Lp Contraction Semigroups for Vector Valued Functions

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We discuss semigroups acting on vector valued functions. We consider a comparison theorem between two semigroups: a semigroup acting on scalar valued functions and a semigroup acting on vector valued functions. The criterion is given by the abstract Kato theorem. By using this, we give a sufficient condition for the criterion in the setting of square field operator. We also consider the essential self-adjointness of a perturbed semigroup.

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1. INTRODUCTION

In this paper, we discuss a theory of Lp contraction semigroups for vector valued functions. The theory of Markovian semigroups is well developed in terms of Dirichlet forms. By virtue of the interpolation theory, the symmetric Markovian semigroup is not only an L2 contraction semigroup but also an Lp contraction semigroup for any p ∈ [1, ∞). Lp setting of the semigroup is useful in applications. But the semigroup acts on scalar valued functions. The Markovian property makes no sense for semigroups acting on vector valued functions. To deal with semigroups acting on vector valued functions, we need a comparison theorem.

To be precise, let (X, B, m) be a σ-finite measure space. Suppose we are given a Markovian semigroup \( \{T_t\} \) on \( L^2 = L^2(X, m) \). Further suppose that \( \{T_t\} \) is a contraction semigroup in \( L^p \) for any \( p \in [1, \infty) \). In addition, we are given a semigroup \( \{\tilde{T}_t\} \) acting on Hilbert space valued functions. We denote the norm of the Hilbert space by \( |\cdot| \). If we have

\[ |\tilde{T}_t u| \leq T_t |u|, \quad \forall u \in L^2, \quad (1.1) \]

then we can see that \( \{\tilde{T}_t\} \) defines a semigroup on \( L^p \) for \( p \in [1, \infty) \). A necessary and sufficient condition for (1.1) is given by the abstract Kato theorem due to Simon [34, 35] (see also [19]). In this paper, we give sufficient conditions for (1.1) in terms of square field operator.

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The organization of this paper is as follows. In Section 2, we consider two semigroups \( \{ T_t \} \), \( \{ T_{\tau} \} \) and review the abstract Kato theorem. In Section 3, we formulate the problem in the framework of the square field operator. We give sufficient conditions for (1.1). The positivity and the derivation property of the square field operator play an important role. As an application, we discuss a perturbed semigroup and consider the essential self-adjointness in Section 4. We give examples in Section 5. Our main interest is in the infinite dimensional case and we consider a submanifold of the Wiener space as an example.

2. COMPARISON THEOREM

Let \( (X, \mathcal{E}) \) be a measurable space and let \( m \) be a \( \sigma \)-finite measure. Suppose we are given a strongly continuous contraction semigroup \( \{ T_t \} \) on \( L^2 \). We suppose that \( T_t \) is Markovian, i.e., if \( f \in L^2 \) satisfies \( 0 \leq f \leq 1 \), then \( 0 \leq T_t f \leq 1 \). Here these inequalities hold a.e. But we do not specify “a.e.” in the remainder of this article, for simplicity. The generator of \( \{ T_t \} \) is denoted by \( A \) and its resolvents by \( G_\lambda \), \( \lambda > 0 \). In addition, we assume that

\[
\int_X T_t f \, dm \leq \int_X f \, dm, \quad f \in L^1 \cap L^2_+,
\]

where + stands for the non-negative functions. Note that if \( \{ T_t \} \) is symmetric, the above property follows from the Markovian property. Thus \( \{ T_t \} \) becomes a contraction semigroup on \( L^1 \).

Now, by the Riesz–Thorin interpolation theorem, \( \{ T_t \} \) is a contraction semigroup on \( L^p \) for \( p \in [1, \infty] \). To be precise, for \( f \in L^2 \cap L^p \), \( T_t f \in L^2 \cap L^p \) and

\[
\| T_t f \|_p \leq \| f \|_p.
\]

Therefore, for \( p \in [1, \infty) \), \( T_t \) can be uniquely extended to a bounded linear operator on \( L^p \). But, for \( p = \infty \), \( T_t \) is defined only on \( L^2 \cap L^\infty \), the completion of \( L^2 \cap L^\infty \) with respect to \( \| \cdot \|_\infty \). We regard \( \{ T_t \} \) as a contraction semigroup on \( L^p \) throughout the paper.

Let \( \{ T^*_t \} \) be the dual semigroup of \( \{ T_t \} \). From the assumption, \( \{ T^*_t \} \) is Markovian. Moreover, by duality, we can see that \( \{ T^*_t \} \) is a contraction semigroup on \( L^p \) for \( p \in [1, \infty) \) as well. In this case, \( T_t : L^\infty \to L^\infty \) can be defined as a dual operator of \( T^*_t : L^1 \to L^1 \). In the sequel, we suppose that \( T_t : L^\infty \to L^\infty \) is defined in this manner. We denote the associated generator and resolvents by \( A^* \) and \( G^*_\lambda \), respectively.
Later, we will consider complex Hilbert spaces and so complex coefficients will be necessary. From now on we assume that \( \{ T_t \} \) acts on complex valued functions in a natural way. It is easy to see that for a complex valued function \( f \),

\[ |T_t f| \leq T_t |f|. \]

In fact, when \( f = \sum a_i 1_{A_i}, A_i \cap A_j = \emptyset \) if \( i \neq j \), we have

\[ |T_t f| = |T_t \left( \sum a_i 1_{A_i} \right)| = \left| \sum a_i T_t 1_{A_i} \right| \leq \left| \sum |a_i| T_t 1_{A_i} \right| = T_t |f|. \]

Hence \( \{ T_t \} \) is also a contraction semigroup on \( L^p(X, m; \mathbb{C}) \). As usual, the inner product in \( L^2(X, m; \mathbb{C}) \) is defined by

\[ (f | g) = \int_X fg \, dm \]

where \(-\) denotes the complex conjugate. We also use this notation as a pairing between \( L^p \) and \( L^q \), \( 1/p + 1/q = 1 \). Note that \(( \cdot | \cdot )\) is anti-linear in the second variable.

The following is a natural generalization of the Hölder inequality:

**Lemma 2.1.** For \( p \in (1, \infty) \) let \( q \) be the conjugate exponent of \( p \): \( 1/p + 1/q = 1 \). Then it holds that

\[ |T_t (fg)| \leq T_t |fg| \leq \left( T_t |f|^p \right)^{1/p} \left( T_t |g|^q \right)^{1/q} \quad \text{for } f \in L^p \text{ and } g \in L^q. \]  

(2.1)

For \( p = 1 \), it holds that

\[ |T_t (fg)| \leq T_t |fg| \leq (T_t |f|) \|g\|_\infty \quad \text{for } f \in L^1 \text{ and } g \in L^\infty. \]  

(2.2)

Furthermore we have, for \( 1 \leq p \leq p' \leq \infty \),

\[ \{ T_t |f|^p \}^{1/p} \leq \left( T_t |f|^p \right)^{1/p'} \quad \text{for } f \in L^p \cap L^{p'}. \]  

(2.3)

**Proof.** We recall the following inequality:

\[ ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad a, b \geq 0. \]  

(2.4)
We fix versions of $T_t |f|^{r}$ and $T_t |g|^{q}$ and consider the set $\{ x : (T_t |f|^{r})(x) = 0 \text{ or } (T_t |g|^{q})(x) = 0 \}$. Then, for $x \in \{ T_t |f|^{r} = 0 \}$ and $n \in \mathbb{N}$,

$$T_t |fg| (x) = T_t |nf/gn| (x) \leq \frac{n^p T_t |f|^{r}(x)}{p} + \frac{T_t |g|^{q}(x)}{n^q}$$

$$\leq \frac{T_t |g|^{q}(x)}{n^q} \quad \text{a.e.}$$

Since $n$ is arbitrary, we have

$$T_t |fg| = 0 \quad \text{a.e. on } \{ T_t |f|^{r} = 0 \}.$$ A similar result holds for the set $\{ T_t |g|^{q} = 0 \}$. Thus we have

$$T_t |fg| \leq \{ T_t |f|^{r} \}^{1/p} \{ T_t |g|^{q} \}^{1/q} \quad \text{a.e. on } \{ T_t |f|^{r} = 0 \text{ or } T_t |g|^{q} = 0 \}.$$ Next for $x \in \{ T_t |f|^{r} \neq 0, T_t |g|^{q} \neq 0 \}$,

$$\frac{|f|}{\{ T_t |f|^{r}(x) \}^{1/p} \{ T_t |g|^{q}(x) \}^{1/q}} \leq \frac{|f|^{r}}{p T_t |f|^{r}(x)} + \frac{|g|^{q}}{q T_t |g|^{q}(x)}.$$ Hence we have

$$\frac{T_t |fg| (x)}{\{ T_t |f|^{r}(x) \}^{1/p} \{ T_t |g|^{q}(x) \}^{1/q}} \leq \frac{T_t |f|^{r}(x)}{p T_t |f|^{r}(x)} + \frac{T_t |g|^{q}(x)}{q T_t |g|^{q}(x)}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

and therefore we have

$$T_t |fg| (x) \leq \{ T_t |f|^{r}(x) \}^{1/p} \{ T_t |f|^{r}(x) \}^{1/q}$$

a.e. on $\{ T_t |f|^{r} \neq 0, T_t |g|^{q} \neq 0 \}$. Combining them, we obtain (2.1).

To show (2.3), let $r$ be the conjugate exponent of $p'/p$. We take $g \in L^{r}$ such that $0 \leq g \leq 1$. Then by (2.1) and the Markov property, we have

$$T_t(|f|^{r} g) \leq \{ T_t |f|^{r} \}^{p'/p} \{ T_t |g|^{r} \}^{1/r} \leq \{ T_t |f|^{r} \}^{p'/r}.$$ By letting $g \uparrow 1$, we obtain (2.3).
We have seen that \( \{ T_t \} \) is a contraction semigroup. Let us derive the strong continuity.

**Proposition 2.2.** \( \{ T_t \} \) (resp. \( \{ T^*_t \} \)) is a strongly continuous semigroup in \( L^p \) for \( p \in [1, \infty) \).

**Proof.** First we show the case \( p = 1 \). Take any \( f \in L^1_+ \). Here, \( + \) means the positive functions and we use this convention without mentioning it. Then by Lemma 2.1, we have

\[
|T_t \sqrt{f}|^2 \leq T_t f
\]

and therefore

\[
\|T_t \sqrt{f}\|_2^2 \leq \|T_t f\|_1 \leq \|f\|_1 = \|\sqrt{f}\|_2.
\]

Since \( \{ T_t \} \) is strongly continuous in \( L^2 \), we have \( \lim_{t \to 0} \|T_t \sqrt{f}\|_2 = \|\sqrt{f}\|_2 \). Combining this with the above fact, we have

\[
\lim_{t \to 0} \|T_t f\|_1 = \|f\|_1. \tag{2.5}
\]

Now if we take \( f \in L^1_+ \cap L^2_+ \), then there exists a sequence \( \{ t_n \} \) such that \( t_n \to 0 \) and \( T_{t_n} f \to f \) a.e. This combined with (2.5) implies that

\[
\lim_{t \to 0} \|T_t f - f\|_1 = 0 \tag{2.6}
\]

(see, e.g., [21, Theorem (13.47)]). Now it is easy to see that (2.6) holds for any \( f \in L^1 \).

To show the general case \( p \in (1, \infty) \), we take \( r > p \) and use the following inequality (an easy application of the Hölder inequality; see, e.g., [21, Theorem (13.19)]): for \( f \in L^1 \cap L^r \),

\[
\|T_t f - f\|_p^p \leq \|T_t f - f\|_r^r \|T_t f - f\|_p^{p(1 - \alpha)} \tag{2.7}
\]

Here \( \alpha \) is taken so that \( p = \alpha + (1 - \alpha) r \). Now the strong continuity follows from (2.6) and the contraction property in \( L^r \).

**Remark 2.1.** Other proofs of the above strong continuity can be found in [28, Theorem X.55] when \( m \) is a finite measure and in [13, Proposition I.2.4.2] when \( p = 1 \).

For later use, we give a basic fact on \( A \). We denote the generator in \( L^p \) by \( A_p \) to specify the acting space. \( \text{Dom}(A_p) \) denotes the domain of \( A_p \). We always assume that \( \text{Dom}(A_p) \) is equipped with the graph norm. It is a Banach space with respect to the graph norm. Similarly, \( \text{Dom}(\delta) \) is equipped with the inner product \( \delta_1(\cdot, \cdot) = \delta(\cdot, \cdot) + \langle \cdot, \cdot \rangle_{L^2} \).
Further, the inner product in $L^2$ is denoted by $\langle \cdot | \cdot \rangle$. We use it as a pairing of $L^p$ and $L^q$, where $1/p + 1/q = 1$. Note that $\langle \cdot | \cdot \rangle$ is $C$-linear in the first variable but anti-$C$-linear in the second variable.

**Lemma 2.3.** We have the following:

1. $G_s(L^1 \cap L^\infty) \subseteq \bigcap_{1 \leq p < \infty} \text{Dom}(A_p)$ and $G_s(L^1 \cap L^\infty)$ is independent of $s$ and dense in $L^p$ for any $p \in [1, \infty)$.

2. For $1 \leq p < \infty$ and $f \in L^p$, suppose that there exists $g \in L^p$ such that
   $$\langle f | \alpha G_s^* h - h \rangle = \langle g | G_s^* h \rangle, \quad \forall h \in L^1 \cap L^\infty.$$
   Then $f \in \text{Dom}(A_p)$ and $A_p f = g$.

3. For $1 \leq p < \infty$ and $f \in L^p$, suppose that there exists $g \in L^p$ such that
   $$\langle f | A^s h \rangle = \langle g | h \rangle, \quad \forall h \in G_s^*(L^1 \cap L^\infty).$$
   Then $f \in \text{Dom}(A_p)$ and $A_p f = g$.

4. For $1 \leq p, q < \infty$, $u \in \text{Dom}(A_p) \cap \text{Dom}(A_q) \Rightarrow A_p u = A_q u$.

5. For $1 \leq p, q < \infty$, $u \in \text{Dom}(A_p) \cap L^q$ and $A_p u \in L^p \Rightarrow u \in \text{Dom}(A_q)$

6. Let $q$, $q' \in (1, \infty]$ be conjugate exponents of $p, p' \in [1, \infty)$, respectively. $1/p + 1/q = 1$, $1/p' + 1/q' = 1$. Then, if $u \in \text{Dom}(A_p^*) \cap L^p$ and $v \in \text{Dom}(A_q^*) \cap L^q$, we have $\langle A_p^* u | v \rangle = \langle u | A_p v \rangle$.

**Proof.** (1) is trivial, e.g., the denseness follows from the fact that for $u \in L^\infty$, $\pi G_s u \to u$ in $L^p$ as $\pi \to \infty$.

Next we show (2). By the assumption, we have
   $$\langle \pi G_s f - f | h \rangle = \langle G_s g | h \rangle.$$  
   This implies $\pi G_s f - f = G_s g$ and therefore $G_s(\pi f - g) = f$. Hence we have $f \in \text{Dom}(A_p)$ and
   $$A_p f = A_p G_s(\pi f - g) = - (\pi f - g) + \pi G_s(\pi f - x) = - (\pi f - g) + \pi f = g$$
   which is the desired result.

(3) easily follows from (2). Set $h = G_s^* u$, $u \in L^1 \cap L^\infty$. Then
   $$\langle g | G_s^* u \rangle = \langle f | A^s G_s^* u \rangle = \langle f | -u + G_s^* u \rangle.$$  
   By (2), we have $A_p f = g$. 

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To see (4), it is enough to note that

\[ A_{\rho} u = \lim_{t \to 0} \frac{T_t u - u}{t} \quad \text{in } L^p. \]

As for (5), since \( u = G_{\lambda}(\tau u - A_{\rho} u) \) and \( \tau u - A_{\rho} u \in L^p \), it follows that \( u \in \text{Dom}(A_{\rho}) \). (6) can be shown as follows:

\[ (A_{\rho} u | v) = \left( \lim_{t \to 0} \frac{T_t u - u}{t} \right) (\lim_{t \to 0} \frac{T_t v - v}{t}) = (u | A_{\rho} v). \]

This completes the proof. 

So far, we have considered the semigroup acting on scalar valued functions. Now we consider another semigroup \( \{T_t\} \) that acts on vector valued functions. Sometimes we have to consider a space of sections of a vector bundle. But only the measurable structure is relevant and so we can regard it as a trivial bundle. We therefore consider the \( L^p \)-space of vector valued functions. Let \( K \) be a real or complex Hilbert space. We denote the inner product and the norm on \( K \) by \( (\cdot | \cdot)_K \) and \( |\cdot|_K \), respectively. For simplicity, we usually write \( |\cdot| \) in place of \( |\cdot|_K \). We assume that \( K \) is separable.

\[ L^p = L^p(X, m; K) \] denotes the space of all \( K \)-valued measurable functions \( u \) with

\[ \|u\|_p = \left\{ \int_X |u(x)|^p \, dm \right\}^{1/p} < \infty. \]

As usual, we identify two functions if they coincide \( m \)-a.e. Suppose we are given a strongly continuous contraction semigroup \( \{T_t\} \) on \( L^2(X, m; K) \). We denote the associated generator, and resolvents by \( A \) and \( G_{\lambda} \), respectively.

We are interested in when the following relation holds:

\[ |T_t u(x)| \leq T_t |u| (x) \quad m \text{-a.e. } x, \quad \forall u \in L^2(X, m; K). \]

For this problem, the following comparison theorem is due to B. Simon [34, 35]:

**Theorem 2.4.** The following three conditions are equivalent to each other:

1. \( |T_t u| \leq e^{\lambda T_t} |u| \),
2. \( |G_{\lambda + \varepsilon} u| \leq G_{\lambda} |u|, \quad \varepsilon > \max\{0, -\lambda\} \),
(3) $A |u| \geq \Re((A - \lambda) u | sgn u)_{K}$, $u \in \text{Dom}(A)$,

$$\text{sgn } u = \begin{cases} \frac{u}{|u|}, & u \neq 0 \\ 0, & u = 0. \end{cases}$$ (2.8)

Proof. We give here a proof for the reader’s convenience. The equivalence of (1) and (2) can be seen easily by noting that

$$G_{x} = \int_{0}^{\infty} e^{-\nu T_{t}} dt, \quad G_{x+\lambda} = \int_{0}^{\infty} e^{-(\nu + \lambda) T_{t}} dt$$

and

$$T_{t} = \lim_{n \to \infty} \left( \frac{n^{\nu}}{t} \right) G_{nT_{t}}, \quad T_{t} = e^{it} \lim_{n \to \infty} \left( \frac{n^{\nu}}{t} \right) G_{nT_{t}+\lambda}.$$ 

Next we derive the implication (1) $\Rightarrow$ (3). For $u \in \text{Dom}(A_{2})$, take any $f \in \text{Dom}(A_{2})$ and $f \not\equiv 0$. Then we have

$$(T_{t} |u| | f) \geq (\Re(T_{t} u | f sgn u)_{K} | f)$$

and hence

$$((|u| | T_{t} f) \geq \Re(T_{t} u | f sgn u).$$

In the case of $t = 0$, the following identity holds evidently:

$$(|u| | f) = (u | f sgn u).$$

Thus we have

$$\left( \frac{T_{t} f - f}{t} \right) \geq \Re \left( \frac{T_{t} u - u}{t} | f sgn u \right).$$

Now, letting $t \to 0$, we have

$$((|u| | A^{*} f) \geq \Re(A u | f sgn u) = (\Re(A u | u)_{K} | f),$$

which implies (3).

We prove the implication (3) $\Rightarrow$ (1). For any $g \in \text{Dom}(A_{2}^{*}+)$, we have

$$(|u| | (A^{*} - \lambda) g) \geq (\Re(A - \lambda - x) u | sgn u)_{K} | g) \geq -(|(A - \lambda - x) u | g).$$
Now we set $u = G_{s + \lambda} v$ and $g = G_s \psi$, $\psi \geq 0$. Then we have
\[
\langle |G_{s + \lambda} v| \mid (A^* - \lambda) G_s \psi \rangle \geq -\langle |(A - \lambda) G_{s + \lambda} v| \mid G_s \psi \rangle.
\]
Hence
\[
\langle |G_{s + \lambda} v| \mid \psi \rangle \leq \langle G_s \mid v \mid \psi \rangle.
\]
Thus we have $|G_{s + \lambda} v| \leq G_s |v|$. 

So far, we have not assumed the symmetry of $T_t$ and $T_t$. If they are symmetric, we can think of the associated quadratic forms. In this case, we denote the quadratic forms associated with $\{T_t\}$ and $\{T_t\}$ by $\mathcal{E}$ and $\mathcal{E}$, respectively.

**Theorem 2.5.** Assume that $T_t$ and $T_t$ are symmetric and one of the conditions in Theorem 2.4 is satisfied. Then we have that if $u \in \text{Dom}(\mathcal{E})$, then $|u| \in \text{Dom}(\mathcal{E})$ and
\[
\mathcal{E}(|u|, |u|) \leq \mathcal{E}(u, u) + \lambda \|u\|^2_L.
\]
Moreover if $g \geq 0$ and $u \in \text{Dom}(\mathcal{E})$ satisfy $gu \in \text{Dom}(\mathcal{E})$, then $g |u| \in \text{Dom}(\mathcal{E})$ and
\[
\mathcal{E}(|u|, g |u|) \leq \mathcal{E}(u, gu) + \lambda \|g |u|\|^2_L.
\]

**Proof.** Assuming (2) in Theorem 2.4, we show (2.9). We note that, for $u \in \text{Dom}(\mathcal{E})$,
\[
\langle (\alpha^2 G_s) |u| \mid |u| \rangle_{L^2} \leq \langle (\alpha^2 G_{s + \lambda}) u \mid u \rangle_{L^2}.
\]
Letting $\alpha \to \infty$,
\[
\lim_{\alpha \to \infty} \langle (\alpha^2 G_s) |u| \mid |u| \rangle_{L^2} = \lim_{\alpha \to \infty} \langle (\alpha^2 G_{s + \lambda}) u \mid u \rangle_{L^2} = \mathcal{E}(u, u) + \lambda \|u\|^2_L.
\]
We therefore have $|u| \in \text{Dom}(\mathcal{E})$ and
\[
\mathcal{E}(|u|, |u|) \leq \mathcal{E}(u, u) + \lambda \|u\|^2_L.
\]

Next we show (2.10). Assume that $gu \in \text{Dom}(\mathcal{E})$. Then, by the above result, we have $g |u| \in \text{Dom}(\mathcal{E})$. Now we notice the following inequality, which follows from (2):
\[
\langle (\alpha^2 G_s) |u| \mid g |u| \rangle_{L^2} \leq \langle (\alpha^2 G_{s + \lambda}) u \mid gu \rangle_{L^2}
\]
Letting $\alpha \to \infty$, we get the desired result (2.10).
The above theorem can be used in the following situation. We assume all conditions in Theorem 2.5. We assume further that the following defective logarithmic Sobolev inequality holds for $E$: there exists $\alpha, \beta > 0$ such that
\[ \int_X f^2 \log(f^2/\|f\|_2^2) \, dm \leq \alpha \mathcal{E}(f, f) + \beta \|f\|_2^2 \quad \text{for } f \in \text{Dom}(\mathcal{E}). \] (2.11)

Then we have a similar inequality for $E$:
\[ \int_X |u|^2 \log(|u|^2/\|u\|_2^2) \, dm \leq \alpha \mathcal{E}(u, u) + (\beta + \lambda) \|u\|_2^2 \quad \text{for } u \in \text{Dom}(\mathcal{E}). \] (2.12)

To show this, set $f = |u|$. By virtue of (2.9), we have
\[ \int_X |u|^2 \log(|u|^2/\|u\|_2^2) \, dm \leq \alpha \mathcal{E}(|u|, |u|) + \beta \||u||_2^2 \]
\[ \leq \alpha \mathcal{E}(u, u) + (\beta + \lambda) \|u\|_2^2. \]
The inequality (2.12) plays an important role in dealing with the Laplacian acting on tensor fields.

**Proposition 2.6.** Suppose that one of the conditions in Theorem 2.4 is fulfilled. Then $\{T_t\}$ (resp. $\{T^*_t\}$) is a strongly continuous semigroup in $L^p$ for $p \in [1, \infty)$.

**Proof.** First we prove that $\lim_{t \to 0} \|T_t u - u\|_1 = 0$ for $u \in L^1 \cap L^2$. Otherwise there exist $\varepsilon > 0$ and a sequence $\{t_n\}$ such that $t_n \to 0$ and
\[ \int_X |T_{t_n} u - u|_1 \, dm \geq 5\varepsilon. \] (2.13)

We may assume that $T_{t_n} u \to u \text{ m.a.e.}$ by taking a subsequence if necessary. By the Fatou lemma, we have
\[ \int_X |u| \, dm \leq \liminf_{n \to \infty} \int_X |T_{t_n} u| \, dm \leq \lim_{n \to \infty} \int_X |T_{t_n} u| \, dm \]
\[ \leq \lim_{n \to \infty} \int_X T_{t_n} |u| \, dm \leq \int_X |u| \, dm \]
which implies that
\[ \lim_{n \to \infty} \int_X |T_{t_n} u| \, dm = \int_X |u| \, dm. \]
We therefore have \(|T_n u| \to |u|\) in \(L^1\) and further, for any \(E \subset X\),

\[
\lim_{n \to \infty} \int_E |T_n u| \, dm = \int_E |u| \, dm
\]

(see [21, Theorem (13.47)]). Now we take \(E \subset X\) such that \(m(E) < \infty\) and

\[
\int_{X \setminus E} |u| \, dm \leq \varepsilon.
\]

We set \(v_n = 1_E T_n u\) and \(v = 1_E u\). Since \(\{|v_n|\}\) is uniformly integrable, we can take \(N > 0\) such that

\[
\int |v_n - 1_{\{|v_n| > N\}}| \, dm < \varepsilon, \quad \forall n.
\]

We may assume that the above inequality holds for \(v\). For such \(N > 0\), we define \(\varphi_N : K \to K\) by

\[
\varphi_N(k) = \begin{cases} k & \text{if } |k| \leq N, \\ Nk/|k| & \text{if } |k| > N. \end{cases}
\]

Set \(v^{(N)}_n = \varphi_N(v_n), \ v^{(N)} = \varphi_N(v)\). Then we have

\[
\int_X |T_n u - u| \, dm \leq \int_X 1_E |T_n u - u| \, dm + \int_{X \setminus E} |T_n u - u| \, dm
\]

\[
\leq \int_X \{|v_n - v^{(N)}_n| + |v^{(N)}_n - v^{(N)}| + |v^{(N)} - v|\} \, dm
\]

\[
+ \int_{X \setminus E} \{|T_n u| + |u|\} \, dm
\]

\[
\leq \int_X |v_n - 1_{\{|v_n| > N\}}| \, dm + \int_X |v^{(N)}_n - v^{(N)}| \, dm
\]

\[
+ \int_X |v - 1_{|v| > N}| \, dm + \int_{X \setminus E} |T_n u| \, dm + \int_X |u| \, dm.
\]

Now by the dominated convergence theorem, we have

\[
\lim_{n \to \infty} \int_X |T_n u - u| \, dm
\]

\[
\leq \varepsilon + \lim_{n \to \infty} \int_X |v^{(N)}_n - v^{(N)}| \, dm + \varepsilon + \lim_{n \to \infty} \int_{X \setminus E} |T_n u| \, dm + \varepsilon \leq 4 \varepsilon
\]
which contradicts (2.13). Now we have proved the strong continuity of \( \{ T_t \} \) in \( L^1 \).

For general \( p \in [1, \infty) \), it is enough to notice the same inequality as (2.7). The proof is completed.

3. SQUARE FIELD OPERATOR AND THE CONTRACTION SEMIGROUP

In this section, we discuss the contraction semigroup in the framework of the square field operator, which is called “opérateur carré du champ” in the French literature. From now on, we assume that \( \{ T_t \} \) is a symmetric semigroup. Following Bouleau and Hirsch [13], we assume the following condition:

\((\Gamma)\) For \( f, g \in \text{Dom}(A_2) \), we have \( f \cdot \hat{g} \in \text{Dom}(A_1) \).

Under the above assumption, we set

\[
\Gamma(f, g) = \frac{1}{2} \{ A_1(f \cdot \hat{g}) - A_2 f \cdot \hat{g} - f \cdot A_2 \hat{g} \}.
\]

(3.1)

The reader can see an extensive discussion of \( \Gamma \) in [13, Chap. 1, Sect. 4] (but should note that our definition of \( \Gamma \) is different from that of [13] up to a constant). We list some fundamental properties of \( \Gamma \) for later use. \( \Gamma \) can be uniquely extended to a continuous sesquilinear form from \( \text{Dom}(\mathcal{E}) \times \text{Dom}(\mathcal{E}) \) into \( L^1 \) and has the following properties. For \( f, g \in \text{Dom}(\mathcal{E}) \),

\[
\Gamma(f, f) \geq 0,
\]

(3.2)

\[
|\Gamma(f, g)|^2 \leq \Gamma(f, f) \Gamma(g, g),
\]

(3.3)

\[
\int_X \Gamma(f, f) \, dm \leq \mathcal{E}(f, f) \leq 2 \int_X \Gamma(f, f) \, dm,
\]

(3.4)

(but see also (3.17)) below). Further for \( f, g \in \text{Dom}(A_2) \),

\[
\mathcal{E}(f, g) = \int_X \Gamma(f, g) \, dm - \frac{1}{2} \int_X A_1(f \cdot \hat{g}) \, dm.
\]

(3.5)

\( \text{Dom}(A_4) \cap L^\infty \) is an algebra and for \( f, g \in \text{Dom}(A_4) \cap L^\infty \), we have

\[
A_4(f \cdot \hat{g}) = A_1 f \cdot \hat{g} + f \cdot A_1 \hat{g} + 2 \Gamma(f, g).
\]

(3.6)

We remark that \( \text{Dom}(A_4) \cap L^\infty \subseteq \text{Dom}(\mathcal{E}) \). We can see this by noting that

\[
\mathcal{E}(f, f) = \lim_{t \to 0} \left( \frac{f - T_t f}{t} \right) = -(A_1 f | f) \leq \| A_1 f \|_1 \| f \|_\infty.
\]
In [13], the reader can find several conditions that are equivalent to \((\Gamma)\). Among them, the following condition is rather practical:

\((\Gamma')\) There exists a subspace \(\mathcal{E} \subseteq \text{Dom}(A_2)\) such that \(\mathcal{E}\) is dense in \(\text{Dom}(\mathcal{E})\) and closed under complex conjugation, and \(|f|^2 \in \text{Dom}(A_1)\) for all \(f \in \mathcal{E}\).

Let us consider the semigroup \(\{T_t\}\). We also assume that \(\{T_t\}\) is symmetric. To define the square field operator for \(\{T_t\}\), we impose the following condition:

\((\Gamma_1)\) For \(u, v \in \text{Dom}(A_2)\), we have \((u \mid v) \in \text{Dom}(A_1)\) and there exists \(\lambda \in \mathbb{R}\) such that

\[
A_1 |u|^2 - 2(A_2 u \mid u) + 2\lambda |u|^2 \geq 0.
\]  

Under the above condition we define \(\Gamma\) by

\[
\Gamma(u, v) = \frac{1}{2} \left\{ A_1(u \mid v) - (A_2 u \mid v) - (u \mid A_2 v) \right\}.
\]  

\(\Gamma\) satisfies

\[
\Gamma(u, u) + \lambda |u|^2 \geq 0 \quad \text{for} \quad u \in \text{Dom}(A_2).
\]  

Now recall that the positivity of \(\Gamma\) corresponds to

\[
|T_t f|^2 \leq T_t |f|^2, \quad \forall f \in L^2.
\]  

The following theorem is a natural generalization of the above fact.

**Theorem 3.1.** Under the assumptions \((\Gamma)\) and \((\Gamma_1)\), we have, for \(u \in L^2\),

\[
|T_t u|^2 \leq e^{2\lambda t} T_t |u|^2
\]  

for \(u \in \text{Dom}(A_2)\) and \(g \in \text{Dom}(\mathcal{E})\) and we have

\[
\int_x \Gamma(u, u) \text{d}m \leq \mathcal{E}(u, u), \quad u \in \text{Dom}(\mathcal{E}).
\]  

**Proof.** By considering \(A - \lambda I\) in place of \(A\), we may assume that \(\lambda = 0\). Take any \(t > 0\) and fix it. For \(u \in \text{Dom}(A_2)\) and \(g \in L^2_+\), define \(\Phi(s), s \in [0, t]\) by

\[
\Phi(s) = (T_{t-s} |T_s u|^2 \mid g).
\]
Noting that $|T_t \cdot u|^2 \in \text{Dom}(A_1)$ and $T_t u \in \text{Dom}(A_2)$ and using (Γ), we have

$$
\Phi'(s) = (2T_{s \cdot} (A_2 T_t u \mid T_t u) - g) - (A_1 T_{s \cdot} |T_t u|^2 \mid g) \\
= (2(A_2 T_t u \mid T_t u) - A_1 |T_t u|^2 \mid T_t u) \\
\leq 0.
$$

Thus we have $\Phi(0) \geq \Phi(t)$, which implies (3.10) for $u \in \text{Dom}(A_3)$. The general case easily follows from this.

For $p \in [2, \infty]$, the contraction property of $\{e^{-\lambda T_t}\}$ on $L^p$ is easily obtained from (2.3):

$$
e^{-\lambda \cdot |T_t u|^2} \leq \{T_t |u|^2\}^{1/2} \leq \{T_t |u|^p\}^{1/p}.
$$

(Recall that when $p = \infty$, $T_t$ is defined on $L^{\infty \cap L^p}$. By duality, $\{e^{-\lambda T_t}\}$ is also a contraction semigroup on $L^p$ for $p \in [1, 2]$.

To see the strong continuity of $\{T_t\}$, we note that

$$
\int_X |u| \, dm \leq \lim_{\rho \to 0} \int_X |T_t u| \, dm \leq \lim_{\rho \to 0} \int_X |T_t u| \, dm \leq \int_X |u| \, dm.
$$

Then the rest is the same as the proof of Proposition 2.6.

To show (3.11), we repeat the argument in [6]. From the definition of $\Gamma$, we see that

$$
\int_X \Gamma(u, v) \, dm = \mathcal{E}(u, v) + \frac{1}{2} \int_X A_1 (u \mid v) \, dm. \quad (3.12)
$$

Further, by the contraction property of $\{T_t\}$, we have

$$
\int_X T_t |u|^2 \, dm \leq \int_X |u|^2 \, dm
$$

and thereby

$$
\int_X A_1 |u|^2 \, dm = \lim_{\rho \to 0} \int_X \frac{T_t |u|^2 - |u|^2}{t} \, dm \leq 0.
$$

Thus we have, for $u \in \text{Dom}(A_3)$,

$$
\int_X \Gamma(u, u) \, dm \leq \mathcal{E}(u, u). \quad (3.13)
$$
On the other hand, by the positivity of (3.9), we have the following Schwarz inequality:

\[ |\Gamma(u, v) + \lambda |u\, v|K| \leq \{ |\Gamma(u, u) + \lambda |u|\}^{1/2} \{ |\Gamma(v, v) + \lambda |v|\}^{1/2}. \quad (3.14) \]

Therefore we have

\[ ||\Gamma(u, v)||_1 \leq \{ E(u, u) + |u|\}^{1/2} \{ E(v, v) + |v|\}^{1/2} + |\lambda| \|u\|_2 \|v\|_2. \]

Now it is easy to see that \( \Gamma \) can be extended to continuous bilinear map from \( \text{Dom}(\mathcal{E}) \times \text{Dom}(\mathcal{E}) \) into \( L^1 \). This completes the proof. \( \square \)

We give a sufficient condition for the assumption of the above theorem.

**Proposition 3.2.** Assume that there exists a subspace \( \mathcal{D} \subseteq \text{Dom}(A_2) \) such that \( \mathcal{D} \) is a core for \( A_2 \) and \( (u \mid v)_K \in \text{Dom}(A_1) \) for \( u, v \in \mathcal{D} \). Further we suppose that there exists \( \lambda \in \mathbb{R} \) such that

\[ \Gamma(u, u) + \lambda |u|^2 \geq 0 \quad \text{for} \quad u \in \mathcal{D}. \quad (3.15) \]

Then the condition \( (\Gamma_K) \) holds.

**Proof.** By the same proof as in Theorem 3.1, we have

\[ \int_X \Gamma(u, u) \, dm \leq \mathcal{E}(u, u), \quad u \in \text{Dom}(\mathcal{E}). \]

From this, it is easy to see that \( \Gamma \) is well-defined on \( \text{Dom}(\mathcal{E}) \times \text{Dom}(\mathcal{E}) \) and \( \Gamma \) is a bounded sesquilinear operator from \( \text{Dom}(\mathcal{E}) \times \text{Dom}(\mathcal{E}) \) into \( L^1 \).

Recall that for \( u, v \in \mathcal{D} \),

\[ A_1(u \mid v)_K = 2\Gamma(u, v) + (A_2 u \mid v)_K + (u \mid A_2 v)_K. \]

Since the right hand side is continuous from \( \text{Dom}(A_2) \times \text{Dom}(A_2) \) to \( L^1 \) and \( \mathcal{D} \) is a core for \( \text{Dom}(A_2) \), we can deduce that \( (u \mid v)_K \in \text{Dom}(A_1) \) for \( u, v \in \text{Dom}(A_2) \). The rest is easy. \( \square \)

From now on we suppose that \( \mathcal{E} \) has the local property in the following sense (see [13, Definition I.5.1.2]):

\[ (L) \quad \text{For any real valued function} \ f \in \text{Dom}(\mathcal{E}), \ F, G \in C^\infty_0(\mathbb{R}), \]

\[ \text{supp} \, F \cap \text{supp} \, G = \emptyset \Rightarrow \mathcal{E}(F(x), G(x)) = 0, \quad (3.16) \]

where \( F_\alpha(x) = F(x) - F(0), \ G_\alpha(x) = G(x) - G(0) \).

The above condition is satisfied as soon as it is satisfied for each element of a dense subset of \( \text{Dom}(\mathcal{E}) \).
Under the condition \((L)\), the following identity holds:

\[
E(f, g) = \int_X F(f, g) \, dm. \tag{3.17}
\]

Moreover the following derivation property of \(\Gamma\) follows (for the proof, see, e.g., [13, Chap. I, Sect. 5]):

1. Let \(F: \mathbb{R}^n \to \mathbb{R}\) be a \(C^1\) function with \(F(0) = 0\). Then for real valued functions \(f_1, \ldots, f_n \in \text{Dom}(\ell) \cap L^\infty\), we have \(F(f_1, \ldots, f_n) \in \text{Dom}(\ell)\) and

\[
\Gamma(F(f_1, \ldots, f_n), g) = \sum_j \partial_j F(f_1, \ldots, f_n) \Gamma(f_j, g), \quad \forall g \in \text{Dom}(\ell). \tag{3.18}
\]

If, in addition, \(\partial_j F\) are bounded, then the above identity holds for \(f_1, \ldots, f_n \in \text{Dom}(\ell)\).

2. Let \(F: \mathbb{R}^n \to \mathbb{R}\) be a \(C^2\) function with \(F(0) = 0\). Then for real valued functions \(f_1, \ldots, f_n \in \text{Dom}(A_1) \cap L^\infty\), we have \(F(f_1, \ldots, f_n) \in \text{Dom}(A_1) \cap L^\infty\) and

\[
A_1 F(f_1, \ldots, f_n) = \sum_j \partial_j F(f_1, \ldots, f_n) A_1 f_j + \sum_{i,j} \partial_i \partial_j F(f_1, \ldots, f_n) \Gamma(f_i, f_j). \tag{3.19}
\]

Moreover if \(\partial_i, \partial_j, \partial_i \partial_j, F, i, j = 1, \ldots, n\) are all bounded and \(F(0) = 0\), then \(F(f_1, \ldots, f_n) \in \text{Dom}(A_1) \cap \text{Dom}(\ell)\) for \(f_1, \ldots, f_n \in \text{Dom}(A_1) \cap \text{Dom}(\ell)\) and the above identity holds.

In particular, the following identity is most commonly used: For \(f, g, h \in \text{Dom}(\ell) \cap L^\infty\),

\[
\Gamma(fg, h) = f \Gamma(g, h) + g \Gamma(f, h) \quad \text{for} \quad \forall h \in \text{Dom}(\ell). \tag{3.20}
\]

Now we can prove the main theorem in this section:

**Theorem 3.3.** Assume conditions \((\Gamma)\) and \((L)\). Moreover we assume the following:

1. If \(u \in \text{Dom}(\ell)\), then \(|u| \in \text{Dom}(\ell)\) and

\[
\ell(|u|, |u|) \leq \ell(u, u) + \lambda \|u\|_2^2. \tag{3.21}
\]

2. If \(u \in \text{Dom}(\ell) \cap L^\infty\) and \(f \in \text{Dom}(\ell) \cap L^\infty\), then \(fu \in \text{Dom}(\ell) \cap L^\infty\).
For $u \in \text{Dom}(\mathcal{E}) \cap L^\infty$ and $g \in \text{Dom}(\mathcal{E}) \cap L^\infty$, it holds that

$$\mathcal{E}(|u|, g |u|) \leq \mathcal{E}(u, gu) + \lambda \zeta(g, |u|^2).$$

(3.22)

Then $A_2 |u| \geq \Re((A - \lambda) u, \text{sgn} u)_X$ in the sense of distributions for $u \in \text{Dom}(A_2)$ and therefore the assumption of Theorem 2.4 is satisfied.

**Proof.** We may assume $\lambda = 0$. Take any $u \in \text{Dom}(A_2) \cap L^\infty$. For $\varepsilon > 0$, set $|u|_\varepsilon = \sqrt{|u|^2 + \varepsilon^2}$. Then $1/|u|_\varepsilon \in \text{Dom}(\mathcal{E})$. For $g \in \text{Dom}(A_2) \cap L^\infty$, Taking $f = g |u|_\varepsilon$ in (3.22), we have

$$\left( A_2 u \mid g |u|_\varepsilon \right) = -\mathcal{E}(u, g |u|_\varepsilon)
\leq -\mathcal{E}(|u|, g |u|_\varepsilon)
= -\int_X \Gamma(|u|, g |u|_\varepsilon) \, dm \quad (\because \text{(3.17)})
= -\int_X \left\{ g |u| \left( -\frac{|u|}{|u|_\varepsilon} \Gamma(|u|, |u|) + \frac{g}{|u|_\varepsilon} \Gamma(|u|, |u|) \right)
+ \frac{|u|}{|u|_\varepsilon} \Gamma(|u|, g) \right\} \, dm
= -\int_X \left\{ g |u| \left( -\frac{|u|}{|u|_\varepsilon} \Gamma(|u|, |u|) + \frac{|u|}{|u|_\varepsilon} \Gamma(|u|, |u|) \right) \right\} \, dm
\leq -\int_X \frac{|u|}{|u|_\varepsilon} \Gamma(|u|, g) \, dm.

Now letting $\varepsilon \to 0$, we have

$$\int_X \frac{|u|}{|u|_\varepsilon} \Gamma(|u|, g) \, dm \to \int_X 1_{\{|u| \neq 0\}} \Gamma(|u|, g) \, dm.$$

On the other hand, by the derivation property of $\Gamma$,

$$\Gamma(|u|^{1+\varepsilon}, |u|^{1+\varepsilon}) = (1 + \varepsilon)^2 |u|^{2\varepsilon} \Gamma(|u|, |u|).$$

Hence $1_{\{|u| \neq 0\}} \Gamma(|u|^{1+\varepsilon}, |u|^{1+\varepsilon}) = 0$. Letting $\varepsilon \to 0$, we have

$$1_{\{|u| = 0\}} \Gamma(|u|, |u|) = 0.$$

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Using the Schwarz inequality $|\Gamma(u, g)| \leq \Gamma(|u|, |g|)^{1/2} \Gamma(|u|, |u|)^{1/2}$, we obtain

$$1_{|u| = 0} \Gamma(|u|, g) = 0.$$ 

Now we eventually have

$$(A_2 u \mid g \text{ sgn } u) \leq -\int_X \Gamma(|u|, g) \, dn = -\mathcal{E}(|u|, g) = (|u| \mid A_2 g).$$

To see that the above inequality holds for $g \in \text{Dom}(A_2) \cap L^2_+$, we note that $\alpha G_n(g \wedge n) \in \text{Dom}(A_2) \cap L^\infty_+$ for $n \in \mathbb{N}$. Letting $n \to \infty$, we get the desired result. This completes the proof.

Now we introduce the following condition on $\Gamma$ and $\mathcal{E}$. It is related to the derivation property of $\Gamma$.

**D** For $u, v \in \text{Dom}(\mathcal{E}) \cap L^\infty$, $f \in \text{Dom}(\mathcal{E}) \cap L^\infty$, it holds that $(u \mid v)_K \in \text{Dom}(\mathcal{E})$, $fu \in \text{Dom}(\mathcal{E})$ and

$$2f\Gamma(u, v) = -\Gamma(f, (v \mid u)_K) + \Gamma(u, \tilde{f}v) + \Gamma(fu, v), \quad (3.23)$$

$$\Gamma(fu, u) = \Gamma(fu, u), \quad (3.24)$$

$$\mathcal{E}(fu, u) = \mathcal{E}(fu, u). \quad (3.25)$$

On the other hand, from the definition of $\Gamma$, for $u, v \in \text{Dom}(A_2) \cap L^\infty$ and $f \in \text{Dom}(\mathcal{E}) \cap L^\infty$,

$$2f\Gamma(u, v) = fA_2(u \mid v)_K - f(A_2u \mid v)_K - f(u \mid A_2v)_K.$$ 

Integrating both sides over $X$, we have

$$\int_X 2f\Gamma(u, v) = -\mathcal{E}(f, (v \mid u)_K) + \mathcal{E}(u, \tilde{f}v) + \mathcal{E}(fu, v). \quad (3.26)$$

The above identity holds for $u, v \in \text{Dom}(\mathcal{E}) \cap L^\infty$ and $f \in \text{Dom}(\mathcal{E}) \cap L^\infty$.

Now we return to (3.23) when $u = v$:

$$2f\Gamma(u, u) = -\Gamma(f, |u|^2) + \Gamma(u, \tilde{f}u) + \Gamma(fu, u)$$

$$= -\Gamma(f, |u|^2) + \Gamma(fu, u) + \Gamma(fu, u)$$

$$= -\Gamma(f, |u|^2) + 2\Gamma(fu, u).$$
We integrate both sides over $X$:

$$\int_X 2f \Gamma(u, u) = -\varepsilon(f, |u|^2) + 2 \int_X \Gamma(fu, u).$$

Comparing this with (3.26) and (3.25), we have

$$\int_X \Gamma(fu, u) = \varepsilon(fu, u).$$

By polarization, we eventually obtain

$$\int_X \Gamma(fu, v) = \varepsilon(fu, v). \quad (3.27)$$

We can give a sufficient condition for Theorem 3.3.

**Theorem 3.4.** Assume conditions $(\Gamma)$, $(L)$, $(\Gamma_\lambda)$, and $(D)$. Then, for $u \in \text{Dom}(\varepsilon)$, we have $|u| \in \text{Dom}(\varepsilon)$ and

$$\Gamma(|u|, |u|) \leq \Gamma(u, u) + \lambda |u|^2. \quad (3.28)$$

In addition, for $f \in \text{Dom}(\varepsilon) \cap L^\infty$ and $u \in \text{Dom}(\varepsilon) \cap L^\infty$, we have

$$\{\Gamma(fu, fu) + \lambda |fu|^2\}^{1/2} \leq |f| \{\Gamma(u, u) + \lambda |u|^2\}^{1/2} + |u| \Gamma(f, f)^{1/2}. \quad (3.29)$$

Furthermore the conditions of Theorem 3.3 are satisfied.

**Proof.** For simplicity, we give a proof in the case $\lambda = 0$. Take $u \in \text{Dom}(\varepsilon) \cap L^\infty$ and $f \in \text{Dom}(\varepsilon) \cap L^\infty$. From the assumption, $v = fu \in \text{Dom}(\varepsilon)$ and we substitute $v$ in (3.23):

$$\Gamma(fu, fu) + \Gamma(u, |f|^2 u) = 2f \Gamma(u, fu) + \Gamma(f, f |u|^2).$$

Hence

$$\Gamma(fu, fu) = -\Gamma(u, |f|^2 u) + 2f \Gamma(u, fu) + \Gamma(f, f |u|^2)$$

$$= -|f|^2 \Gamma(u, u) - \frac{1}{2} \Gamma(|f|^2, |u|^2) + f \{2 \Gamma(u, u) + \Gamma(f, |u|^2)\}$$

$$+ \Gamma(f, f |u|^2)$$

$$= -|f|^2 \Gamma(u, u) - \frac{1}{2} f \Gamma(f, |u|^2) - \frac{1}{2} f \Gamma(f, |u|^2) + 2 |f|^2 \Gamma(u, u)$$

$$+ f \Gamma(f, |u|^2) + \Gamma(f, f |u|^2) + |u|^2 \Gamma(f, f)$$

$$= |f|^2 \Gamma(u, u) + \frac{1}{2} f \Gamma(f, |u|^2) + \frac{1}{2} f \Gamma(f, |u|^2) + |u|^2 \Gamma(f, f).$$
In particular, if we take \( f = |u|^2 \), we get
\[
\Gamma(|u|^2, u, |u|^2) = |u|^4 \Gamma(u, u) + 2 |u|^2 \Gamma(|u|^2, |u|^2). \tag{3.30}
\]

On the other hand, substituting \( v = u \) and \( f = |u|^2 \) in (3.23), we have
\[
2 |u|^2 \Gamma(u, u) + \Gamma(|u|^2, |u|^2) = 2 \Gamma(|u|^2, u, u).
\]

Taking the square,
\[
4 |u|^4 \Gamma(u, u)^2 + 4 |u|^2 \Gamma(u, u) \Gamma(|u|^2, |u|^2) + \Gamma(|u|^2, |u|^2)^2
\leq 4 \Gamma(|u|^2, u, u)^2 \quad \text{(by the Schwarz inequality)}
\]
\[
= 4 \Gamma(|u|^2, u, u)^2 + 2 |u|^2 \Gamma(|u|^2, |u|^2) \Gamma(u, u) \quad \text{(... (3.30))}
\]
\[
= 4 |u|^4 \Gamma(u, u)^2 + 8 |u|^2 \Gamma(|u|^2, |u|^2) \Gamma(u, u).
\]

Thus we have
\[
\Gamma(|u|^2, |u|^2) \leq 4 |u|^2 \Gamma(u, u). \tag{3.31}
\]

Now for \( \varepsilon > 0 \), set \( \varphi_\varepsilon(t) = \sqrt{t} + \varepsilon - t \). Then by the derivation property and (3.31),
\[
\Gamma(\varphi_\varepsilon(|u|^2), \varphi_\varepsilon(|u|^2)) \leq \frac{1}{4(|u|^2 + \varepsilon)} \Gamma(|u|^2, |u|^2)
\leq \frac{4 |u|^2}{4(|u|^2 + \varepsilon)} \Gamma(u, u) \leq \Gamma(u, u).
\]

Letting \( \varepsilon \to 0 \), we easily obtain that \( |u| \in \text{Dom}(\delta) \) and
\[
\Gamma(|u|, |u|) \leq \Gamma(u, u).
\]

In fact, it is enough to take a sequence \( \{\varepsilon_n\} \) such that the Cesaro mean of \( \{\varphi_\varepsilon(|u|^2)\} \) converges to \( |u| \) in \( \text{Dom}(\delta) \).

Now we return to \( \Gamma(\mathcal{F}, \mathcal{F}) \):
\[
\Gamma(\mathcal{F}, \mathcal{F}) = |f|^2 \Gamma(u, u) + \frac{1}{2} f \Gamma(f, |u|^2) + \frac{1}{2} f \Gamma(f, |u|^2) + |u|^2 \Gamma(f, f)
\leq |f|^2 \Gamma(u, u) + |f|^2 \Gamma(|u|^2, |u|^2) + |u|^2 \Gamma(f, f)
\leq |f|^2 \Gamma(u, u) + 2 |f| |u| \Gamma(u, u)^{1/2} \Gamma(f, f)^{1/2} + |u|^2 \Gamma(f, f)
\leq \{|f|^2 \Gamma(u, u)^{1/2} + |u|^2 \Gamma(f, f)^{1/2}\}^2,
\]
which shows (3.29).
Next we check the conditions of Theorem 3.3. (1) is clear from (3.28). (2) is already assumed. We show (3). Take \( g \in \text{Dom}(\mathcal{E}) \cap L^\infty_x \) and \( u \in \text{Dom}(\mathcal{E}) \cap L^\infty_x \). Then, by (D) and (3.28),

\[
\mathcal{E}(u, gu) = \int \Gamma(u, gu) \, dm \quad (\therefore (3.27))
\]

\[
= \int \left\{ g \Gamma(u, u) + \frac{1}{2} \Gamma(g, |u|^2) \right\} \, dm
\]

\[
\geq \int \left\{ g \Gamma(|u|, |u|) + |u| \Gamma(g, |u|) \right\} \, dm
\]

\[
= \int \Gamma(g |u|, |u|) \, dm
\]

\[
= \mathcal{E}(g |u|, |u|).
\]

This completes the proof.

As before, it is sufficient to assume (D) for elements of a core. We state it as a proposition. We also include an analogue of (F').

**Proposition 3.5.** Assume that the assumptions (F') and (L) hold. Furthermore, assume that there exist an algebra \( \mathcal{E} \subseteq \text{Dom}(\mathcal{E}) \cap L^\infty_x \) and a subspace \( \mathcal{D} \subseteq \text{Dom}(A_2) \cap L^\infty_x \) such that for \( f \in \mathcal{E} \) and \( u, v \in \mathcal{D} \), we have

\[
\Gamma(fu, v) + \Gamma(u, fv) = 2f \Gamma(u, v) + \Gamma(f, (u | v)_x)
\]

(3.32)

\[
\Gamma(fu, u) = \overline{\Gamma(fu, u)}
\]

(3.33)

\[
\mathcal{E}(fu, u) = \overline{\mathcal{E}(fu, u)}
\]

(3.34)

\[
\Gamma(u, u) + \lambda |u|^2 \geq 0.
\]

(3.35)

Finally, we assume that \( \mathcal{E} \) is closed under complex conjugation and a core for \( \mathcal{E} \) and \( \mathcal{D} \) is a core for \( \mathcal{E} \). Then (F') and (D) are satisfied.

**Proof.** We may suppose \( \lambda = 0 \). Take \( u \in \mathcal{D} \) and \( f \in \mathcal{E} \). From the assumption, \( v = fu \in \mathcal{D} \). Now by the same argument as in the proof of Theorem 3.4, we have

\[
\Gamma(fu, fu) = f^2 \Gamma(u, u) + \frac{1}{2} f \Gamma(f, |u|^2) + \frac{1}{2} \overline{f} \Gamma(f, |u|^2) + |u|^2 \Gamma(f, f) \\
\leq f^2 \Gamma(u, u) + |f| \Gamma(f, f)^{1/2} \Gamma(|u|^2, |u|^2)^{1/2} + |u|^2 \Gamma(f, f).
\]
Clearly this shows that
\[ E(fu, fu) \leq 2 \int_X \Gamma(fu, fu) \, dm \]
\[ \leq 2 \|f\|^2 \int_X \Gamma(u, u) \, dm + 2 \|f\| \int_X \Gamma(f, f)^{1/2} \Gamma(|u|^2, |u|^2)^{1/2} \, dm \]
\[ + 2 \|u\|^2 \int_X \Gamma(f, f) \, dm \]
\[ \leq 2 \|f\|^2 \int_X \Gamma(u, u) \, dm \]
\[ + 2 \|f\| \left( \int_X \Gamma(f, f)^{1/2} \Gamma(|u|^2, |u|^2)^{1/2} \, dm \right)^{1/2} \]
\[ + 2 \|u\|^2 \int_X \Gamma(f, f) \, dm \]
\[ \leq 2 \|f\|^2 \, E(u, u) + 2 \|f\| \, E(f, f)^{1/2} \, E(|u|^2, |u|^2)^{1/2} + 2 \|u\|^2 \, E(f, f). \]

We claim that if \( f \in \text{Dom}(\mathcal{E}) \cap L^\infty, u \in \mathcal{D}, \) we have \( fu \in \text{Dom}(\mathcal{E}) \) and (3.32), (3.36) hold. To show this, take any real valued function \( f \in \mathcal{E}. \) Then, for any \( C^1 \)-function \( F: \mathbb{R} \to \mathbb{R}, \) we take a sequence of polynomials \( \{P_n\} \) such that \( P_n \to F \) uniformly on any compact sets up to the first derivative.

Since (3.32) holds for \( u \in \mathcal{D}, \) \( P_n(f) u \) is bounded and \( P_n(f) u \to F(f) u \) in \( L^2. \) Thus we have \( F(f) u \in \text{Dom}(\mathcal{E}) \) and

\[ 2F(f)(u, v) = \Gamma(F(f), (u \mid v)_w) - \Gamma(u, F(f) v) - \Gamma(F(f) u, v) \]
\[ \leq 2 \|F(f)\| \, E(u, u) + 2 \|F(f)\| \, E(F(f), f, f)^{1/2} \, E(|u|^2, |u|^2)^{1/2} \]
\[ + 2 \|u\|^2 \, E(F(f), f, f). \]

Now for any \( f \in \text{Dom}(\mathcal{E}) \cap L^\infty, \) we take a bounded \( C^1 \) function \( \varphi \) such that \( \varphi(t) = t \) for \( |t| \leq \|f\| \). Let \( \{f_n\} \subseteq \mathcal{E} \) be a sequence converging to \( f \) in \( \text{Dom}(\mathcal{E}). \) Clearly \( g_n = \varphi(3f_n) + \sqrt{-1} \varphi(3f_n) \to f \) weakly in \( \text{Dom}(\mathcal{E}) \) and \( \{g_n\} \) is uniformly bounded. Moreover, by (3.36) we have \( \sup_n E(g_n u, g_n u) < \infty. \) Hence we can extract a subsequence \( \{g_n u\} \) whose Cesaro mean converges strongly in \( \text{Dom}(\mathcal{E}). \) Together with the fact that \( g_n u \to fu \) in \( L^2, \) we can see that for \( f \in \text{Dom}(\mathcal{E}) \cap L^\infty \) and \( u \in \mathcal{D}, \) we have \( fu \in \text{Dom}(\mathcal{E}) \) and (3.32) hold.

Since (3.32) holds for \( u \in \mathcal{D} \) and \( f \in \text{Dom}(\mathcal{E}) \cap L^\infty, \) we can take \( f = |u|^2. \) By repeating the argument in the proof of Theorem 3.4, for \( u \in \mathcal{D} \) and \( f \in \text{Dom}(\mathcal{E}) \cap L^\infty, \) we have \( fu \in \text{Dom}(\mathcal{E}) \) and
\[ \Gamma(fu, fu)^{1/2} \leq |f| \Gamma(u, u)^{1/2} + |u| \Gamma(f, f)^{1/2} \]  
(3.38)

\[ \Gamma(|u|, |u|) \leq \Gamma(u, u). \]  
(3.39)

Since \( D \) is dense in \( \text{Dom}(\mathcal{E}) \), it is easy to see that (3.39) holds for \( u \in \text{Dom}(\mathcal{E}) \).

Now we can prove (3.32) for \( f \in \text{Dom}(\mathcal{E}) \cap L^\infty \) and \( u \in \text{Dom}(\mathcal{E}) \cap L^\infty \). Take any \( u \in D \). For \( \varepsilon > 0 \), get

\[ \psi_\varepsilon(t) = \frac{1}{1 + \varepsilon t}, \quad t \in [0, \infty). \]

Note that \( \psi_\varepsilon' = -\varepsilon/(1 + \varepsilon t)^2 \). Since \( \psi_\varepsilon(|u|) \in \text{Dom}(\mathcal{E}) \), we have \( \psi_\varepsilon(|u|) u \in \text{Dom}(\mathcal{E}) \), and

\[
\Gamma(f\psi_\varepsilon(|u|) u, f\psi_\varepsilon(|u|) u)^{1/2} \\
\leq |f| \psi_\varepsilon(|u|) \Gamma(u, u)^{1/2} + |u| \Gamma(f\psi_\varepsilon(|u|), f\psi_\varepsilon(|u|))^{1/2} \\
\leq |f| \psi_\varepsilon(|u|) \Gamma(u, u)^{1/2} + |u| \Gamma(f, f) \\
+ 2\psi_\varepsilon(|u|) \psi'_\varepsilon(|u|) f\Gamma(|u|, f) + f^2 \psi'_\varepsilon(|u|)^2 \Gamma(|u|, |u|)^{1/2} \\
\leq |f| \psi_\varepsilon(|u|) \Gamma(u, u)^{1/2} + |u| \Gamma(f, f) \\
+ 2\psi_\varepsilon(|u|) \psi'_\varepsilon(|u|) f\Gamma(|u|, |u|)^{1/2} \Gamma(f, f)^{1/2} \\
+ f^2 \psi'_\varepsilon(|u|)^2 \Gamma(|u|, |u|)^{1/2} \\
\leq |f| \psi_\varepsilon(|u|) \Gamma(u, u)^{1/2} + |u| \psi_\varepsilon(|u|) \Gamma(f, f)^{1/2} \\
+ |u| \frac{1}{|u|} |\psi_\varepsilon(|u|)| \Gamma(u, u)^{1/2}.
\]

Now for any \( v \in \text{Dom}(\mathcal{E}) \cap L^\infty \), we take a sequence \( \{u_n\} \subseteq D \) converging to \( v \) in \( \text{Dom}(\mathcal{E}) \). Then by the above estimate, we can see that

\[ \sup_n \mathcal{E}(f\psi_\varepsilon(|u_n|) u_n, f\psi_\varepsilon(|u_n|) u_n) < \infty. \]

Hence, by virtue of the Banach–Saks theorem, we have, by taking a subsequence if necessary, the Cesaro mean of \( \{f\psi_\varepsilon(|u_n|) u_n\} \) converges to \( f\psi_\varepsilon(|u|) u \) in \( \text{Dom}(\mathcal{E}) \). Moreover, note that

\[
\Gamma\left(\frac{1}{n} \sum_{k=1}^{n} f\psi_\varepsilon(|u_k|) u_k, \frac{1}{n} \sum_{k=1}^{n} f\psi_\varepsilon(|u_k|) u_k\right)^{1/2} \\
\leq \frac{1}{n} \sum_{k=1}^{n} \Gamma(f\psi_\varepsilon(|u_k|) u_k, f\psi_\varepsilon(|u_k|) u_k)^{1/2} \\
\leq \frac{1}{n} \sum_{k=1}^{n} \left( |f| \psi_\varepsilon(|u_k|) \Gamma(u_k, u_k)^{1/2} + |u_k| \psi_\varepsilon(|u_k|) \Gamma(f, f)^{1/2} + |u_k| \frac{1}{|u_k|} |\psi_\varepsilon(|u_k|)| \Gamma(u_k, u_k)^{1/2}\right).
\]
Letting \( n \to \infty \), we have for \( v \in \text{Dom}(\mathcal{E}) \cap L^\infty \),
\[
\Gamma(f\psi | (|v| \psi v) v)^{1/2} \leq [f | \psi (|v| \psi v) \Gamma(v, v)^{1/2} + |v| \psi \Gamma(f, f)]^{1/2}.
\]
\[ + |v| \Gamma(f, f) \]
Next, letting \( \varepsilon \to 0 \), we eventually obtain that \( f \psi, f \psi \in \text{Dom}(\mathcal{E}) \) and
\[
\Gamma(f \psi, f \psi)^{1/2} \leq [f | \psi (|v| \psi v) \Gamma(v, v)^{1/2} + |v| \psi \Gamma(f, f)]^{1/2}.
\]
Next, we see that (3.32) holds for \( u, v \in \text{Dom}(\mathcal{E}) \cap L^\infty \) and \( f \in \text{Dom}(\mathcal{E}) \). To see this, we first note the following identity: for \( u, v \in \mathcal{E} \) and \( f, \psi \in \text{Dom}(\mathcal{E}) \),
\[
2\Gamma(\psi u, v) = \Gamma(f \psi, \psi u) - \Gamma(\psi u, f) - \Gamma(f \psi, v).
\]
This identity is clear from the assumption if \( \psi \in C \). By an approximating argument, we can see it for \( \psi \in \text{Dom}(\mathcal{E}) \cap L^\infty \). Using this, we can repeat the above argument and get the desired result.

So far, we have obtained that \( \text{Dom}(\mathcal{E}) \cap L^\infty \) is a \( \text{Dom}(\mathcal{E}) \cap L^\infty \)-module. Using this fact, we shall show (1).

Take \( u, v \in \mathcal{D} \) and \( f \in \text{Dom}(A_2) \cap L^\infty \). Then
\[
(A_2 f \ | (u | v)_{\lambda}) = (f \ | A_1 (u | v)_{\lambda})
\]
\[ = (f, (A_2 u | v)_{\lambda} + (A_2 v | f u) + (f | 2 \Gamma(u, v))
\]
\[ = -\delta(f, u) + (A_2 v | f u) + (f | 2 \Gamma(u, v)).
\]
Now, for \( u \in \text{Dom}(A_2) \cap L^\infty \), we take a sequence \( \{u_n\} \subseteq \mathcal{D} \) that converges to \( u \) in \( \text{Dom}(\mathcal{E}) \). Then, for \( v \in \mathcal{D} \) and \( f \in \text{Dom}(A_2) \cap L^\infty \),
\[
(A_2 f \ | (u | v)_{\lambda}) = \lim_{n \to \infty} (A_2 f \ | (u_n | v)_{\lambda})
\]
\[ = -\lim_{n \to \infty} \delta(f, u_n) + (A_2 v | f u_n)
\]
\[ + \lim_{n \to \infty} (f | 2 \Gamma(u_n, v))
\]
\[ = -\delta(f, u) + (A_2 v | f u) + (f | 2 \Gamma(u, v))
\]
\[ = (fv | A_2 u) - \delta(v, f u) + (f | 2 \Gamma(u, v)).
\]
Second for \( v \in \text{Dom}(A_2) \cap L^\infty \), we take a sequence \( \{v_n\} \subset \mathcal{D} \) that converges to \( v \) in \( \text{Dom}(\mathcal{E}) \). Then, for \( u \in \text{Dom}(A_2) \cap L^\infty \) and \( f \in \text{Dom}(A_2) \cap L^\infty \),

\[
(A_2 f \mid (u \mid v)_K) = \lim_{n \to \infty} (A_2 f, (u \mid v_n)_K) \\
= \lim_{n \to \infty} (A_2 u \mid f v_n) - \lim_{n \to \infty} \mathcal{E}(v_n | f u) + \lim_{n \to \infty} (f, 2 \Gamma(u, v_n)) \\
= (A_2 u | f v) - \mathcal{E}(v, f u) + (f | 2 \Gamma(u, v)) \\
= (A_2 u, f v) + (A_2 v | f u) + (f | 2 \Gamma(u, v)) \\
= (f \mid (A_2 u | v)_K + (u | A_2 v)_K + 2 \Gamma(u, v)).
\]

Hence we have, for \( u, v \in \text{Dom}(A_2) \cap L^\infty \),

\[
A_1(u \mid v)_K = (A_2 u | v) + (u | A_2 v) + 2 \Gamma(u, v).
\]

Noticing that \( \text{Dom}(A_2) \cap L^\infty \) is dense in \( \text{Dom}(A_2) \), we get the desired result.

This completes the proof. \( \Box \)

4. PERTURBATION OF A CONTRACTION SEMIGROUP

So far, our basic semigroup is a contraction Markovian semigroup. In this section, we discuss perturbed semigroups. Let \( \{T_t\} \) and \( \{T'_t\} \) be two semigroups as in the previous section. Our basic assumption is that \( \{T_t\} \) satisfies hyperboundedness. Under this assumption, the perturbation theory was developed by Segal, Høegh-Krohn, Simon (see [20, 32, 36]). We shall deal with the semigroup \( \{T'_t\} \) acting on vector valued functions. We mainly follow the Segal method and make use of comparison between semigroups.

Suppose that a real valued function \( V \) and \( \mathcal{S}(K) \)-valued function \( R \) are given. Here, \( \mathcal{S}(K) \) is the set of all bounded symmetric operators on \( K \) and the norm in \( \mathcal{S}(K) \) is the operator norm, which we denote by \( \| \cdot \|_{op} \). We always assume that \( V \) and \( R \) are measurable.

We consider the quadratic forms

\[
\mathcal{E}^V(f, g) = \mathcal{E}(f, g) + \int_X Vfg \, dm \tag{4.1}
\]

This completes the proof. \( \Box \)
and
\[ \mathcal{E}^R(u, v) = \mathcal{E}(u, v) + \int_X (Ru \, | \, v) \, dm, \quad (4.2) \]

where \( \mathcal{E} \) and \( \mathcal{E}^R \) are the quadratic forms associated with semigroups \( \{T_t\} \) and \( \{T^R_t\} \), respectively.

First we consider the case where \( V \) and \( R \) are bounded. In this case, \( \mathcal{E}^V \) and \( \mathcal{E}^R \) are closed quadratic forms with the same domains as \( \mathcal{E} \) and \( \mathcal{E}^R \), respectively.

**Proposition 4.1.** Assume that \( V \) and \( R \) are bounded and
\[ V(x) \, |k|^2 \leq (R(x) \, k \, k)_x, \quad \forall k \in K \quad a.e. \ x. \quad (4.3) \]

Then \( T^V_t \) is a positivity preserving semigroup and
\[ |T^R_t u| \leq T^V_t |u|. \quad (4.4) \]

**Proof.** We can easily see that \( \mathcal{E}^V(\{f\}, \{f\}) \leq \mathcal{E}^V(\{f\}, \{f\}) \) and hence \( T^V_t \) is positivity preserving (see, e.g., [29, Theorem XIII.50]).

Next we prove (4.4). By the Trotter product formula, we have
\[ T^V_t f = \lim_{n \to \infty} (e^{-tV_n}T^{V_n})^n f \]
and
\[ T^R_t u = \lim_{n \to \infty} (e^{tR_n}T^{R_n})^n u. \]

By noting that
\[ |T_t u| \leq T_t |u|, \]
and
\[ \|e^{-tR(x)}\|_{op} \leq e^{-tV(x)} \quad a.e. \ x, \]
we have
\[ |T^R_t u| = \lim_{n \to \infty} |(e^{-tR_n}T^{R_n})^n u| \leq \lim_{n \to \infty} (e^{-tV_n}T^{V_n})^n |u| = T^V_t |u|. \]

This completes the proof.
We further assume the following condition (DLS) (the defective logarithmic Sobolev inequality):

\[(DLS) \quad \text{There exist } \alpha > 0, \beta \geq 0 \text{ such that}\]
\[
\int_X f^2 \log(f^2/\|f\|_2^2) \, dm \leq \alpha \mathcal{E}(f, f) + \beta \|f\|_2^2 \quad \text{for } f \in \text{Dom}(\mathcal{E}). \quad (4.5)
\]

Then, as we mentioned after Theorem 2.5, we have a similar inequality for $\mathcal{E}$:

\[
\int_X |u|^2 \log(|u|^2/\|u\|_2^2) \, dm \leq \alpha \mathcal{E}(u, u) + \beta \|u\|_2^2 \quad \text{for } u \in \text{Dom}(\mathcal{E}). \quad (4.6)
\]

Under the above assumption, we have, for $t > 0$ and $1 < p < q < \infty$ with $(q - 1)/(p - 1) \leq e^{\alpha t}$,

\[
\| T_t \|_{p \to q} \leq \exp \left\{ \beta \left( \frac{1}{p} - \frac{1}{q} \right) \right\} \| e^{-V} \|_p e^{St},
\]

(see [14, Theorem 6.1.14; 18, Lemma 5.5]).

**Proposition 4.2.** Assume that (DLS) holds. Then for $t > 0$ and $1 < p < q < \infty$ such that $(q - 1)/(p - 1) \leq e^{\alpha t}$, with $x' > x$, we have

\[
\| T_t^x \|_{p \to q} \leq \exp \left\{ \beta \left( \frac{1}{p} - \frac{1}{q} \right) \right\} \| e^{-V} \|_p e^{St},
\]

where

\[
r = \frac{1}{4} \left( \frac{1}{x} - \frac{1}{x'} \right)^{-1} \left( \frac{p^2}{p - 1} \vee \frac{q^2}{q - 1} \right), \quad S = \frac{4\beta}{\alpha},
\]

where $\vee$ denotes the maximum. In particular, for any $t > 0$,

\[
\| T_t^x \|_{p \to p} \leq \| e^{-V} \|_{\mathfrak{p}^q; \mathfrak{p}^{-1}} e^{S p t}. \quad (4.9)
\]

**Proof.** Define $x''$ so that $(q - 1)/(p - 1) = e^{\alpha t}$. For $n \in \mathbb{N}$, we define a sequence \( \{p_0, p_1, \ldots, p_n\} \) inductively as follows:

\[
p_0 = p, \quad e^{4t\alpha} = \frac{p_{k+1} - 1}{p_k - 1}, \quad k = 0, \ldots, n - 1.
\]
Clearly $p_n = q$. Moreover, define $q_k$ and $r_k$ as
\[
 e^{4/n} = \frac{q_k - 1}{p_k - 1}
\]
and
\[
 \frac{1}{p_{k+1}} = \frac{1}{r_k} \frac{1}{q_k}.
\]
Then
\[
 \|e^{-tV_n} T_{i,n} f\|_{p_{k+1}} \leq \|e^{-tV_n} r_k \|_{p_k} f_{p_k} \|\|_{p_k}
\]
\[
 \leq \|e^{-tV_n} r_k \|_{p_k} \|T_{i,n} f\|_{q_k} \|f\|_{p_k}
\]
\[
 \leq \|e^{-tV_n} r_k\|_{p_k} \exp \left\{ 4\beta \left( \frac{1}{p_k} - \frac{1}{q_k} \right) \right\} \|f\|_{p_k}.
\]
On the other hand
\[
 \frac{1}{r_k} = \frac{1}{p_{k-1}} - \frac{1}{q_k}
\]
\[
 = \frac{1}{e^{4/n} (p_k - 1) + 1 - e^{4/n} (p_k - 1) + 1}
\]
\[
 = \frac{1}{(e^{4/n} (p_k - 1) + 1)(e^{4/n} (q_k - 1) + 1)}.
\]
In general, for $0 < a < b$, set
\[
 f(x) = \frac{1}{ax + 1} - \frac{1}{bx + 1}, \quad x \geq 0.
\]
Then,
\[
 f'(x) = -\frac{a}{(ax + 1)^2} + \frac{b}{(bx + 1)^2}
\]
\[
 = \frac{(b-a)(1-abx^2)}{(ax + 1)^2 (bx + 1)^2}.
\]
f reaches its maximum at \( x = (ab)^{-1/2} \). Note that

\[
\begin{align*}
\sqrt{b} - \sqrt{a} &= \frac{\sqrt{b}}{\sqrt{a + \sqrt{b}}} - \frac{\sqrt{a}}{\sqrt{a + \sqrt{b}}} \\
&= \frac{\sqrt{b} - \sqrt{a}}{\sqrt{a + \sqrt{b}}}
\end{align*}
\]

Hence we have

\[
\begin{align*}
\frac{(e^{4i/n} - e^{4i/n^*})}{(p - 1)}& \left( e^{4i/n}(p - 1) + 1 \right) \\
& \leq \frac{1}{r_k} \leq \frac{e^{2i/n} - e^{2i/n^*}}{e^{2i/n} + e^{2i/n^*}}
\end{align*}
\]

and therefore

\[
\begin{align*}
& P_n := \frac{(e^{4i/n}(p - 1) + 1)(e^{4i/n}(p - 1) + 1)}{(e^{4i/n} - e^{4i/n^*})(p - 1)} \\
& \geq r_k \\
& Q_n := \frac{e^{2i/n} - e^{2i/n^*}}{e^{2i/n} + e^{2i/n^*}} \geq \frac{1}{r_k}.
\end{align*}
\]

Further,

\[
\begin{align*}
\sum_{k=0}^{n-1} \left( \frac{1}{p_k - q_k} \right) &= \sum_{k=0}^{n-1} \left( \frac{1}{p_k - p_{k+1} + 1} + \frac{1}{r_k} \right) \\
&= \frac{1}{p} + \sum_{k=0}^{n-1} \frac{1}{q_k} + \frac{1}{r_k} \\
&\leq \frac{1}{p} + \frac{1}{q} + nQ_n.
\end{align*}
\]
Now
\[
\| (e^{-iV/n}T_{v/n})^{n} \|_{p \to q} \leq \prod_{k=0}^{n-1} \| e^{-iV/n}T_{v/n} \|_{p_{k} \to p_{k+1}}
\]
\[
\leq \prod_{k=0}^{n-1} \| e^{-iV/n} \|_{p_{k}} \exp \left\{ 4\beta \left( \frac{1}{p_{k}} - \frac{1}{q_{k}} \right) \right\}
\]
\[
\leq \| e^{-iV/n} \|_{p_{k}} \exp \left\{ 4\beta \left( \frac{1}{p} - \frac{1}{q} + nQ_{n} \right) \right\}
\]
\[
\leq \exp \left\{ 4\beta \left( \frac{1}{p} - \frac{1}{q} \right) \right\} \left\{ \int_{X} e^{-iPr_{n}dn} \right\}^{n/P_{n}} e^{4\beta nQ_{n}}
\]

Letting \( n \to \infty \), we have
\[
P_{n} = \frac{(e^{4\beta/n}(p-1)+1)(e^{4\beta/n}(p-1)+1)}{n(e^{4\beta/n}-e^{4\beta/n})(p-1)}
\]
\[
\sqrt{\frac{(e^{4\beta/n}(q-1)+1)(e^{4\beta/n}(q-1)+1)}{n(e^{4\beta/n}-e^{4\beta/n})(q-1)}}
\]
\[
- \frac{1}{4t} \left\{ \frac{1}{x} - \frac{1}{x^2} \right\}^{-1} \left( \frac{p^2}{p-1} \vee \frac{q^2}{q-1} \right) \leq U/t
\]
and
\[
4\beta nQ_{n} = \frac{4\beta n(e^{2\beta/n} - e^{2\beta/n})}{e^{2\beta/n} + e^{2\beta/n}} \to 4\beta t \left\{ \frac{1}{x} - \frac{1}{x^2} \right\} \leq St.
\]

Thus we have
\[
\| T_{v/n}^{n} \|_{p \to q} \leq \exp \left\{ 4\beta \left( \frac{1}{p} - \frac{1}{q} \right) \right\} \| e^{-V} \|_{p} e^{St}.
\]

Lastly, we note that when \( q \) tends to \( p \), \( x^* \) tends to \( \infty \) and hence we have (4.9). \( \blacksquare \)

**Proposition 4.3.** Suppose that there exists \( \tilde{x} > x \) such that \( \exp \{ \tilde{x} \| R \|_{\text{sup}} \} \in L^{1} \). Then
\[
\mathcal{E}^{R}(u, v) := \mathcal{E}(u, v) + \int_{X} (Ru \mid v)_{\kappa} dn
\]
is well defined for \( u, v \in \text{Dom}(\mathcal{E}) \) as a lower semibounded closed quadratic form in \( L^{2}(m) \). Moreover if \( \mathcal{D} \subseteq \text{Dom}(\mathcal{E}) \) is dense in \( \text{Dom}(\mathcal{E}) \), then it is dense in \( \text{Dom}(\mathcal{E}^{R}) \) also.
Proof. By the Hausdorff–Young inequality $st \leq s \log s + e'$, $s > 0$, $t \in \mathbb{R}$, we have for $u \in \text{Dom}(\mathcal{E})$ with $\|u\|_2 = 1$,

$$\int_X |(Ru\mid u)_{X}| \, dm \leq \int_X \|R\|_{\text{op}} |u|^2 \, dm$$

$$= \frac{1}{2} \int_X \{|u|^2 (\tilde{\alpha} \|R\|_{\text{op}})\} \, dm$$

$$= \frac{1}{2} \int_X \{ |u|^2 \log |u|^2 - |u|^2 + \exp(\tilde{\alpha} \|R\|_{\text{op}})\} \, dm$$

$$= \frac{1}{2} \{ \pi E(u, u) + \beta - 1 + \|\exp(\tilde{\alpha} \|R\|_{\text{op}})\|_1 \}.$$ 

For general $u$, we have

$$\int_X |(Ru\mid u)_{X}| \, dm = \frac{\pi}{2} E(u, u) + \frac{\beta}{2} - 1 + \|\exp(\tilde{\alpha} \|R\|_{\text{op}})\|_1 \|u\|_2.$$ 

Now, by the KLMN theorem, the assertion follows.

We denote the semigroups associated with $\mathcal{E}^R$ and $\mathcal{E}^V$ by $\{T^R_t\}$ and $\{T^V_t\}$, respectively. We have the following comparison theorem.

**Theorem 4.4.** We have

$$|T^R_t u| \leq T^V_t |u|. \quad (4.10)$$

To prove this theorem, we need an approximation argument. We set

$$X_n = \{x \in X; |R(x)|_{\text{op}} \leq n, |V(x)| \leq n\}$$

and define

$$R_n = 1_{X_n} R, \quad V_n = 1_{X_n} V.$$ 

We shall show the convergence of $\mathcal{E}^R$. To do this, we use the convergence of quadratic forms. We say that a sequence of quadratic forms $\mathcal{E}^n$ converges to $\mathcal{E}$ in the sense of Mosco (see, e.g., [27]) if

$$(M.J.) \quad \text{For any sequence } \{u_n\} \text{ that converges to } u \text{ weakly, it holds that}$$

$$\mathcal{E}(u, u) \leq \lim_{n \to \infty} \mathcal{E}^n(u_n, u_n). \quad (4.11)$$
(M.2) For any \( u \in H \), there exists a sequence \( \{ u_n \} \) converging to \( u \) strongly, such that

\[
\mathcal{E}(u, u) \geq \lim_{n \to \infty} \mathcal{E}''(u_n, u_n). \tag{4.12}
\]

**Proposition 4.5.** Suppose that there exists \( \alpha' < \alpha \) such that \( \exp\{2\alpha' \| R\|_{op} \} \in L^1 \). Then \( \mathcal{E}_N \) converges to \( \mathcal{E}^\alpha \) in the sense of Mosco.

**Proof.** We first show (M.1). Let \( \{ u_n \} \) be a sequence that converges to \( u \) weakly. We may assume that \( \{ \mathcal{E}_N(u_n, u_n) \} \) is a converging sequence. Since there exist constants \( a, b > 0 \) (independent of \( n \)) such that

\[
\mathcal{E}_N(u, u) \leq a \mathcal{E}^\alpha(u, u) + b \| u \|^2_2.
\]

Hence \( \{ u_n \} \) is a bounded sequence in \( \text{Dom}(\mathcal{E}^\alpha) \) and \( u \in \text{Dom}(\mathcal{E}) \). By lower semi-continuity,

\[
\mathcal{E}(u, u) + \int_X (Ru | u)_K \, dm - \int_X 1_{X \setminus X_N} \| R\|_{op} |u|^2 \, dm
\]

\[
\leq \lim_{n \to \infty} \left\{ \mathcal{E}(u_n, u_n) + \int_X (Ru_n | u_n)_K \, dm - \int_X 1_{X \setminus X_N} \| R\|_{op} |u_n|^2 \, dm \right\}
\]

\[
\leq \lim_{n \to \infty} \left\{ \mathcal{E}(u_n, u_n) + \int_X (Ru_n | u_n)_K \, dm \right\} - \lim_{n \to \infty} \int_X 1_{X \setminus X_N} \| R\|_{op} |u_n|^2 \, dm.
\]

Hence

\[
\lim_{n \to \infty} \int_X \| R\|_{op} 1_{X \setminus X_N} |u_n|^2 \, dm
\]

\[
\leq \mathcal{E}(u, u) + \int_X (Ru | u)_K \, dm - \int_X 1_{X \setminus X_N} \| R\|_{op} |u|^2 \, dm
\]

\[- \lim_{n \to \infty} \left\{ \mathcal{E}(u_n, u_n) + \int_X (Ru_n | u_n)_K \, dm \right\}.
\]

Since the left hand side is independent of \( N \), by letting \( N \to \infty \), we have

\[
\lim_{n \to \infty} \int_X 1_{X \setminus X_N} \| R\|_{op} |u_n|^2 \, dm
\]

\[
\leq \mathcal{E}(u, u) + \int_X (Ru | u)_K \, dm - \lim_{n \to \infty} \left\{ \mathcal{E}(u_n, u_n) + \int_X (Ru_n | u_n)_K \, dm \right\}.
\]
We now turn to $E^R_n(u, u)$:

$$
\lim_{n \to \infty} E^R_n(u, u) \\
= \lim_{n \to \infty} \left\{ E(u, u) + \int_X (R_n u_n | u_n) \, dm \right\} \\
= \lim_{n \to \infty} \left\{ E(u, u) + \int_X (R_n u_n | u_n) \, dm + \int_X (1_{X \setminus X_n} R_n u_n | u_n) \, dm \right\} \\
\geq \lim_{n \to \infty} \left\{ E(u, u) + \int_X (R_n u_n | u_n) \, dm - \int_X 1_{X \setminus X_n} |R||u_n|^2 \, dm \right\} \\
= \lim_{n \to \infty} \left\{ E(u, u) + \int_X (R_n u_n | u_n) \, dm \right\} - \lim_{n \to \infty} \int_X 1_{X \setminus X_n} |R||u_n|^2 \, dm \\
\geq \lim_{n \to \infty} \left\{ E(u, u) + \int_X (R_n u_n | u_n) \, dm \right\} + E(u, u) + \int_X (R u | u) \, dm \\
- \lim_{n \to \infty} \left\{ E(u, u) + \int_X (R_n u_n | u_n) \, dm \right\} \\
= E(u, u) + \int_X (R u | u) \, dm.
$$

Next we prove (M.2). For any $u \in \text{Dom}(E) = \text{Dom}(E^R)$, we take $u_n = u$.

Then, by the dominated convergence theorem, we have

$$E^R_n(u, u) = E(u, u) + \int_X (R_n u | u) \, dm \to E(u, u) + \int_X (R u | u) \, dm = E^R(u, u).$$

This completes the proof. \[\square\]

**Proof of Theorem 4.4.** By Proposition 4.1, we have

$$|T^{R_n}u| \leq T^{R_n} |u|.$$  

By letting $n \to \infty$, we can get the desired result. \[\square\]

We have proved the strong convergence of $\{ T^{R_n} \}$. But we can even obtain convergence in norm.

**Proposition 4.6.** Suppose that $\exp\{ |R||u_n| \} \in L^\infty$. Then $\{ T^{R_n} \}$ is a strongly continuous semigroup in $L^p$ for any $p > 1$.

Moreover, $\{ T^{R_n} \}$ converges to $\{ T^R \}$ in norm sense as an operator in $L^p$.  

---

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Proof. We follow the Segal method. We take $V = -\|R\|_{\infty}$ and let $R_n$ be the same as before. We first note the following Duhamel formula:

$$T_t^R u = T_t^{R_n} u + \int_0^t T_s^{R_m} (R_m - R_n) T_s^{R_n} u \, ds.$$ 

We choose $q' < p < q$ such that $(q - 1)/(p - 1) < e^{2\alpha}$, $(p - 1)/(q' - 1) < e^{2\alpha}$. By Proposition 4.2, there exist constants $C > 0$ and $M > 0$ such that for $s \geq t/2$,

$$\|T_s^{R_n}\|_{p \to q} \leq CM^s,$$

$$\|T_s^{R_n}\|_{q' \to p} \leq CM^s,$$

and for all $s > 0$,

$$\|T_s^{R_n}\|_{q' \to p} \leq M^s.$$ 

We choose $r > 1$ such that

$$\frac{1}{q} \geq \frac{1}{p} + \frac{1}{r},$$

$$\frac{1}{r} \geq \frac{1}{q} + \frac{1}{q').$$

Then, by the Hölder inequality, we have

$$\|T_t^{R_n} - T_t^{R_m} u\|_{p} \leq \int_0^{t/2} \|T_t^{R_m} - R_m\|_r \|T_s^{R_m}\|_{p \to p} \|u\|_p \, ds + \int_{t/2}^t \|T_s^{R_m} - R_m\|_r \|T_s^{R_m} u\|_{p \to q} \|u\|_p \, ds.$$ 

Therefore

$$\|T_t^{R_n} - T_t^{R_m}\|_{p \to p} \leq CM^t \|R_m - R_n\|.$$ 

Now it is easy to see that $T_t^{R_n}$ is a Cauchy sequence with respect to the operator norm in $L^p$. But $T_t^{R_n}$ converges to $T_t^R$ strongly in $L^2$. Hence the limit of $T_t^{R_n}$ is $T_t^R$. This completes the proof. 

Now we give a sufficient condition for essential self-adjointness of $A^R$.

**Theorem 4.7.** For any $p \in (1, \infty)$, a subspace $D$ is dense in $\text{Dom}(A^R_p)$ if it is dense in $\text{Dom}(A^R_q)$ for some $q > p$. 
Proof. We mimic the method of Wu [38, Theorem 2.5]. Let $q'$ be the conjugate exponent of $q$: $1/q + 1/q' = 1$. We first show that for any $r < q$,\[ \text{Dom}(A_q) \subseteq \text{Dom}(A_{q'}) \text{, and for } u \in \text{Dom}(A_q), \]
\[ A_{q'}^r u = A_q u - Ru. \] (4.13)
To show this, we first note that
\[ G_{q'}^r(A_q u - Ru) = G_{q'}^r(A_q u - R_s u + R_n u - Ru) \]
\[ = - u + \pi G_{q'}^r u + G_{q'}^r(R_n u - Ru). \]
By the Hölder inequality
\[ \|G_{q'}^r(R_n u - Ru)\|_x \leq \|G_{q'}^r\|_{p_1 \rightarrow p_2} \|R_n - R\|_x \|x\|_q \rightarrow 0. \]
Here we take $r$ so that $1/r = 1/q' + 1/s$. We recall that $G_{q'}^r$ converges to $G_{q'}^s$ in norm sense by Proposition 4.6. Hence, by letting $n \rightarrow \infty$, we have
\[ G_{q'}^r(A_q u - Ru) = - u + \pi G_{q'}^s u \]
which means (4.13).
We show that $(x - A_{q'})(D)$ is dense in $L^p$. To see this, let $p'$ and $q'$ be the conjugate exponents of $p$ and $q$, respectively: $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. We take $v \in L^{p'}$ perpendicular to $(x - A_{q'})(D)$. For any $w \in D$,
\[ (v + G_s(Rv) | (x - A_q) w) = (v | (x - A_p) w) + (Rv | G_s(x - A_q) w) \]
\[ = (v | (x - A_p + R) w) \]
\[ = 0. \]
From the assumption, $(x - A_q)(D)$ is dense in $L^s$ and therefore
\[ v = - G_s(Rv). \]
Since $Rv \in L^{p'}$, $G_{q'}^s(Rv) \in \text{Dom}(A_q)$, we have $v \in \text{Dom}(A_q)$ and
\[ (x - A_q) v = - Rv. \]
Hence, by (4.13), we have
\[ (x - A_{q'}) v = 0. \]
Since $x - A_{q'}$ is injective, we have $v = 0$. This completes the proof. \[\Box\]
5. EXAMPLES

In this section, we give examples.

Our main interest is in infinite dimensional spaces and so we first take an abstract Wiener space. Let \( (B, H, \mu) \) be an abstract Wiener space, i.e., \( B \) is a real separable Banach space and \( \mu \) is a Gaussian measure with a reproducing kernel Hilbert space \( H \). Let \( F = (F^1, \ldots, F^d) \) be a smooth function in the sense of Malliavin. We further assume that \( F \) is non-degenerate in the following sense: setting \( _\sigma = (\sigma^{i,j}) \), \( \sigma^{i,j} = (DF^i, DF^j) \), we have that \( (\det \sigma)^{-1} \in L^{\infty} \). Take any point \( a \in \mathbb{R}^d \) and set \( S = \{ x \in B; F(x) = a \} \). We regard \( S \) as a submanifold of \( B \). Since \( F \) is non-degenerate, we can define a measure on \( S \) in the following manner:

\[
m(dx) := \sqrt{\det(\sigma)} \delta_a(F).
\]

Here, on the right hand side, \( \delta_a \) denotes the Dirac measure at \( a \) and \( \delta_a(F) \) stands for the composite of a distribution \( \delta_a \) and a Wiener function \( F \) in the sense of Watanabe (see, e.g., [37]). We regard \( m \) as a reference measure. Let \( W^{m,-} \) be the set of all real-valued smooth functions in the sense of Malliavin. We identify two functions \( f, g \in W^{m,-} \) if they coincide \( m \)-a.e. To be precise, denoting the above equivalence relation by \( \sim \), we set \( \mathcal{F}(S) := W^{m,-} / \sim \) and we regard an element of \( \mathcal{F}(S) \) as a smooth function on \( S \). Clearly, \( \mathcal{F}(S) \) is an algebra.

Next let us define tensor bundles. Set \( T(S)_x = \{ h \in H; \langle DF^i(x), h \rangle = 0, i = 1, \ldots, d \} \). Then \( T(S) = \bigcup_{x \in S} T(S)_x \) is the tangent bundle of \( S \). The cotangent bundle and tensor bundle can be defined similarly. We denote the set of all smooth tensor fields of type \((l,m)\) by \( \mathcal{T}^{(l,m)}(S) \). The restriction of the inner product of \( H \) defines a metric on \( T(M) \). It can be regarded as a Riemannian metric, denoted by \( (\cdot, \cdot) \). We can also define a metric on the tensor bundle.

We introduce a Dirichlet form on \( L^2(m) \). The gradient operator on \( S \) is denoted by \( \nabla \). Thus, \( \nabla \) defines a Dirichlet form as follows:

\[
\mathcal{E}(f, g) = \int_S (\nabla f \cdot \nabla g) \, dm.
\]

Let \( \{ T_t \} \) be the associated semigroup. The generator \( A \) is given by \( A = -\nabla \nabla \). In this case, \( \Gamma \) is expressed as

\[
\Gamma(f, g) = (\nabla f \cdot \nabla g)
\]

and we have

\[
\mathcal{E}(f, g) = \int_S \Gamma(f, g) \, dm.
\]
It is easy to see that $I$ has the derivation property and so conditions (1) and (D) in Section 3 are fulfilled.

Aida [3] considered the operator $A$ when $F$ is given as the solution of a stochastic differential equation on a compact Riemannian manifold. He showed that $A$ is essentially self-adjoint on $W^{\infty,\infty}$. Moreover, in the $L^p$ setting, he obtained the denseness of $W^{\infty,\infty}$ in $\text{Dom}(A)$ and therefore we can apply our perturbation method discussed in Section 4.

We can also define a covariant derivative $V$ acting on tensor fields on $S$ (see, e.g., [22]). Then the quadratic form

$$\mathcal{E}(u, v) = \int_S (\nabla u \mid \nabla v) \, dm \quad u, v \in \Gamma(T^s_m(S))$$

induces a contraction semigroup $\{T_t\}$ in $L^2(\Gamma(T^s_m(S)))$ where $L^2(\Gamma(T^s_m(S)))$ is the set of all measurable sections $u$ with

$$\|u\|_2 = \left( \int_S |u|^2 \, dm \right)^{1/2} < \infty.$$

The generator is given by $A = -V^*V$ and $\Gamma$ is given by

$$\Gamma(u, v) = (\nabla u \mid Vv).$$

Now the positivity of $\Gamma$ is clear.

Next let us check that the assumptions in Proposition 3.5 are all satisfied. We first recall that for $f \in \mathcal{F}(S)$ and $u \in \Gamma(T^s_m(S))$, we have $fu \in \Gamma(T^s_m(S))$ and

$$\nabla (fu) = \nabla f \otimes u + f \nabla u.$$

Hence we have

$$(\nabla (fu) \mid Vv) + (u \mid \nabla (fv)) = (\nabla f \otimes u + f \nabla u \mid Vv) + (\nabla u \mid \nabla f \otimes v + f \nabla v)$$

$$= (\nabla f \mid (u \mid Vv)) + (\nabla u \mid v)) + 2(f(\nabla u \mid v))$$

$$= (\nabla f \mid \nabla (u \mid v)) + 2(f(\nabla u \mid v)).$$

Here, in the last line, we used

$$\nabla (u \mid v) = (\nabla u \mid v) + (u \mid \nabla v).$$

Thus we have that $I$ and $\Gamma$ satisfy (3.32). Thus, we can apply Theorem 3.4 and obtain

$$|T_t u| \leq T_t |u|.$$
We now turn to the finite dimensional case. Let $M$ be a compact Riemannian manifold. We denote the Laplacian by $\Delta$ and the Levi-Civita covariant derivative by $\nabla$. Let us consider the Hodge-Kodaira Laplacian $\Box = dd^* + d^*d$, where $d$ is the exterior differential and $d^*$ is its dual. On the space of 1-forms, the Weitzenböck formula says that

$$\Box u = \nabla^* \nabla u + R(\cdot, u).$$

Here $R$ denotes the Ricci curvature. We consider two semigroups $\{T_t\}$ and $\{T'_t\}$: $\{T_t\}$ is generated by $\Delta$ and $\{T'_t\}$ is generated by $-\Box$. The $\Gamma$ associated with $\Box$ is given by

$$2\Gamma(u, v) := \Delta(u \mid v) + (\Box u \mid v) + (u \mid v) + 2R(u, v).$$

Since $M$ is compact, there exists $\lambda \in \mathbb{R}$ such that

$$R(u, u) \geq -\lambda |u|^2,$$

which implies that

$$\Gamma(u, u) + \lambda |u|^2 \geq 0.$$

Moreover, we can easily check that (3.23) holds for $C^\infty$ sections. Then, by Proposition 3.5 and Theorem 3.4, we have

$$|T_t u| \leq e^{\lambda t T_t} |u|.$$  

REFERENCES

5. S. Aida, Essential selfadjointness of Ornstein-Uhlenbeck operators on loop groups, preprint.


