application of the sub-equation method to the time FSKI equation*

We now implement sub-equation method to Eq. (2). Firstly, we make use of the travelling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x + ct,$$

where $c$ is a constant and transform Eq. (2) into a nonlinear fractional ODE

$$c^\alpha D_\xi^\alpha u + 252u^3 u_\xi + 63u_\xi^3 + 378uu_\xi u_{\xi\xi} + 126u^2 u_{\xi\xi\xi} + 63u_{2\xi}u_{3\xi} + 42u_\xi u_{4\xi} + 21uu_{5\xi} + u_{7\xi} = 0. \tag{43}$$

We now suppose that Eq. (43) has the solution in the form

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \phi(\xi)^i, \tag{44}$$

where $a_i$ $(i = 1, \ldots, n)$ are constants to be determined and $\phi$ satisfies the fractional Riccati equation (41). Balancing the highest order derivative terms with nonlinear terms in Eq. (43), we get $n = 2$ and hence

$$u(\xi) = a_0 + a_1\phi + a_2\phi^2. \tag{45}$$

We then substitute Eq. (45) along with Eq. (41) into Eq. (43) then collect the coefficients of $\phi^i$ and equate them to zero. We obtain a set of algebraic equations in $c, a_0, a_1$ and $a_2$. Solving these algebraic equations with the help of Maple, we get the following two cases:

**Case 1:**

$$a_1 = 0, \quad a_2 = -2, \quad c = (-252a_0^3 - 1008a_0^2\sigma - 1344a_0\sigma^2 - 608\sigma^3)^{\frac{1}{\alpha}}.$$

Thus, from (42), we obtain five types of exact travelling wave solutions of Eq. (2), namely

$$u_1(x,t) = a_0 - 2\left[-\sqrt{-\sigma}\tanh_\alpha\{\sqrt{-\sigma}(x+ct)\}\right]^2, \quad \sigma < 0,$$
$$u_2(x,t) = a_0 - 2\left[-\sqrt{-\sigma}\coth_\alpha\{\sqrt{-\sigma}(x+ct)\}\right]^2, \quad \sigma < 0,$$
$$u_3(x,t) = a_0 - 2\left[\sqrt{\sigma}\tan_\alpha\{\sqrt{\sigma}(x+ct)\}\right]^2, \quad \sigma > 0,$$
$$u_4(x,t) = a_0 - 2\left[-\sqrt{\sigma}\cot_\alpha\{\sqrt{\sigma}(x+ct)\}\right]^2, \quad \sigma > 0,$$
$$u_5(x,t) = a_0 - 2\left(-\frac{\Gamma(1+\alpha)}{(x+ct)^\alpha + \omega}\right)^2, \quad \omega = \text{const.},\ \sigma = 0,$$

where $a_0$ is an arbitrary constant.

**Case 2:**

$$a_0 = -8\sigma/3, \quad a_1 = 0, \quad a_2 = -4, \quad c = (-256\sigma^3/3)^{\frac{1}{\alpha}}.$$

Likewise for this case, we also obtain five types of exact travelling solutions of Eq. (2) given by

$$u_1(x,t) = \frac{-8\sigma}{3} - 4\left[-\sqrt{-\sigma}\tanh_\alpha\{\sqrt{-\sigma}(x+ct)\}\right]^2, \quad \sigma < 0,$$
$$u_2(x,t) = \frac{-8\sigma}{3} - \left[-\sqrt{-\sigma}\coth_\alpha\{\sqrt{-\sigma}(x+ct)\}\right]^2, \quad \sigma < 0,$$
$$u_3(x,t) = \frac{-8\sigma}{3} - 4\left[\sqrt{\sigma}\tan_\alpha\{\sqrt{\sigma}(x+ct)\}\right]^2, \quad \sigma > 0,$$
$$u_4(x,t) = \frac{-8\sigma}{3} - 4\left[-\sqrt{\sigma}\cot_\alpha\{\sqrt{\sigma}(x+ct)\}\right]^2, \quad \sigma > 0,$$
$$u_5(x,t) = \frac{-8\sigma}{3} - 4\left(-\frac{\Gamma(1+\alpha)}{(x+ct)^\alpha + \omega}\right)^2, \quad \omega = \text{const.},\ \sigma = 0.$$

## 6. Concluding remarks

In this work, we applied the Lie group analysis to time FSKI equation (2) based in the sense of Riemann–Liouville derivative. We obtained two– dimensional Lie symmetry algebra for (2). Using the nontrivial Lie point symmetry generator, we have shown that time FSKI equation can be transformed into a NFODE. Next, we constructed conservation laws for the time FSKI equation via Ibragimov's nonlocal conservation theorem adapted to the time fractional partial differential equations. In addition, using the sub-equation method, we obtained hyperbolic, trigonometric and rational solutions to time FSKI equation (2). The exact solutions obtained here can be used as benchmarks against the numerical simulations. Furthermore, the exact solutions and conservation laws obtained in this paper might be very useful in various areas of applied mathematics in interpreting some physical phenomena.

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