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NOTE

On the Parity of Colourings and Flows

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We extend a result of Tarsi and show that the chromatic polynomial and flow polynomial evaluated at $1+k$ are up to sign the same modulo k^2 for any integer k such that $|k| \geqslant 2$. © 2002 Elsevier Science (USA)

the set of nowhere zero 3-flows are of the same parity. In this note it is shown that this is just a special case of a more general result.

As pointed out by Tarsi, the set $C_3(G)$ of proper 3-colourings has the property that $|C_3(G)|$ is always divisible by 6 (permutations of the 3 colours), while flows come in pairs (obtained by reversing the entire orientation). Thus, if $NZF_3(G)$ denotes the set of nowhere zero 3-flows on G, what Tarsi actually shows (his Theorem 1.3) is that

$$|C_3(G)| \equiv |NZF_3(G)| \bmod 4. \tag{1}$$

We change notation to that of [2] and write $P(G; \lambda)$ for the chromatic polynomial and $F(G; \lambda)$ for the flow polynomial. Thus (1) can be rewritten as

$$P(G; 3) \equiv F(G; 3) \bmod 4. \tag{2}$$

To prove our results we use the following properties of a graph $G = (V, E)$ with $k(G)$ components and rank $r(E) = |V| - k(G)$,

$$P(G; \lambda) = (-1)^{r(E)} \lambda^{k(G)} T(G; 1-\lambda, 0), \tag{3}$$

$$F(G; \lambda) = (-1)^{|E|-r(E)} T(G; 0, 1-\lambda), \tag{4}$$

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where T is the Tutte polynomial of G, denoted by

$$T(G; x, y) = \sum t_{i,j} x^i y^j. \tag{5}$$

Two particular properties of the Tutte polynomial are noted: for $|E| > 0$ we have $t_{0,0} = 0$ and for $|E| > 1$ we have $t_{1,0} = t_{0,1}$.

From (3) and (4)

$$P(G; \lambda) = (-1)^{r(E)} \lambda^{k(G)} \sum_{i \geqslant 1} t_{i,0} (1-\lambda)^i \tag{6}$$

and

$$F(G; \lambda) = (-1)^{|E| - r(E)} \sum_{j \geqslant 1} t_{0,j} (1-\lambda)^j. \tag{7}$$

THEOREM. *For graph $G = (V, E)$, $|E| \geqslant 2$, and integer λ, $|\lambda| \geqslant 2$,*

$$P(G; 1+\lambda) \equiv (-1)^{|E|} F(G; 1+\lambda) \bmod \lambda^2.$$

Proof. Using (6) and (7) we have

$$P(G; 1+\lambda) \equiv (-1)^{r(E)} (1+\lambda)^{k(G)} (-\lambda)\, t_{1,0} \bmod \lambda^2,$$

and

$$F(G; 1+\lambda) \equiv (-1)^{|E| - r(E)} (-\lambda)\, t_{0,1} \bmod \lambda^2.$$

The result follows since $(1+\lambda) F(G; 1+\lambda) \equiv F(G; 1+\lambda) + \lambda F(G; 1+\lambda) \equiv F(G; 1+\lambda) \bmod \lambda^2$ and $t_{1,0} = t_{0,1}$. ∎

Putting $\lambda = 2$ gives as a corollary (2) above. Since $P(G; 3)$ and $F(G; 3)$ are even and $2 \equiv -2 \bmod 4$ the sign $(-1)^{|E|}$ of the theorem is redundant in this instance.

REFERENCES

1. M. Tarsi, The graph polynomial and the number of proper vertex colourings, *Ann. Inst. Fourier, Grenoble* **49** (1999), 1089–1093.
2. D. J. A. Welsh, "Complexity: Knots, Colourings and Counting," London Mathematical Society Lecture Notes Series, Vol. 186, Cambridge Univ. Press, Cambridge, UK, 1993.