1. Introduction

Let $G = (V, E)$ be a graph with $n$ vertices and minimum degree $\delta(G)$. Several recent papers have considered problems associated with using mobile guards to defend $G$ against an infinite sequence of attacks; see for instance [1,2,4–6,9–11,13]. Denote the open and closed neighborhoods of a vertex $x \in V$ by $N(x)$ and $N[x]$, respectively. That is, $N(x) = \{v | xv \in E\}$ and $N[x] = N(x) \cup \{x\}$.

A dominating set of $G$ is a set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $x \in D$ adjacent to $u$. A dominating set $D$ is called a connected dominating set if the subgraph $G$ induced by $D$ is connected. The minimum cardinality amongst all dominating sets of $G$ is the domination number $\gamma(G)$, while the minimum cardinality amongst all connected dominating sets is the connected domination number $\gamma_c(G)$. An excellent treatment of domination theory can be found in [8].

A vertex cover of $G$ is a set $C \subseteq V$ such that for each edge $uv \in E$, at least one of $u$ and $v$ is in $C$. Let $\alpha(G)$ denote the vertex cover number of $G$, the size of a minimum vertex cover of $G$. An independent set of $G$ is a set $I \subseteq V$ with the property that no two vertices in $I$ are adjacent. The maximum cardinality amongst all independent sets is the independence number $\beta(G)$. It is well known that $n - \beta(G) = \alpha(G)$ for all graphs $G$ (see e.g. [3, Theorem 9.12]).

An independent set of edges of $G$ is a set of edges, no two of which have a common end-vertex. The edge independence number $\beta_1(G)$ is the maximum cardinality among the independent sets of edges of $G$. It is well known that $\alpha(G) \geq \beta_1(G)$ for all graphs $G$, and that equality holds for bipartite graphs. The latter result is known as Königs’s theorem (see e.g. [3, Theorem 9.13]).

Let $D_i \subseteq V, i \geq 1$, be a set of vertices with a guard located on each vertex of $D_i$. In this paper we allow at most one guard to be located on a vertex. The set $D_i$ is also called a configuration of guards. The problems we study can be modeled as two-player games between a defender and an attacker: the defender chooses each $D_i, i \geq 1$, while the attacker chooses the locations of the attacks $r_1, r_2, \ldots,$ depending on the configuration $D_1, D_2, \ldots$ of guards. Each attack $r_i$ is handled by the defender by choosing the next $D_i$ subject to some constraints that depend on the particular game (see below). The defender wins the game if they can successfully defend against any series of attacks, the attacker wins otherwise.
A vertex is protected if there is a guard on the vertex. An attack at vertex $v$ is defended if a guard is sent to $v$ from a neighboring vertex, or trivially, if a guard occupied $v$ prior to the attack.

In the eternal dominating set problem, each $D_i$, $i \geq 1$, is required to be a dominating set, called an eternal dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and $D_{i+1}$ is obtained from $D_i$ by moving one guard to $r_i$ from a vertex $v \in D_i$, $v \notin N(r_i)$. The size of a smallest eternal dominating set of $G$ is denoted $\gamma^\infty(G)$. This problem was first studied in [2].

In the $m$-eternal dominating set problem, each $D_i$, $i \geq 1$, is required to be a dominating set, called an $m$-eternal dominating set, $r_i \in V$ (again assume without loss of generality $r_i \notin D_i$), and $D_{i+1}$ is obtained from $D_i$ by moving guards to neighboring vertices. That is, any number of guards in $D_i$ may move to an adjacent vertex. It is required that $r_i \in D_{i+1}$. The size of a smallest $m$-eternal dominating set of $G$ is denoted $\gamma^\infty_m(G)$. This "all-guards move" version of the problem was introduced in [5]. It is clear that $\gamma^\infty(G) \geq \gamma^\infty_m(G) \geq \gamma(G)$ for all graphs $G$.

It is obvious that for any graph $G$ without isolated vertices, $\alpha(G) \geq \gamma(G)$. In this paper we are interested in comparing the vertex cover number of a graph with the $m$-eternal domination number. Our initial motivation was the classic result of Hartnell and Rall [7] characterizing graphs without isolated vertices having equal vertex cover number and domination number.

In the eternal vertex cover problem, a vertex cover must be maintained eternally in the face of a series of attacks on edges. In this problem, a guard must move across an attacked edge. This problem was introduced in [12] and compared with the $m$-eternal domination number in [13]. All guards are allowed to move when an attack occurs in the eternal vertex cover problem. To formalize, each $D_i$, $i \geq 1$, is required to be a vertex cover, $r_i \in E$, and $D_{i+1}$ is obtained from $D_i$ by moving guards to neighboring vertices. That is, each guard in $D_i$ may move to an adjacent vertex. It is required that in moving from $D_i$ to $D_{i+1}$, a guard move across edge $r_i$ (we assume without loss of generality that at least one vertex of $r_i$ is not in $D_i$). The size of a smallest $m$-eternal vertex cover for $G$ is denoted $\alpha^\infty_m(G)$. Obviously $\gamma^\infty_m(G) \leq \alpha^\infty_m(G)$ for any graph $G$ without isolated vertices and $\alpha^\infty_m(G) \geq \alpha(G)$ for all graphs $G$.

In Section 2, we determine which trees have $\alpha(T) = \gamma^\infty_m(T)$ and which have $\gamma^\infty_m(T) = 2\alpha(T)$. Using this result, we characterize graphs that satisfy $\gamma^\infty_m(G) = 2\alpha(G)$ in Section 3. In Section 4, we prove that the $m$-eternal domination number is less than the vertex cover number of any graph of minimum degree at least two and girth equal to seven or girth greater than or equal to nine.

2. Trees

In this section we bound the $m$-eternal domination numbers of trees in terms of their vertex cover numbers. For $V_i \subseteq V$, let $(V_i)$ denote the subgraph induced by $V_i$.

A neo-colonization is a partition $\Pi = \{V_1, V_2, \ldots, V_t\}$ of the vertex set of graph $G$ such that each $(V_i)$ is a connected graph. A part $V_i$ is assigned a weight $w(V_i) = 1$ if it induces a clique and $w(V_i) = 1 + \gamma_c((V_i))$ otherwise. Define the weight $w(\Pi)$ by $w(\Pi) = \sum_{i=1}^{t} w(V_i)$. Then the clique-connected cover number $\theta_c(G)$ of $G$ is the minimum weight of any neo-colonization of $G$. Goddard et al. [5] defined this parameter and proved that $\gamma^\infty_m(G) \leq \theta_c(G)$.

Theorem 1 ([10]). For any tree $T$, $\theta_c(T) = \gamma^\infty_m(T)$.

A diametrical path (abbreviated $d$-path) of a tree $T$ is a path of maximum length. A stem of $T$, also sometimes called a support vertex, is a vertex of degree at least two that is adjacent to a leaf. A vertex of $T$ that is not a leaf is called an internal vertex. We partition the internal vertices of $T$ into loners, weak stems and strong stems depending on whether they are adjacent to no, exactly one or at least two leaves. Denote the set of leaves and the set of internal vertices of $T$ by $L(T)$ and $\text{Int}(T)$, respectively, and let $\ell = |L(T)|$, $\text{int}(T) = |\text{Int}(T)|$. Obviously, $\gamma_c(T) = n - \ell = \text{int}(T)$ for any tree $T$ of order $n \geq 3$.

We state some results from [12, 13].

Theorem 2 ([12]). For any nontrivial, connected graph $G$, $\alpha(G) \leq \alpha^\infty_m(G) \leq 2\alpha(G)$.

Theorem 3 ([12]). For any tree $T$ of order $n$, $\alpha^\infty_m(T) = \text{int}(T) + 1$.

The formula in Theorem 3 comes from initially locating guards on all internal vertices of $T$ and one guard on an arbitrary leaf.

Theorem 4 ([13]). Let $T$ be a tree of order $n \geq 3$. Then $\gamma^\infty_m(T) < \alpha^\infty_m(T)$ if and only if $\exists$ there exists a set $E' \subseteq E(T)$, where each edge in $E'$ is incident with a loner, such that each component of $T - E'$ contains a strong stem and has a leaf that is a loner of $T$.

Proposition 5 ([13]). For any connected graph $G$, $\gamma^\infty_m(G) \leq 2\gamma(G)$, and the bound is sharp for all values of $\gamma(G)$.

We need another lemma before stating the main theorem of this section. A star is a complete bipartite graph $K_{1,m}$, $m \geq 1$. We say a vertex is $M$-saturated if it belongs to matching $M$ and $M$-unsaturated otherwise.

Lemma 6. Every minimum-weight neo-colonization $\{V_1, V_2, \ldots, V_m\}$ of a tree $T$ satisfies $(V_i) = K_2$ for each $i$ if and only if $T$ has a perfect matching of size $m$. 


that is an o-colonization of $K$.

Theorem 7. Let $T = \{V_1, V_2, \ldots, V_m\}$ be any minimum-weight neo-colonization of $T$ and assume without loss of generality that $u \in V_1$.

If $V_1 = \{u, v\}$, then $\Pi' = \{V_2 - \{u, v\}, \ldots, V_m\}$ is a neo-colonization of $T'$. If $w(T') > \theta(T')$, let $\{U_1, \ldots, U_m\}$ be a minimum-weight neo-colonization of $T$. Then $|V_1| \leq |U_1|$, hence $\Pi' = \{U_1, \ldots, U_m\}$ is a neo-colonization of $T$.

Suppose $V_1 = \{u\}$ and assume without loss of generality that $v \in V_2$. By the minimality of $\Pi$, $|V_2| \geq 2$, hence $x \in V_2$. Since $v$ is a leaf of $\{V_2\}$, $\Pi' = \{V_2 - \{v\}, \ldots, V_m\}$ is a neo-colonization of $T'$ and $w(\Pi') \geq w(T') + 1$. If $w(T') > \theta(T')$, let $\Pi' = \{U_1, \ldots, U_m\}$ be a minimum-weight neo-colonization of $T$. Then $\Pi' = \{\{u, v\}, U_1, \ldots, U_m\}$ is a neo-colonization of $T$ such that $w(\Pi') = w(\Pi') + 1 < w(T') + 1 \leq w(T')$, contradicting the minimality of $\Pi$. Hence $w(T') = \theta(T')$ and, by the induction hypothesis, $t = m$ and $\Pi' = \{U_1, \ldots, U_m\}$ is a neo-colonization of $T$.

Finally, if $\{u, v\} \subseteq V_1$, then $\Pi' = \{V_2 - \{u, v\}, V_3, \ldots, V_m\}$ is a neo-colonization of $T'$. Since $\deg_T v = 2$, $\deg_{V_1} v = 2$ also, and $x \in V_1$. Hence $w(\Pi') \geq w(V_1) + w(\{u, v\}) + 1$, with equality if and only if $V_1 = \{u, v\}$ or $\{V_1 - \{u, v\}\}$ is a star of order at least three centered at $x$. As above, it follows that $w(\Pi') = \theta(T')$. By the induction hypothesis, $t = m - 1$ and $\Pi' = \{V_2 - \{u, v\}\} = \Pi_2$ for $i = 2, \ldots, m - 1$. But then $w(\Pi') \geq w(V_1 - \{u, v\}) + 2$ and so $\{u, v\}, V_1 - \{u, v\}, V_2, \ldots, V_m$ is a neo-colonization of $T$ of weight less than $w(\Pi')$, a contradiction.

Theorem 7. For any nontrivial tree $T$, $\alpha(T) \leq \gamma_m^\infty(T) \leq 2\alpha(T)$.

Equality holds in the lower bound if and only if $T$ has a perfect matching. Equality holds in the upper bound if and only if

$P1$: $T \neq K_2$, each stem of $T$ is a strong stem, no two loners are adjacent and no two vertices of degree at least three are adjacent.

Proof. We prove by induction on the order of $T$ that $\alpha(T) \leq \gamma_m^\infty(T)$ and that equality holds if and only if $T$ has a perfect matching. The statement is clearly true for $K_2$. Assume it is true for all trees with fewer than $k$ vertices and let $T$ be a tree of order $k \geq 3$. Let $u$ be a stem that at most one neighbor of $v$ is not a leaf, let $u$ be a leaf adjacent to $v$, and let $\Pi = \{V_1, V_2, \ldots, V_m\}$ be a minimum-weight neo-colonization of $T$ such that $v$ is a minimum. Assume without loss of generality that $u \in V_1$. If $|V_1| = 1$ for some $i$, say $V_i = \{v\}$, let $w'$ be any vertex adjacent to $u$ and assume without loss of generality that $w' \in V_1$. Since $T[V_i \cup \{u\}]$ has at most one more internal vertex than $T[V_{i-1}, w']$, $w(V_{i-1} \cup \{u\}) \leq w(V_{i-1} + 1)$, hence the partition $\Pi' = \{V_1, V_2, \ldots, V_{i-1} \cup \{u\}\}$ is a neo-colonization of $T$ such that $w(\Pi') \leq w(\Pi')$ and $\Pi'$ has fewer parts than $\Pi$, contradicting the choice of $\Pi$. Hence $|V_i| \geq 2$ for each $i$.

If $v$ is adjacent to a leaf $x \neq u$, then $T$ does not have a perfect matching. Since $|V_1| \geq 2$ for each $i$, $\{u, v, x\} \subseteq V_1$. Let $T' = T - x$. Then $\Pi' = \{V_1 - \{x\}, V_2, \ldots, V_m\}$ is a neo-colonization of $T'$. Consider any minimum vertex cover $A$ of $T'$. Then $A \cap \{u, v\} = 1$, and we may assume without loss of generality that $v \in A$. But then $A$ is a vertex cover of $T$, so that by the induction hypothesis, $\gamma_m^\infty(T') \geq \gamma_m^\infty(T') \geq \alpha(T)$, where the second inequality is strict if $T'$ does not have a perfect matching. Hence assume $T'$ has a perfect matching. Recall that by Theorem 1, $\gamma_m(T) = \theta(T)$. If $w(T') > \theta(T')$, then $\gamma_m^\infty(T) = \theta(T) = w(T) \geq w(T') > \theta(T') = \alpha(T)$ as required, and if $w(T') = \theta(T')$, then by Lemma 6, $V_1 - \{v\} = K_2$, so $w(V_1) \geq 2 > w(V_1 - \{x\})$, and it follows that $\gamma_m^\infty(T) = w(T') = w(T') = \alpha(T)$.

Now assume $u$ is the only leaf adjacent to $v$. Then $\deg v = 2$. Consider the tree $T'' = T - \{u, v\}$ and note that $\alpha(T'') \geq \alpha(T) - 1$. By the induction hypothesis, $\gamma_m(T'') \geq \alpha(T)$. If $V_1 = \{u, v\}$, then $\Pi'' = \{V_2, \ldots, V_m\}$ is a neo-colonization of $T''$ and $w(T'') \geq \theta(T'') = \gamma_m^\infty(T'') \geq \alpha(T)$, so that $w(T) = \theta(T) \geq \theta(T'') + 1 \geq \alpha(T'') + 1 \geq \alpha(T)$ and the bound follows from Theorem 1. If $V_1 = \{u\}$, assume that $v \in V_2$ and note that $|V_2| \geq 2$ by the minimality of $\Pi$. In this case $\{V_2 - \{v\}, \ldots, V_m\}$ is a neo-colonization of $T''$ and the bound follows similarly. The result also follows easily if $\{u, v\} \subseteq V_1$. Thus the lower bound holds.

If $T$ has a perfect matching $M$, then by Lemma 6, $t = |M|$ and $|V_1| = K_2$ for each $i$, so that $w(T) = |M| = \alpha(T)$ and thus (Theorem 1) $\gamma_m(T) = \alpha(T)$.

Assume $T$ satisfies Property P1 and say $T$ has $k$ stems. Since $T \neq K_2$, $k \geq 1$. If $k = 1$, then $T$ is a star, $\alpha(T) = 1$ and $\gamma_m(T) = 2$, so assume $k \geq 2$. Since each stem is strong, each stem has degree at least three. Since each loner is adjacent only to stems and no two vertices of degree at least three are adjacent, each loner is adjacent to exactly two stems and thus has degree two. Let $H$ be the tree obtained from $T$ by deleting all leaves and contracting each path $u, v, w$, where $v$ is a loner and $u$ and $w$ are stems, to the edge $uw$. Then $H$ has $k$ vertices and thus $k - 1$ edges. It follows that $T$ has $k - 1$ loners, so
that $T$ has $2k - 1$ internal vertices. Moreover, $T$ has a bipartition $(U, U')$, where $U$ consists of all stems and $U'$ consists of all loners and leaves. Hence $\alpha(T) \leq k$, and since a matching of size $k$ can be found by matching each stem to a leaf, $\alpha(T) = k$.

We now use Theorems 3 and 4 to show that $\gamma_\infty^m(T) = 2\alpha(T)$. Let $u_1, v_1$ be any edge of $T$ joining a stem $u_1$ to a loner $v_1$, and let $T_{u_1}$ and $T_{v_1}$ be the subtrees of $T - u_1v_1$ that contain $u_1$ and $v_1$, respectively. Then $T_{u_1}$ contains no leaf that is a loner of $T$. Similarly, let $u_2, v_2$ be any edge of $T_{u_1}$ that joins a stem $u_2$ to a loner $v_2$, and let $T_{u_2}$ and $T_{v_2}$ be the subtrees of $T_{u_1} - u_2v_2$ that contain $u_2$ and $v_2$, respectively. Then $T_{u_2}$ contains no leaf that is a loner of $T$. This process can be repeated any number of times. At each stage of the process there is a component that contains no leaf that is a loner of $T$. Hence no set $E' \subseteq E(T)$ satisfies Condition C1 of Theorem 4 and it follows that $\gamma_\infty^m(T) = \alpha_m(T)$. However, by Theorem 3, $\alpha_m(T) = \text{int}(T) + 1$. But $\text{int}(T) = 2k - 1$ and so $\gamma_\infty^m(T) = \alpha_m(T) = 2k = 2\alpha(T)$.

Conversely, assume $T$ does not have Property P1. If $T = K_2$ then $\gamma_\infty^m(T) < 2\alpha(T)$, so suppose $T$ has at least one stem.

- If $T$ has a weak stem $u$ that is adjacent to the leaf $v$, let $T' = T - \{u, v\}$. Then $T'$ has no isolated vertices and so $\gamma_\infty^m(T') \leq 2\alpha(T')$. Now $\gamma_\infty^m(T) \leq \gamma_\infty^m(T') + 1$ and $\alpha(T) = \alpha(T') + 1$, therefore $\gamma_\infty^m(T) \leq \gamma_\infty^m(T') + 1 \leq 2\alpha(T') + 1 < 2\alpha(T)$.

- Similarly, if $T$ has two adjacent loners $x$ and $y$, then $T - \{x, y\}$ has no isolated vertices and it follows as above that $\gamma_\infty^m(T) < 2\alpha(T)$.

- If $T$ has two adjacent vertices $u_1$ and $u_2$ of degree at least three, let $T_1$ and $T_2$ be the two subtrees of $T - u_1u_2$. By Theorems 2 and 3, $\alpha_m(T_i) = \text{int}(T_i) + 1 \leq 2\alpha(T_i)$ for each $i$. But $\alpha(T) \geq \alpha(T_1) + \alpha(T_2)$ and $\text{int}(T) = \text{int}(T_1) + \text{int}(T_2)$, so

\[
\gamma_\infty^m(T) \leq \gamma_\infty^m(T_1) + \gamma_\infty^m(T_2) \leq \alpha_m(T_1) + \alpha_m(T_2) = \text{int}(T_1) + \text{int}(T_2) + 2 = \text{int}(T) + 2 \leq 2\alpha(T_1) + \alpha(T_1) \leq 2\alpha(T).
\]

Hence $\text{int}(T) + 2 \leq 2\alpha(T)$ and thus $\gamma_\infty^m(T) \leq \alpha_m(T) = \text{int}(T) + 1 < 2\alpha(T)$. □

3. Graphs

If $G$ is a nontrivial connected graph, then $\gamma(G) \leq \alpha(G)$ and thus Proposition 5 implies that $\gamma_\infty^m(G) \leq 2\alpha(G)$. We use Theorem 2 to characterize the extremal graphs for this bound. We first prove a lemma.

**Lemma 8.** Let $H$ be a graph obtained from $C_{2m} = v_0, v_1, \ldots, v_{2m-1}, v_0$, $m \geq 3$, by adding any number of new endvertices to any subset of $\{v_0, v_2, \ldots, v_{2m-2}\}$. Then $\gamma_\infty^m(H) < 2m$.

**Proof.** Let $A = \{v_i : i \text{ is even}\}$ and $B = \{v_i : i \text{ is odd}\}$. The result is obvious if $H = C_{2m}$, hence assume that some $v_{2i} \in A$ is adjacent to an endvertex. A placement of $2m - 1$ guards, one on each vertex in $A$ and on all but one vertex in $B$, is called an $X$-configuration of guards. A placement of $2m - 1$ guards, one on each vertex in $A$, an endvertex $w_{2i}$ adjacent to $v_{2i}$, and all vertices in $B - \{v_{2i-1}, v_{2i+1}\}$, where the arithmetic in the subscripts is performed modulo $2m$, is called a $Y$-configuration of guards.

While the guards are in an $X$-configuration, defend an attack on an unprotected vertex $u \in B$ by moving a guard from an adjacent vertex $a \in A$ to $u$ and another guard from the other neighbor $b \in B - \{u\}$ of $a$ to $a$. This results in another $X$-configuration of guards. Defend an attack on an unprotected endvertex $w$ adjacent to some $v_{2j}$ by moving the guard on $v_{2j}$ to $w$, and the other guards in such a way that each vertex in $A \cup B - \{v_{2j-1}, v_{2j+1}\}$ contains a guard. This forms a $Y$-configuration.

While the guards are in a $Y$-configuration, note that each unprotected vertex $b \in B$ is adjacent to a vertex $v_{2j}$ such that $v_{2j}$ and an endvertex $w$ adjacent to it contains guards. Defend $b$ by moving the two guards to $b$ and $v_{2j}$, respectively, which results in an $X$-configuration. Still while the guards are in a $Y$-configuration, defend an unprotected endvertex adjacent to $v_{2j}$ in the obvious way. Defend an attack on an endvertex $w'$ adjacent to $v_{2j}, j \neq i$, by moving a guard on $b' \in B$ adjacent to $v_{2j}$ to $v_{2j}$, the guard on $v_{2j}$ to $w'$, the guard on $w$ to $v_{2i}$, and all other guards so that each vertex in $A \cup B - \{v_{2j-1}, v_{2j+1}\}$ contains a guard. This forms another $Y$-configuration.

This strategy defends any sequence of attacks and thus $\gamma_\infty^m(H) < 2m$. □

Let $T^+ \subseteq T$ be a tree with $\gamma_\infty^m(T) = 2\alpha(T)$. By Property P1 of Theorem 2, a loner in $T$ has degree two. Let $T^+$ consist of all graphs that can be obtained from $T$ by a sequence of zero or more applications of the following operation.

**Parallel operation.** Let $v$ be a loner of $T$; say $v$ is adjacent to $u$ and $w$. Add a new vertex $v'$ to $T$, joining $v'$ to $u$ and $w$.

We say $T$ is the underlying tree of $G \in T^+$. If $v \in V(T)$ and $v'$ is created by a parallel operation from $v$, we say $v'$ is parallel to $v$. Define the class $\tilde{g}^+$ of graphs by

\[\tilde{g}^+ = \bigcup\{T^+ : T \text{ is a tree that satisfies Property P1}\}.
\]

**Theorem 9.** For any connected graph $G$, $\gamma_\infty^m(G) = 2\alpha(G)$ if and only if $G \in \tilde{g}^+$. 

**Proof.** Let $G \in \tilde{g}^+$, let $T$ be the underlying tree of $G$ and let $V' = V(G) - V(T)$. As shown in the proof of Theorem 7, the set $C$ of strong stems of $T$ is a minimum vertex cover of $T$. Then $C$ is a vertex cover of $G$ and therefore $\alpha(C) \leq \alpha(T)$. Since $\alpha(C) \leq \alpha(G)$ for any spanning subgraph $H$ of $G$, $\alpha(G) \leq \alpha(T)$. Thus, to show that $\gamma_\infty^m(G) = 2\alpha(G)$, it is sufficient to show that $\gamma_\infty^m(G) = \gamma_\infty^m(T)$.

First, to show that $\gamma_\infty^m(G) \leq \gamma_\infty^m(T)$, we partition $G$ into $\alpha(G) = \alpha(T)$ stars of order at least three, each of which has a strong stem as its center. Two guards on each star protect it against any sequence of attacks. Hence $\gamma_\infty^m(G) \leq 2\alpha(T) = \gamma_\infty^m(T)$. 


Now suppose \( \gamma_m^\infty(G) < \gamma_m^\infty(T) \). Then there is a protection strategy for \( G \) that is not a protection strategy for \( T \). This is only possible if attacks on the vertices of \( G - V' \) can be defended by a strategy that involves guard movements to or from, or placements on, a loner of \( T \) as well as one of its parallel vertices. Consider such a strategy that involves as few of these parallel pairs as possible. For this strategy, begin with an initial placement of guards using as few vertices of degree two as possible. Subject to this constraint, begin with guards on as many loners of \( T \) (instead of their parallel vertices) as possible, and maintain this requirement throughout. In particular, whenever a loner of \( T \) and one of its parallel vertices both contain guards, and exactly one guard needs to move, assume without loss of generality that the guard on the parallel vertex moves. Also assume without loss of generality that the protection strategy is accomplished with the minimum number of guard movements.

Suppose that an attack on \( G - V' \) is defended by a step \( S \) that involves guard movements to or from, or placements on, a loner \( v \) of \( T \) as well as a parallel vertex \( v' \). Let \( u \) and \( w \) be the stems of \( T \) adjacent to \( v \). We first show that

if there is exactly one guard on \( \{ v, v' \} \), then this guard is on \( u \).

Suppose, to the contrary, that there is a guard on \( v' \) but not on \( v \). If \( v' \) contained a guard in the initial configuration, then so did \( v \); if only one guard had to move, then \( v' \) should have moved. If \( v' \) did not have a guard initially, then a guard moved to \( v' \) during a previous step. Since only attacks on \( G - V' \) are considered, \( v' \) was not attacked, hence a guard moved to \( v' \) in response to an attack elsewhere. According to our strategy, this guard should have moved to \( v \) instead. Hence (1) holds.

Now consider step \( S \).

• Suppose neither \( u \) nor \( v' \) contains a guard before step \( S \). Since \( S \) involves \( v \) and \( v' \), both \( v \) and \( v' \) contain guards after \( S \), and thus both \( u \) and \( w \) contain guards before \( S \). This guard movement is only required if \( v \) is attacked. Any such attack can be defended by moving a guard to \( v \) only. This guard can, in future, move to either \( u \) or \( w \) if required to do so. Hence \( S \) does not involve the minimum number of guard movements.

• Suppose exactly one of \( v \) and \( v' \) contains a guard before \( S \). By (1), there is a guard on \( v \), and \( u \) or \( w \) also contains a guard. Since \( S \) involves \( v \) and \( v' \), another guard moves to \( v' \) during \( S \). By (1), both \( v \) and \( v' \) contain guards after \( S \). Therefore, in a minimum guard movement strategy, a guard \( g \) moved from \( u \) or \( w \) to \( v' \). Since \( v' \) was not attacked, \( g \) moved in response to an attack elsewhere, to be ready for future attacks. But the guard already on \( v \) can take care of any such future attacks. Hence \( S \) does not involve the minimum number of guard movements.

• Suppose both \( v \) and \( v' \) contain guards immediately before step \( S \). By the previous cases this is only possible if \( v \) and \( v' \) contained guards in the initial configuration. Then at least four vertices in \( N[u] \cup N[w] \) contain guards. If both guards on \( \{ v, v' \} \) move during step \( S \), then one moves to \( u \) and the other one to \( w \). Any attack on \( G \) defended in this way can also be defended by guards on \( u \) and \( w \), and on a vertex in each of \( N(u) - \{ v, v' \} \) and \( N(w) - \{ v, v' \} \). A similar statement holds if neither guard or exactly one guard moves during \( S \). This is contrary to the condition that the initial placement of guards used as few vertices of degree two as possible.

We conclude that each protection strategy for \( G \) reduces to a protection strategy for \( T \) and hence that \( \gamma_m^\infty(G) = \gamma_m^\infty(T) \), as required.

For the converse, assume that \( \gamma_m^\infty(G) = 2\alpha(G) \). We show that \( G \in \mathcal{G}^+ \). Let \( T \) be any spanning tree of \( G \). Then any \( m \)-eternal dominating set of \( T \) is an \( m \)-eternal dominating set of \( G \), and any vertex cover of \( G \) is a vertex cover of \( T \). Hence \( \gamma_m^\infty(G) \leq \gamma_m^\infty(T) \) and \( \alpha(T) \leq \alpha(G) \), so that \( 2\alpha(T) \leq 2\alpha(G) = \gamma_m^\infty(G) \leq \gamma_m^\infty(T) \). By Theorem 7, \( \gamma_m^\infty(T) = 2\alpha(T) \), hence equality holds throughout and \( T \) satisfies Property P1.

As shown in the proof of Theorem 7, the set \( C \) of strong stems of \( T \) is a minimum vertex cover of \( T \). Let \( C = \{ u_1, \ldots, u_\ell \} \), where \( \alpha = \alpha(T) = |C| \), and partition \( T \) into \( \alpha \) stars \( S_1, \ldots, S_\ell \). of order at least three such that \( u_i \) is the center of \( S_i \); a loner incident with two strong stems \( u_i, u_j \) is assigned arbitrarily to either \( S_i \) or \( S_j \). Note that an initial configuration of a guard on \( u_i \) and another one on a leaf of \( S_i \) is an \( m \)-eternal dominating set of \( S_i \); together the \( 2\alpha \) guards form an \( m \)-eternal dominating set of \( T \) and also of \( G \).

If \( u \in C \) and \( D \) is a vertex cover of \( T \) such that \( u \not\in D \), then \( D \) contains all leaves of \( T \) adjacent to \( u \). Since \( u \) is a strong stem it follows that \( |D| > |C| \). Hence \( C \) is the unique minimum vertex cover of \( T \). Since \( \alpha(G) = \alpha(T) \) and any vertex cover of \( G \) is a vertex cover of \( T \), \( C \) is also the unique minimum vertex cover of \( G \). Therefore each edge \( e \in E(G) - E(T) \) is incident with a vertex in \( C \). We consider five cases, depending on the other endvertex of \( e \).

Case 1 \( e = u_iu_j \) where \( u_i, u_j \in C \). Consider an initial configuration of two guards on each \( S_k, k \neq i, j \), and one guard on \( u_i \). The guards on \( S_k, k \neq i, j \), protect \( S_i \) against any sequence of attacks. The three guards on \( S_i \) and \( S_j \) protect \( S_i \cup S_j \) against any attack sequence. Hence the initial configuration forms an \( m \)-eternal dominating set of \( G \) containing \( \gamma_m^\infty(T) = 1 \) guards, contradicting \( \gamma_m^\infty(G) = \gamma_m^\infty(T) \).

Case 2 \( e = u_iw \) where \( w \) is a loner or a leaf of \( T \) such that, in \( G \), \( w \) is adjacent to \( u_j, u_k, j \neq i \). Assume without loss of generality that \( w \in V(S_i) \). Place one guard on each of \( u_i, u_j, u_k \), and another guard on a leaf of \( S_i \). These five guards protect \( S_i \cup S_j \cup S_k \) against any attack sequence, and, together with two guards on each of the other stars, form an \( m \)-eternal dominating set of \( G \) containing \( \gamma_m^\infty(T) = 1 \) guards, a contradiction as above.

Case 3 \( e = u_ix \) where \( x \) is a leaf of \( T \) such that \( d(u_i, x) > 3 \). Say \( x \) is adjacent in \( T \) to \( u_j \in C \). Then \( d(u_i, u_j) > 2 \). Assume without loss of generality that \( i < j \) and that \( P = u_i, x_1, u_{i+1}, x_{i+1}, \ldots, x_{j-1}, u_j \) is the \( u_i - u_j \) path in \( T \). Then each \( x_j \) is a loner of \( T \), hence \( P \) together with all the leaves of \( T \) adjacent to \( u_i, u_{i+1}, \ldots, u_j \), and the edge \( e \), form a graph \( H \) as defined in
Proof. Let \( G \) be a connected graph with \( \delta(G) \geq 3 \). Similar constructions can be done for \( v \) with the same property. For example, take two copies of \( G \). Let \( S_1^i = K_1 \) or \( K_2 \), depending on whether \( u_i \) is adjacent to zero or one endvertex of \( G \), and for each \( k \neq j \), let \( S_k^j \) consist of \( S_k \) together with loners or leaves of \( T \) adjacent to \( u_i \) that are adjacent to \( u_k \) in \( G \). (The \( S_k^j \) are not necessarily disjoint but that does not matter.) Then \( G \) can be protected by one guard on \( S_1^i \) and two guards on each \( S_k^j, k \neq j \), another contradiction.

Case 5 None of Cases 1–4 holds. Then each \( e \in E(G) - E(T) \) joins a strong stem \( u_i \) to a leaf \( w \) of \( T \), where \( w \) is adjacent in \( T \) to the strong stem \( u_j \), \( d(u_i, u_j) = 2 \), \( \deg_G w = 2 \), and each strong stem of \( T \) is adjacent to at least two endvertices of \( G \). Let \( T_1 \) be the subtree of \( T \) obtained by deleting all leaves of \( T \) that are not endvertices of \( G \). Then \( G \in T_1 \) and thus \( G \in \tilde{g}^+ \), as required. \( \square \)

4. Girth

4.1. Basic bounds

A fundamental bound for \( \gamma_m^\infty \) was proved in [5].

**Theorem 10 ([5]).** For any graph \( G \), \( \gamma_m^\infty(G) \leq \beta(G) \).

The following bound is the primary motivation for our study in this section. We include the proof from [13] for completeness.

**Proposition 11 ([13]).** If \( G \) is a connected graph with \( \delta(G) \geq 2 \), then \( \gamma_m^\infty(G) \leq \alpha(G) \).

**Proof.** If \( \delta(G) \geq 2 \), then obviously \( \alpha(G) \geq 2 \). Let \( D \) be a minimum vertex cover of \( G \) and initially place a guard on each vertex in \( D \). Note that \( V - D \) is independent and thus each vertex in \( V - D \) is adjacent to at least two vertices in \( D \). In response to an attack on a vertex \( u \in V - D \), move a guard from an adjacent vertex \( v \) to \( u \). For each subsequent attack, say on a vertex \( w \), move the guard on \( u \) back to \( v \), and (if \( w \neq v \)) move a guard on a vertex \( x \in N(w) \cap (D - \{v\}) \) to \( w \); this is possible since \( \delta(G) \geq 2 \). \( \square \)

4.2. Example graphs

Recall that the girth of a graph \( G \) is the length of a shortest cycle of \( G \). We next give some examples where the bound in Proposition 11 is sharp for girths three and four.

**Girth 4** \( K_{2,m}, m \geq 2 \), is an infinite family of bipartite graphs with girth four and \( \gamma_m^\infty(G) = \alpha(G) = 2 \).

**Girth 3** Let \( L_{2,m} \) be the graph obtained by adding an edge between the two vertices of \( K_{2,m}, m \geq 2 \), in the part of size two. Then \( L_{2,m} \) has girth three and \( \gamma_m^\infty(L_{2,m}) = \alpha(L_{2,m}) \). One can “paste” copies of \( L_{2,m} \) together to obtain larger graphs with the same property. For example, take two copies of \( L_{2,3} \), add a new vertex \( v \), and join a vertex of degree \( m + 1 \) from each copy to \( v \). This graph \( G \) has girth three, \( \alpha(G) = 4 \) and \( \gamma_m^\infty(G) = 4 \).

**Girth 4** non-bipartite Construct the graph \( G \) by adding new vertices \( u \) and \( v \) to the 5-cycle \( v_1, v_2, v_3, v_4, v_5, v_1 \), joining \( u \) to \( v_1 \) and \( v_2 \), and \( v \) to \( v_3 \) and \( v_4 \). It is easy to see that \( G \) has girth four, vertex cover number three, and \( m \)-eternal domination number three. Similar constructions can be done for \( C_m, m > 5 \).

4.3. Main results on girth

**Theorem 12.** Let \( G \) be a connected graph with \( \delta(G) \geq 2 \) and girth at least nine. Then \( \gamma_m^\infty(G) < \alpha(G) \).

**Proof.** Let \( C = v_1, v_2, \ldots, v_k, v_1 \) be a shortest cycle of \( G \). Observe that for all \( k \geq 9 \), \( \gamma_m^\infty(C_k) = \gamma(C_k) = \left[ \frac{k}{3} \right] < \left[ \frac{k}{2} \right] = \alpha(C_k) \). Moreover, if \( k \neq 10 \), then \( \gamma_m^\infty(C_k) < \alpha(C_k) - 1 \), while \( C_{10} - \{v_1\} \) can be dominated by \( 3 = \alpha(C_{10}) - 2 \) vertices for any \( i \). Denote the graph \( V(G) - V(C) \) by \( G - C \).

For any \( i, j \) with \( 1 \leq i < j \leq k \), suppose that \( P \) is a \( v_i - v_j \) path internally disjoint from \( C \). Since the distance on \( C \) between \( v_i \) and \( v_j \) is at most \( \left[ \frac{k}{2} \right] \), \( P \) has length at least \( \left[ \frac{k}{2} \right] \), otherwise \( G \) has a shorter cycle than \( C \). Since \( k \geq 9 \), \( P \) has length at least five. In particular, each vertex of \( G - C \) is adjacent to at most one vertex of \( C \), and since \( \delta(G) \geq 2 \), it follows that \( \delta(G - C) \geq 1 \). If \( \delta(G - C) \geq 2 \), then we are done, since then

\[
\gamma_m^\infty(G) \leq \gamma_m^\infty(G - C) + \gamma_m^\infty(C) < \gamma_m^\infty(G - C) + \alpha(C) \leq \alpha(G - C) + \alpha(C) \quad \text{by Proposition 11}
\]

\[
\leq \alpha(G).
\]
Hence we assume that $\delta(G - C) = 1$. Let $I = \{v \in V(G - C) : \deg v = 1\} \neq \emptyset$, $G' = G - C - I$ and $V' = V(G')$. Then each vertex in $I$ has exactly one neighbor in $C$ and exactly one neighbor in $G'$. Since each $v_i - v_j$ path internally disjoint from $C$ has length at least five, $d(u, v) \geq 3$ for all distinct $u, v \in I$. In particular, $I$ is independent and each vertex of $G'$ is adjacent to at most one vertex in $I$.

Let $D$ be a minimum vertex cover of $G - C$. If $u \in D \cap I$ and $v$ is the unique neighbor of $u$ in $G - C$, then $(D - \{u\}) \cup \{v\}$ is also a vertex cover of $G - C$. Hence we may assume that $D \subseteq V'$. Also, each vertex in $I$ is adjacent to exactly one vertex in $D$, and each vertex in $V' - D$ has degree at least two in $G'$ and therefore is adjacent to at least two vertices in $D$. Consider the following dominating configurations of $\alpha(C) + \alpha(G - C) - 1$ guards on $G$.

**U-configuration.** Place a guard on each vertex in $D$, and on a dominating set of cardinality $\alpha(C) - 1$ of $C$.

To define the next three configurations, assume that there are guards on $G$ in a $U$-configuration.

**W-configuration.** For some $d \in D - N(I)$ and some neighbor $w \in V' - D$ of $d$, move the guard on $d$ to $w$.

**X-configuration.** For some $d \in D \cap N(v)$, where $v \in I$, and some neighbor $x \in V' - D$ of $d$, move the guard on $d$ to $x$ and move the guards on $C$ so that there is a guard on the neighbor $z$ of $v$ in $C$.

**Y-configuration.** For some $d \in D \cap N(v)$, where $v \in I$, move the guard on $d$ to $v$.

Note that each configuration can be restored to a $U$-configuration in a single step such that an arbitrarily chosen vertex of $C$ contains a guard. Hence, from any configuration, any attack on $V(C) \cup D$ can be defended by a guard movement that results in a $U$-configuration. We show that attacks on $I$ and $V' - D$ can be defended from any configuration by a guard movement that results in one of these configurations. Since we have $\alpha(C) + \alpha(G - C) - 1 \leq \alpha(G) - 1$ guards, the result follows.

Suppose the guards are in a $U$-configuration.

- **Defend an attack on $v \in I$ by moving the guard on the neighbor $d \in D$ of $v$ to $v$, thus forming a Y-configuration.**
- **Defend an attack on $w \in V' - D$ by moving the guard on a neighbor $d \in D$ of $w$ to $w$.** If $d$ is adjacent to $v \in I$, also move the guards on $C$ so that there is a guard on the neighbor $z$ of $v$ on $C$. Now we have a $W$-or $X$-configuration.

Suppose the guards are in a $W$-configuration.

- **Defend an attack on $v \in I$ by moving the guard on the neighbor $d' \in D$ of $v$ to $v$, while moving the guard on $w$ back to $d$.** Move the guards on $C$ so that there is a guard on the neighbor $z$ of $v$ in $C$. This forms an $X$-configuration.
- **Defend an attack on $w' \in V' - D$ by moving the guard on a neighbor $d' \in D$, $d' \neq d$, of $w'$ to $w''$, and the guard on $w$ back to $d$.** If $d'$ is adjacent to $v \in I$, also move the guards on $C$ so that there is a guard on the neighbor $z$ of $v$ on $C$. Again we have a $W$-or $X$-configuration.

Suppose the guards are in an $X$-configuration.

- **Defend an attack on $v$ by moving the guard on the neighbor $z$ of $v$ in $C$ to $v$, the guards on $C$ to form a dominating set of $C$ or $C - \{z\}$, and the guard on $x$ back to $d$, resulting in a $Z$-configuration.** Defend an attack on $v' \in I - \{v\}$ by returning the guards to a $U$-configuration and proceeding as before—this can be done in one step.
- **Defend an attack on $x' \in V' - D$ similar to the defense when there is a $W$-configuration, ending in a $W$-or $X$-configuration as before.**

Suppose the guards are in a $Y$-configuration.

- **Defend an attack on $v' \in I - \{v\}$ by moving the guard on $v$ back to $d$, and the guard on $d' \in D$ adjacent to $v'$ to $v'$, which results in another $Y$-configuration.**
- **Defend an attack on $w \in V' - D$ by moving the guard on a neighbor $d'$ of $w$, $d' \in D - \{d\}$, to $w$, and the guard on $v$ back to $d$.** If $d'$ is adjacent to $v' \in I$, also move the guards on $C$ so that there is a guard on the neighbor $z$ of $v'$ on $C$. Again we have a $W$-or $X$-configuration.

Suppose the guards are in a $Z$-configuration.

- **Defend an attack on $v' \in I - \{v\}$ by moving the guard on $v$ back to $z \in V(C)$ and the guard on $d' \in D \cap N(v')$ to $v'$, thus forming a $Y$-configuration.**
- **Defend an attack on $w \in V' - D$ by moving the guard on $v$ back to $z \in V(C)$, the guard on $d \in D \cap N(w)$ to $w$, and the guards on $C$ appropriately to form a $W$-or $X$-configuration (depending on whether or not $d$ is adjacent to a vertex in $I$). 

**Proposition 13.** Let $G$ be a connected graph with $\delta(G) \geq 2$ and girth seven. Then $\gamma_\infty^m(G) < \alpha(G)$.

**Proof.** The proof is similar to, and uses the terminology of the proof of Theorem 12 with one exceptional case which we detail below. Let $C = v_1, \ldots, v_7, v_1$ and note that $\gamma_\infty^m(C) = \gamma(C) = 3 < \alpha(C) = 4$, and for any $v_i \in V(C)$, $\gamma(C - v_i) = \gamma(P_6) = 2$.

The exceptional case occurs when a vertex in $D$ is adjacent to $a, b \in I$. Then the distance between the neighbors of $a$ and $b$ on $C_7$ is exactly three, otherwise $G$ contains a cycle of length less than seven. It follows that each $d \in D$ is adjacent to at most two vertices in $I$. Say $a$ is adjacent to $v_i$ for some $i$, and $b$ is adjacent to $v_j$, where $i - j \equiv \pm 3 \pmod{7}$. We define three
more configurations of $\alpha(C) + \alpha(G - C) - 1$ guards on $G$. For the description of the first two configurations, again assume the guards are in a $U$-configuration.

**XX-configuration.** For some $d \in D \cap N(a) \cap N(b)$, where $a, b \in I$, and some neighbor $x \in V' - D$ of $d$, move the guard on $d$ to $x$ and move the guards on $C$ so that there are guards on the neighbors $v_i$ and $v_j$ of $a$ and $b$ in $C$.

**YY-configuration.** For some $d \in D \cap N(a) \cap N(b)$, where $a, b \in I$, move the guard on $d$ to $a$ and move the guards on $C$ so that there is a guard on the neighbor $v_j$ of $b$ in $C$.

**Z-configuration** (restatement). For some $v \in I$, let $N(v) \cap V(C) = \{z\}$, let $S$ be a dominating set of $C - \{z\}$, and place guards on each vertex in $\{v\} \cup S \cup D$.

Defend attacks on $G$ as before. In addition, when $d \in D$ is adjacent to $a, b \in I$, defend attacks as follows. Suppose the guards are in a $U$-configuration. Defend an attack on $a$ by moving the guard on $d$ to $a$, and the guards on $C$ so that there is a guard on $v_i \in V(C) \cap N(b)$, thus forming a YY-configuration. Defend an attack on $w \in V' - D$ adjacent to $d$ by moving the guard $d$ to $w$. Also move the guards on $C$ so that there are guards on the neighbors $v_i$ and $v_j$ of $a$ and $b$ in $C$. Now we have an XX-configuration.

If the guards are in a $W$-configuration, defend attacks as in the proof of Theorem 12 but with the guards forming an XX-configuration afterwards. If the guards are in an XX-configuration, an attack on $a$ or $b$ is defended as before, as is an attack on $x' \in V' - D$, except that the guards may form an XX-configuration if required.

If the guards are in a YY-configuration, defend an attack on $b$ by moving the guard on $v_j$ to $b$, the guards on $C$ to dominate $C - \{v_i\}$, and the guard on $a$ back to $d$. Now the guards are in a Z-configuration. Finally, if the guards are in a Z-configuration we proceed as before. □

We conclude with the following question.

**Question 1.** Let $G$ be a graph with $\delta(G) \geq 2$ and girth equal to five, six or eight. Is it true that $\gamma_m^\infty(G) < \alpha(G)$?

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**References**


