Hamiltonian group actions on symplectic Deligne–Mumford stacks and toric orbifolds

Eugene Lerman *, Anton Malkin 1

Department of Mathematics, University of Illinois, Urbana, IL 61801, United States

Received 26 August 2009; accepted 18 October 2011
Available online 25 October 2011
Communicated by the Managing Editors of AIM

Abstract

We develop differential and symplectic geometry of differentiable Deligne–Mumford stacks (orbifolds) including Hamiltonian group actions and symplectic reduction. As an application we construct new examples of symplectic toric DM stacks.

© 2011 Published by Elsevier Inc.

Keywords: Deligne–Mumford stack; Symplectic geometry; Symplectic reduction; Toric

1. Introduction

We have three goals in this paper. The most fundamental is to write down in a consistent form the basics of differential and symplectic geometry of orbifolds thought of as Deligne–Mumford (DM) stacks over the category of smooth manifolds. This includes descriptions of the tangent and cotangent bundles, vector fields, differential forms, Lie group actions, and symplectic reduction. Most if not all of these notions are well-known on the level of being “analogous to manifolds”. Recall that in the original approach of Satake [14] an orbifold is a topological space which is locally a quotient of a vector space by a finite group action. Smooth functions invariant under these local group actions form the structure sheaf. A more recent incarnation of this idea, largely due to Haefliger, is to think of an atlas on an orbifold as a proper étale Lie groupoid [11]. This approach makes it easy to define local geometric structures such as vector...
fields, differential forms, symplectic structures and Morse functions. However global structures such as Lie group actions are awkward to work with in an étale atlas. One of our observations is that global structures look much simpler in suitable non-étale atlases. So we prefer to think of orbifolds as a Deligne–Mumford (DM) stacks and compute in arbitrary atlases. The downside is that in an arbitrary groupoid atlas vector fields and differential forms look more complicated.

We show that there are consistent descriptions of all such geometric structures on a DM stack. More specifically, given a DM stack $X$ there is a presentation $X_1 \Rightarrow X_0 \rightarrow X$ so that any geometric structure on $X$ is given by a compatible pair of the corresponding structures on $X_1$ and $X_0$. For example a vector field (differential form, function) is a compatible pair of vector fields (differential forms, functions) on $X_1$ and $X_0$. Similarly given a Lie group action on $X$ there is an atlas $X_0 \rightarrow X$ so that the action can be described by a pair of free actions on $X_1$ and $X_0$. Such a presentation of a group action is useful even in the case of manifolds, where it can be thought of as a stacky version of replacing a $G$-manifold $M$ with $EG \times_G M$. Consequently the quotient of $X$ with respect to a $G$-action is represented by the quotients of $X_1$ and $X_0$. Similar statements hold for symplectic quotients, etc. A reader not comfortable with the abstract stack theory can safely take these pair-based descriptions as definitions. This is perfectly fine for applications, since the actual calculations are always done in atlases. However to show that the definitions make sense one should either prove that they are atlas-independent (i.e., Morita-invariant) or convince oneself that there is an abstract definition in terms of the stack $X$, the approach taken in this paper.

The paper is organized as follows. In Section 2 we discuss vector fields and forms on DM stacks and provide their description in non-étale atlases. We end the section with a definition of a symplectic DM stack.

In Section 3 we review group actions on stacks following Romagny [13], define Hamiltonian actions and prove an analogue of Marsden–Weinstein–Meyer reduction theorem for DM stacks.

In Section 4 we relate group actions on quotient stacks to group extensions. We then describe its symplectic analogue, which may be thought of as the stacky version of reduction in stages.

In Section 5 we take up symplectic toric DM stacks. Recall that symplectic toric manifolds are analogues of toric varieties in algebraic geometry, though symplectic-algebraic correspondence is not 1–1. Compact connected symplectic toric manifolds were classified by Delzant [4]. Delzant’s classification was extended to compact orbifolds by Lerman and Tolman [9]. However the class of orbifolds is not as natural as the class of DM stacks. For example it is not closed under taking substacks. For this reason we feel it is preferable to work with symplectic toric DM stacks rather than orbifolds.

In algebraic geometry the corresponding notion of a toric DM stack is still evolving. To the best of our knowledge, it first appeared in the work of Borisov, Chen, and Smith [2] as a construction. Later Iwanari [8] proposed the definition of a toric triple as an effective DM stack with an action of an algebraic torus having a dense orbit isomorphic to the torus. Recently, Fantechi, Mann and Nironi [5] gave a new definition of a smooth toric DM stack as DM stack with an action of a DM torus $T$ having a dense open orbit isomorphic to $T$. According to [5], a DM torus is a Picard stack isomorphic to $T \times B\Gamma$ where $T$ is an algebraic torus and $B\Gamma$ is the classifying stack of a finite abelian group $\Gamma$.

We define a symplectic toric DM stack as a symplectic DM stack with an effective Hamiltonian action of a compact torus. Then, generalizing a construction in [4], we produce a large class of examples of symplectic toric DM stacks as symplectic quotients of the form $(\mathbb{C}^N \times B\Gamma)\!/cA$, where $\Gamma$ is an arbitrary finite group and $A$ is a closed subgroup of $\mathbb{R}^N/\mathbb{Z}^N$ (symplectic quotients of DM stacks are defined in Theorem 3.13 below). From the point of view of symplectic
geometry the restriction that $\Gamma$ is abelian is unnatural. Note additionally that though we start with a trivial gerbe $\mathbb{C}^N \times B\Gamma$ over $\mathbb{C}^N$, the resultant toric DM stack $((\mathbb{C}^N \times B\Gamma)/cA)$ may be non-trivial as a gerbe over $\mathbb{C}^N/cA$.

2. Differential and symplectic geometry of DM stacks

2.1. Groupoids and stacks. All stacks in this paper are stacks over the category of smooth manifolds with the submersion Grothendieck topology [1,6,10]. Recall that a stack $\mathcal{X}$ is differentiable if there is an atlas (representable surjective submersion) $X_0 \to \mathcal{X}$, where $X_0$ is a manifold. Given an atlas one has a presentation $X_1 \times X_0 \to X_0 \to \mathcal{X}$ of $\mathcal{X}$ by a Lie groupoid. We use the notation $X_1 \rightrightarrows X_0$ for a Lie groupoid with the space of objects $X_0$ and the space of arrows $X_1$ (cf. [12]). The two maps from $X_1$ to $X_0$ are the source and target maps denoted by $s$ and $t$ respectively; we suppress the rest of the structure maps of the groupoid. Different presentations of the same stack are Morita equivalent as groupoids [1,6,10]. So one can think of differentiable stacks, to first approximation, as Morita-equivalence classes of Lie groupoids (this doesn’t quite work when group actions enter the picture).

The main goal of this section is to describe various geometric structures on DM stacks. As mentioned in the introduction, the common feature of these descriptions is that there is always a presentation $X_1 \times X_0 \Rightarrow X_0 \to \mathcal{X}$ in which the structure on $\mathcal{X}$ is given by a compatible pair of the corresponding structures on $X_1$ and $X_0$. A reader not comfortable with the abstract stack theory can safely take these pair-based descriptions as definitions. This is perfectly fine for applications, since the actual calculations are always done in atlases. However to show that the definitions make sense one should either prove that they are atlas-independent (i.e., Morita-invariant) or convince oneself that there is an abstract definition in terms of the stack $\mathcal{X}$.

Recall that a stack $\mathcal{X}$ is Deligne–Mumford (DM) if there is a presentation $U_1 \rightrightarrows U_0 \to \mathcal{X}$ such that the groupoid $U_1 \rightrightarrows U_0$ is étale (i.e., $s$ and $t$ are local diffeomorphisms) and proper (i.e., the map $s \times t : U_1 \to U_0 \times U_0$ is proper). Note that even if $\mathcal{X}$ is DM it is often useful and sometimes necessary to consider non-étale atlases of $\mathcal{X}$. See examples below.

2.2. Remark. The classical Satake definition of an orbifold can be reformulated as that of a DM stack with an étale atlas $U_1 \rightrightarrows U_0$ such that the stabilizers in $U_1$ of points in an open dense subset of $U_0$ are trivial. This condition is not preserved under taking substacks. So we consider a more general notion of a DM stack. In such a stack every point could have a non-trivial finite stabilizer group (also called inertia group). An example is the classifying stack $B\Gamma$ of a finite group $\Gamma$, which can be presented by the groupoid $\Gamma \Rightarrow pt$.

2.3. Lie algebroids and non-étale presentations of DM stacks. Recall (cf. [12]) that the Lie algebroid of a Lie groupoid $U_1 \rightrightarrows U_0$ is a vector bundle map called the anchor $A \to TX_0$, where $A = \ker ds|_{X_0}$, and $a = dt$.

The following theorem describes DM stacks in terms of their arbitrary (not necessarily étale) atlases. Such atlases will be crucial later when we study Lie group actions on DM stacks.

2.4. Theorem. Let $\mathcal{X}$ be a stack over the category of smooth manifolds. The following conditions are equivalent:

(2.4.i) $\mathcal{X}$ is DM, i.e., has a proper étale presentation $U_1 \rightrightarrows U_0 \to \mathcal{X}$;

(2.4.ii) $\mathcal{X}$ has a proper presentation $X_1 \rightrightarrows X_0 \to \mathcal{X}$ such that the anchor map of the corresponding Lie algebroid is injective;
(2.4.iii) $\mathcal{X}$ has a proper presentation $X_1 \Rightarrow X_0 \to \mathcal{X}$ such that the subbundles $\ker ds$ and $\ker dt$ of $TX_1$ are transverse;

(2.4.iv) $\mathcal{X}$ is proper differentiable (has a proper atlas) and for any presentation $X_1 \Rightarrow X_0 \to \mathcal{X}$ the anchor map of the corresponding Lie algebroid is injective;

(2.4.v) $\mathcal{X}$ is proper differentiable and for any presentation $X_1 \Rightarrow X_0 \to \mathcal{X}$ the subbundles $\ker ds$ and $\ker dt$ of $TX_1$ are transverse;

(2.4.vi) $\mathcal{X}$ is proper differentiable (has a proper atlas) and the inertia group of any point of $\mathcal{X}$ is finite.

**Proof.** This is essentially Theorem 1 of [3]. We sketch the main ideas of the proof. To go from an étale presentation $U_1 \Rightarrow U_0 \to \mathcal{X}$ to an arbitrary presentation $X_1 \Rightarrow X_0 \to \mathcal{X}$ consider the pullback diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\times} & U \\
\downarrow & & \downarrow \\
X_0 & \xrightarrow{\times} & \mathcal{X}.
\end{array}
$$

Note that the top row is a presentation of the manifold $U$ and hence the anchor map of the corresponding algebroid is injective. Now, since the vertical maps are étale, the same holds for the bottom row algebroid.

In the opposite direction, given a presentation $X_1 \Rightarrow X_0 \to \mathcal{X}$ such that inertia groups are all finite or equivalently the corresponding Lie algebroid has injective anchor map, the action of $X_1$ defines a foliation of $X_0$. One constructs an étale atlas of $\mathcal{X}$ using transverse slices to this foliation. We refer the reader to [3] for details. □

### 2.5. Differential forms and vector fields.

A smooth manifold $X$ comes equipped with the tangent $TX$ and the cotangent $T^*X$ bundles, the sheaf of vector fields $\text{Vect}_X$, and the de Rham complex of sheaves of differential forms $\Omega^\bullet_X$. Moreover $\text{Vect}_X$ is the sheaf $C^\infty_X$ of smooth sections of $TX \to X$, and $\Omega^\bullet_X$ is the sheaf $C^{\infty}_{\Lambda^*T^*X}$ of smooth sections of $\Lambda^*T^*X \to X$. The only part of this story compatible with pullbacks and hence defined for stacks is the de Rham complex. Namely, given an arbitrary stack $\mathcal{X}$ the de Rham complex of sheaves on $\mathcal{X}$ is defined as follows (see, for example, [1]): for an object $\nu \in \mathcal{X}$ over a manifold $U$ (in other words, a map $\nu : U \to \mathcal{X}$) one has $\Omega^\bullet_\mathcal{X}(\nu) = \Omega^\bullet_U(U)$, the de Rham complex on $U$. A differential form of degree $k$ on $\mathcal{X}$ is a global section of the sheaf $\Omega^k_\mathcal{X}$, i.e., a homomorphism from the trivial sheaf on $\mathcal{X}$ to $\Omega^k_\mathcal{X}$. However the sheaf $\Omega^k_\mathcal{X}$ is not the sheaf of sections of a vector bundle on $\mathcal{X}$ even if $\mathcal{X}$ is a DM stack. Moreover, though one can define tangent stack $T\mathcal{X}$ for an arbitrary stack $\mathcal{X}$ [7], the projection $\pi : T\mathcal{X} \to \mathcal{X}$ is not a vector bundle. Rather it is a 2-vector bundle. Hence sections of $\pi$ form not a sheaf of sets but a sheaf of groupoids. We refer the reader to [1] and [7] for discussions on sections of a map of stacks.

The situation is much better in the case of DM stacks, which is the reason naïve definitions used in orbifold theory work well for many purposes. However since we need to use arbitrary and not only étale atlases for DM stacks, we explain the concepts associated with tangent and cotangent bundles and their sections in some detail. A reader familiar with foliations or locally free group actions should recognize many constructions, such as presentations of transverse tangent bundles via Lie algebroids.
The crucial properties of an étale map (local diffeomorphism) \( f : M \to N \) is that one can pullback vector fields along \( f \) and that the pullback of the (co)tangent bundle is the (co)tangent bundle (i.e., \( f^*(TN) = TM \), \( f^*(T^*N) = T^*M \)). Hence the following definitions make sense for a DM stack \( \mathcal{X} \). Consider an étale presentation \( \mathcal{U}_1 \Rightarrow \mathcal{U}_0 \to X \). Then we have \( s^*(TU_0) = t^*(TU_0) = TU_1 \) and hence the bundle \( TU_0 \to U_0 \) descends to a vector bundle \( T\mathcal{X} \to \mathcal{X} \) called the tangent bundle of \( \mathcal{X} \). It is easy to see that the bundle \( T\mathcal{X} \to \mathcal{X} \) does not depend (up to equivalence) on the choice of the étale atlas. One defines the cotangent bundle \( T^*\mathcal{X} \to \mathcal{X} \) in a similar way. These are the usual definitions in the orbifold theory phrased in a fancy way.

Let us now consider an arbitrary presentation \( X_1 \to X_0 \xrightarrow{\xi} \mathcal{X} \) of our DM stack \( X \). We would like to describe pullbacks \( \xi^*(TX) \) and \( \xi^*(T^*X) \) of the tangent and the cotangent bundle \( T\mathcal{X} \). Let \( A \to TX_0 \) be the Lie algebroid of the Lie groupoid \( X_1 \to X_0 \), that is \( A = \ker ds|_{X_0} \), \( a = dt \). Since \( \mathcal{X} \) is DM, Theorem 2.4 implies that \( \ker ds \) and \( \ker dt \) are transverse as subbundles of \( TX_1 \) and the anchor map \( a \) is injective. We identify \( A \) with its image in \( TX_0 \) in what follows.

2.6. Proposition. Let \( X_1 \to X_0 \xrightarrow{\xi} \mathcal{X} \) be a presentation of a DM stack \( \mathcal{X} \) with the associated algebroid \( A \to TX_0 \). Then

(2.6.i) \( \xi^*(T\mathcal{X}) = TX_0/A \) as a vector bundle on \( X_0 \);
(2.6.ii) \( s^*(\xi^*(T\mathcal{X})) = t^*(\xi^*(T\mathcal{X})) = TX_1/(\ker ds + \ker dt) \) as a vector bundle on \( X_1 \);
(2.6.iii) \( \xi^*(T^*\mathcal{X}) = A^\perp \), where \( A^\perp \subset T^*X_0 = (TX_0)^* \) is the annihilator of \( A \);
(2.6.iv) \( s^*(\xi^*(T^*\mathcal{X})) = t^*(\xi^*(T^*\mathcal{X})) = (\ker ds)^\perp \cap (\ker dt)^\perp \subset T^*X_1 \).

Proof. Let \( U \to \mathcal{X} \) be an étale atlas of \( \mathcal{X} \). Then in the diagram

\[
\begin{array}{ccc}
U \times \mathcal{X} X_1 & \xrightarrow{\sim} & U \times \mathcal{X} X_0 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\sim} & X_0 \\
\downarrow & & \downarrow \\
\mathcal{X} & & \\
\end{array}
\]

vertical maps are étale. Hence it is enough to prove (2.6.i)–(2.6.iv) for the first row (a presentation of the manifold \( U \)), which a standard exercise in differential geometry of fibrations. \( \Box \)

2.7. Definition. We define sheaves of vector fields and differential forms on a DM stack \( \mathcal{X} \) as sheaves of smooth sections \( \mathcal{C}^\infty_{T\mathcal{X}} \) and \( \mathcal{C}^\infty_{A^\perp T^*\mathcal{X}} \) of the tangent and the exterior powers of the cotangent bundle respectively.

Sections of these vector bundles do form sheaves of vector spaces (as opposed to sheaves of groupoids) as follows from the following explicit description (cf. [7]). The space of sections \( \mathcal{C}^\infty_{T\mathcal{X}}(\nu) \) on an étale map \( U \xrightarrow{\nu} \mathcal{X} \), i.e., on an object \( \nu \) of \( \mathcal{X} \) over \( U \), is just \( \mathcal{C}^\infty_{TU}(U) \) – the space of vector fields on \( U \). For a pullback \( f \) of étale maps

\[
\begin{array}{ccc}
U & \xrightarrow{f} & \mathcal{X} \\
\downarrow & & \downarrow \\
V & & \\
\end{array}
\]
the pullback \( f^* : C^\infty_TX(V) \to C^\infty_TX(U) \) of sections is the pullback of vector fields under the étale map \( f \). Similarly \( C^\infty_{A^\ast T^\ast X}(U) = C^\infty_{A^\ast T^\ast U}(U) = \Omega^0_U(U) = \Omega^\ast_X(U) \) for an étale map \( U \xrightarrow{\varphi} \mathcal{X} \).

2.8. **Definition.** We define vector fields and differential forms on a stack \( \mathcal{X} \) as global sections of the corresponding sheaves.

Recall that the vector space of global sections of a sheaf \( F \) of vector spaces on \( \mathcal{X} \) is the vector space of homomorphisms from the trivial sheaf \( 1_\mathcal{X} \) to \( F \). Thus, given a presentation \( X_1 \xrightarrow{\xi} X_0 \xrightarrow{\varphi} \mathcal{X} \), the space of global sections of \( F \) is the equalizer of the two pullback maps \( F(X_0) \xrightarrow{\varphi^*} F(X_1) \).

The following proposition describes vector fields and differential forms in an atlas. Recall that for a surjective submersion \( f : Y \to X \) of manifolds, a vector field \( v_Y \in \text{Vect}(Y) \) and a vector field \( v_X \in \text{Vect}(X) \), one says that \( v_Y \) is \( f \)-related to \( v_X \) if

\[
df(v_Y) = v_X \circ f.
\]

Note that this is a relation, not a map \( \text{Vect}(Y) \to \text{Vect}(X) \). Given \( v_X, v_Y \) is determined up to a section of the bundle \( \ker df \subseteq TY \).

2.9. **Proposition.** Let \( X_1 \xrightarrow{\xi} X_0 \xrightarrow{\varphi} \mathcal{X} \) be a presentation of a DM stack \( \mathcal{X} \) with the associated algebroid \( A \hookrightarrow TX_0 \). Then:

(2.9.i) The Lie algebra \( \text{Vect}(\mathcal{X}) := C^\infty_TX(\mathcal{X}) \) of vector fields on \( \mathcal{X} \), i.e., of global sections of \( TX \), is isomorphic to the Lie algebra \( C^\infty_TX_0/A(X_0)^{X_1} \) of \( X_1 \)-invariant sections of the bundle \( TX_0/A \). Explicitly, vector fields on \( \mathcal{X} \) are equivalence classes of pairs consisting of a vector field \( v_0 \) on \( X_0 \) and a vector field \( v_1 \) on \( X_1 \), which are both \( s \)- and \( t \)-related:

\[
\text{Vect}(\mathcal{X}) \cong \{(v_1, v_0) \in \text{Vect}(X_1) \times \text{Vect}(X_0) \mid ds(v_1) = v_0 \circ s, \ dt(v_1) = v_0 \circ t\} / \{(v_1, v_0) \mid ds(v_1) = v_0 \circ s, \ dt(v_1) = v_0 \circ t, \ v_1 \in (\ker ds + \ker dt)\}.
\]

(2.9.ii) The de Rham complex \( \Omega^\ast(\mathcal{X}) := C^\infty_{A^\ast T^\ast X}(\mathcal{X}) \) of differential forms on \( \mathcal{X} \), i.e., of global sections of \( \Lambda^\ast T^\ast \mathcal{X} \), is isomorphic to the complex \( C^\infty_{A^\ast A^\perp}(X_0)^{X_1} \) of \( X_1 \)-invariant forms on \( X_0 \):

\[
\Omega^\ast(\mathcal{X}) \cong \{\tau \in \Omega^\ast(X_0) \mid s^\ast \tau = t^\ast \tau\},
\]

which can be also expressed as a set of pairs:

\[
\Omega^\ast(\mathcal{X}) \cong \{(\sigma_1, \sigma_0) \in \Omega^\ast(X_1) \times \Omega^\ast(X_0) \mid s^\ast \sigma_0 = \sigma_1, \ t^\ast \sigma_0 = \sigma_1\}.
\]

(2.9.iii) The contraction of vector fields and forms on \( \mathcal{X} \) is induced by the contraction of vector fields and forms on \( X_0 \) and \( X_1 \):

\[
\iota_{(v_1, v_0)}(\sigma_1, \sigma_0) = (\iota_{v_1} \sigma_1, \iota_{v_0} \sigma_0).
\]
Proof. By definition of the sheaf of sections of a vector bundle, pullbacks of sections are sections of pullback. Now the proposition follows from the explicit description 2.6 of the pullbacks of the tangent and the cotangent bundles to $X_0$. □

2.10. Remark. One can consider Proposition 2.9 above as a definition of vector fields and differential forms on a DM stack $\mathcal{X}$. Abstract definitions in terms of global sections of vector bundles on $\mathcal{X}$ ensure atlas-independence. One can also check the atlas-independence directly, without any reference to stacks.

2.11. Remark. Let $X_0 \to \mathcal{X}$ be an arbitrary (as opposed to étale) submersion. Then sections on $X_0$ of the sheaf $\mathcal{C}_\mathcal{X} \mathcal{T}^{\bullet}$ are differential forms on $X_0$ vanishing on the corresponding Lie algebroid $A \hookrightarrow TX_0$. Now recall that sections of the abstract de Rham complex $\Omega_{\mathcal{X}}^•$ (defined for arbitrary stacks, cf. [1]) are arbitrary forms on $X_0$. Hence $\mathcal{C}_\mathcal{X} \mathcal{T}^{\bullet}$ and $\Omega_{\mathcal{X}}^•$ are not isomorphic as sheaves. In particular $\Omega_{\mathcal{X}}^k$ is not a sheaf of sections of a vector bundle on $\mathcal{X}$. However the spaces of global sections (differential forms on $X_0$) and hypercohomology groups (de Rham cohomology of $X_0$) of these two sheaves are isomorphic. For example the space of global sections of either $\mathcal{C}_\mathcal{X} \mathcal{T}^{\bullet}$ or $\Omega_{\mathcal{X}}^•$ is isomorphic to the space of differential forms $\tau$ on $X_0$ satisfying $s^* \tau = t^* \tau$ on $X_1$.

2.12. Symplectic DM stacks. Given the above abstract definitions and explicit descriptions of vector fields and differential forms on a DM stack, it is now straightforward to define symplectic forms on DM stacks.

A 2-form $\omega \in \Omega^2(\mathcal{X})$ on a DM stack $\mathcal{X}$ is non-degenerate if the contraction with $\omega$ induces an isomorphism $\text{Vect}(\mathcal{X}) \to \Omega^1(\mathcal{X})$. In a presentation $X_1 \rightrightarrows X_0 \to \mathcal{X}$, the 2-form $\omega$ is given (represented) by a pair $(\omega_1, \omega_0)$, and it is non-degenerate iff $\ker \omega_0 = A \subset TX_0$, the Lie algebroid of $X_1 \rightrightarrows X_0$ or, equivalently, iff $\ker \omega_1 = \ker ds + \ker dt \subset TX_1$.

A symplectic form $\omega$ on a DM stack $\mathcal{X}$ is a non-degenerate closed 2-form $\omega \in \Omega^2(\mathcal{X})$. A symplectic DM stack is a pair $(\mathcal{X}, \omega)$, where $\mathcal{X}$ is a DM stack and $\omega$ is a symplectic form on $\mathcal{X}$.

3. Hamiltonian group actions on symplectic DM stacks

3.1. Group actions on stacks. (Following [13].) Group actions on stacks are more complicated than actions on manifolds because stacks are categories and the collection of all stacks forms a 2-category. Thus when a group acts on a stack, group elements act as functors but the composition of functors representing two group elements can differ from the functor representing their product by a natural transformation. In the case of Lie group $G$-action on a differentiable stack $\mathcal{X}$ we would like the action to be “smooth” so we represent it as a map $a : G \times \mathcal{X} \to \mathcal{X}$ instead of an action homomorphism from $G \to Aut(\mathcal{X})$. The fact that $a$ is an action is encoded in the 2-commutativity of the diagrams
where \( m \) is the multiplication map in \( G \), and \( e_G \) is the identity inclusion. The natural transformations \( \alpha \) and \( \epsilon \), which are part of the data defining the action, should satisfy further compatibility conditions [13]. The whole definition may be thought of as describing a stack over \( BG \).

Given an action \( a : G \times \mathcal{X} \to \mathcal{X} \) and an atlas \( X_0 \to \mathcal{X} \) we consider the composition

\[
G \times X_0 \to G \times \mathcal{X} \xrightarrow{a} \mathcal{X}
\]

to obtain a new atlas \( \widetilde{X}_0 = G \times X_0 \) of \( \mathcal{X} \). Then it follows from the definition of the action of \( G \) on \( \mathcal{X} \) that \( G \)-translations on \( G \times \mathcal{X} \) induce free \( G \)-actions on both \( \widetilde{X}_0 \) and \( \widetilde{X}_1 = \widetilde{X}_0 \times \mathcal{X} \times \widetilde{X}_0 \) and these actions commute with the structure maps of the presentation \( \widetilde{X}_1 \Rightarrow \widetilde{X}_0 \) of \( \mathcal{X} \).

More precisely, \( G \)-stacks \( [X_0/X_1] \) and \( [\widetilde{X}_0/\widetilde{X}_1] \) are isomorphic via an equivalence of fibered categories involving the natural transformations \( \alpha \) and \( \epsilon \) from the definition of the \( G \)-action on \( \mathcal{X} \). We refer the reader to [13] for details of this construction (called strictification in [13]) and summarize this discussion in the following proposition.

3.2. Proposition. Suppose a Lie group \( G \) acts on a differentiable stack \( \mathcal{X} \). Then \( \mathcal{X} \) has a presentation, called a \( G \)-presentation, \( X_1 \Rightarrow X_0 \to \mathcal{X} \) in which the \( G \)-action is given by free \( G \)-actions on \( X_1 \) and \( X_0 \) compatible with structure maps of the groupoid \( X_1 \Rightarrow X_0 \).

Proof. This is essentially Proposition 1.5 of [13]. The freeness of the actions is not stated there but follows from the proof.

One can define a \( G \)-stack as a stack represented by a Lie groupoid with free \( G \)-actions on the set of objects and arrows. This is a natural definition if one thinks of \( G \)-stacks as stacks over \( BG \), that is, over the site of principal \( G \)-bundles.

3.3. Quotient of a stack. Let \( G \) be a compact Lie group. Given a \( G \)-action on a differentiable stack \( \mathcal{X} \) with a \( G \)-presentation \( X_1 \Rightarrow X_0 \to \mathcal{X} \), there is a differentiable quotient stack \( \mathcal{X}/G \) defined by a universal property with respect to maps to manifolds with the trivial \( G \)-action [13]. A presentation of \( \mathcal{X}/G \) is given by the Lie groupoid \( X_1/G \Rightarrow X_0/G \to \mathcal{X}/G \). Note that \( X_1/G \) and \( X_0/G \) are manifolds since \( G \)-actions on \( X_0 \) and \( X_1 \) are free and proper.

This is the only place in the paper we use properness/compactness conditions. One can consider non-proper DM stacks and arbitrary Lie group actions with the quotient stack represented by the semidirect product groupoid \( G \times X_1 \Rightarrow X_0 \). We would like to stick to our philosophy that every structure/procedure (in particular, a \( G \)-action and quotient) should be represented by a pair

\[ (g, \gamma) \mapsto (g \gamma, \gamma) \]

for all composable arrows \( (g, \gamma), (g', \gamma') \) in \( G \times X_1 \).
of the corresponding structures/procedures on objects and arrows of an appropriate presentation. So we restrict ourselves to proper actions.

3.4. Example. Let a compact Lie group $G$ act on the classifying stack $BH$ of a compact Lie group $H$. The stack $BH$ has a presentation $H \ni pt \to BH$. Hence a $G$-presentation of $BH$ is given by $X_0 = G \times pt = G$, $X_1 = G \times_{BH} G = K \times G$, where $K$ is a principal $H$-bundle over $G$ and also a group. Therefore $G$-actions on $BH$ correspond to Lie group extensions $1 \to H \to K \to G \to 1$: an action of $G$ on $BH$ defines an extension of $G$ by $H$ and an extension of $G$ by $H$ defines an action of $G$ on $BH$. The $G$-presentation of $BH$ corresponding to such an extension is given by the action groupoid $K \times G \ni G \to BH$ for the right action of $K$ on $G$, and $G$-action on $BH$ in this presentation comes from the left action of $G$ on $G$. The quotient stack $BH/G$ is equivalent to $BK$. See 4.1 for a generalization of this example and of the example below.

3.5. Example. Suppose a Lie group $G$ acts on a manifold $M$. Then we have a $G$-atlas $X_0 = M \times G$ of $M$, with the $G$-presentation given by the action groupoid $G \times (M \times G) \ni M \times G \to M$ corresponding to the following action of $G$ on $M \times G$: $g \cdot (m, g') = (g \cdot m, g^{-1}g')$. The action of $G$ on $M$ in this presentation is given by left translations on $G$: $g \cdot (m, g') = (m, gg')$. Hence, even in the case of manifolds, stacky point of view has advantages: one can replace arbitrary action on a manifold by a free action on a groupoid (i.e., “resolve” of the original action). The quotient stack is $M/G$ with the presentation $G \times M \ni M \to M/G$.

3.6. Infinitesimal actions. Given an action $a : G \times x \to x$ of a Lie group $G$ on a DM stack $X$, we obtain the derived map (infinitesimal action)

$$da : (g, 0) \mapsto \text{Vect}(G \times x) \to \text{Vect}(x),$$

where we think of the Lie algebra $g$ as the space of right-invariant vector fields on $G$. Moreover, though we don’t use it in this paper, infinitesimal actions on DM stacks have all the usual properties of infinitesimal actions on manifolds. For example, they are homomorphisms of Lie algebras. The proofs are identically the same as in the manifold case since in the case of DM stacks the natural transformations involved in the definition of the action act trivially on tangent spaces and their maps. Roughly speaking, it is hard to describe a Lie group action on an orbifold, but a Lie algebra action is just given by vector fields.

Now suppose $X_1 \ni X_0 \to x$ is a $G$-presentation of $X$. Differentiating free actions of $G$ on $X_1$ and $X_0$ we get a pair of vector fields $(v_1(\varepsilon), v_0(\varepsilon)) \in \text{Vect}(X_1) \times \text{Vect}(X_0)$ for every $\varepsilon \in g$. Moreover, since the structure maps of $X_1 \ni X_0$ commute with the $G$-actions on $X_1$ and $X_0$, we have $ds(v_1(\varepsilon)) = v_0(\varepsilon) \circ s$ and $dt(v_1(\varepsilon)) = v_0(\varepsilon) \circ t$. Hence (cf. (2.9.i)) the pair $(v_1, v_0)$ defines a vector field on the stack $x$. This vector field is $da(\varepsilon)$ in the presentation $X_1 \ni X_0$. One can consider this description as the definition of $da(\varepsilon)$. The abstract definition above ensures atlas-independence.

Note that the vector field $da(\varepsilon)$ is nonvanishing if $v_0(\varepsilon)$ is a nowhere zero section of $TX_0/A \to X_0$ (and $v_1(\varepsilon)$ is a nowhere zero section of $TX_1/(\ker ds + \ker dt) \to X_1$); compare with Proposition 2.6.

3.7. Locally free actions. Similar to the manifold case we say that an action $a : G \times x \to x$ of a Lie group $G$ on a DM stack $X$ is locally free if the corresponding action of the Lie algebra is free, i.e., $da(\varepsilon)$ is a nonvanishing vector field for every $0 \neq \varepsilon \in g$. One should not confuse
this condition with the freeness of $G$-action in an atlas. For example, in a $G$-presentation $X_1 \rightrightarrows X_0 \to \mathcal{X}$ the group $G$ acts freely on both $X_1$ and $X_0$, and such a presentation exists for arbitrary $G$-action on $\mathcal{X}$.

3.8. Lemma. Let $\alpha : G \times \mathcal{X} \to \mathcal{X}$ be an action of a compact Lie group $G$ on a DM stack $\mathcal{X}$, with a $G$-presentation $X_1 \rightrightarrows X_0 \to \mathcal{X}$. In particular, we have free $G$-actions on $X_1$ and $X_0$ and the quotient stack $\mathcal{X}/G$ is represented by $X_1/G \rightrightarrows X_0/G \to \mathcal{X}/G$. Let $B_1 \subset TX_1$ and $B_0 \subset TX_0$ be the subbundles spanned by infinitesimal vector fields generating the (free) actions of the Lie algebra $\mathfrak{g}$ on $X_1$ and $X_0$ respectively, and $A \to TX_0$ the Lie algebroid of $X_1 \rightrightarrows X_0$. Then the following conditions are equivalent:

- (3.8.i) The $G$-action on $\mathcal{X}$ is locally free;
- (3.8.ii) The subbundles $B_1$, $\ker ds$ and $\ker dt$, of $TX_1$ are transverse;
- (3.8.iii) The subbundles $B_0$ and $A$ of $TX_0$ are transverse;
- (3.8.iv) The quotient stack $\mathcal{X}/G$ is DM.

Proof. The explicit description (2.9.i) of vector fields on a differentiable stack implies that conditions (3.8.i), (3.8.ii), and (3.8.iii) are equivalent. The source $s$ and the target $t$ maps of the quotient groupoid $X_1/G \rightrightarrows X_0/G$ are induced by the source and target maps of the original groupoid $X_1 \rightrightarrows X_0$. Hence $\ker(ds)$ and $\ker(dt)$ are transverse (equivalently, $\mathcal{X}/G$ is DM, cf. Theorem 2.4) iff (3.8.ii) is satisfied.

3.9. DM stacks given by equations. Let $\mathcal{X}$ be a DM stack, $f : \mathcal{X} \to \mathbb{R}^n$ a function, and $\xi \in \mathbb{R}^n$. Consider the substack $f^{-1}(\xi)$. In general this is not a differentiable stack. Let $X_1 \rightrightarrows X_0 \to \mathcal{X}$ be a presentation of $\mathcal{X}$. Then $f = (f_1, f_0)$, where $f_0 : X_0 \to \mathbb{R}^n$ and $f_1 = f_0 \circ s = f_0 \circ t : X_1 \to \mathbb{R}^n$. We say that $\xi$ is a regular value of $f$ if it is a regular value of $f_0$, that is, if the differential $(df_0)_x$ is surjective at any point $x \in f_0^{-1}(\xi)$. Note that the surjectivity condition on $df_0$ is preserved under precomposition with submersions. Hence the regularity condition on $f$ does not depend on the choice of a presentation of $\mathcal{X}$.

Assume $\xi$ is a regular value of $f : \mathcal{X} \to \mathbb{R}^n$. Then $f_1^{-1}(\xi) \rightrightarrows f_0^{-1}(\xi)$ is a Lie subgroupoid of $X_1 \rightrightarrows X_0$ representing $f^{-1}(\xi)$. Hence $f_1^{-1}(\xi)$ is a differentiable stack. If we assume that $\mathcal{X}$ is DM then $f^{-1}(\xi)$ is also DM (for example, by the above argument in an étale presentation of $\mathcal{X}$). We record these observations as a lemma.

3.10. Lemma. Let $\mathcal{X}$ be a differentiable stack, $f : \mathcal{X} \to \mathbb{R}^n$ be a function, and $\xi$ a regular value of $f$. Then $f^{-1}(\xi)$ is a differentiable stack. If $\mathcal{X}$ is DM then so is $f^{-1}(\xi)$.

3.11. Hamiltonian actions. Let $\mathcal{X}$ be a DM stack, $\omega \in \Omega^2(\mathcal{X})$ a closed 2-form, and $\alpha : G \times \mathcal{X} \to \mathcal{X}$ an action of a Lie group $G$ on $\mathcal{X}$. If $\omega$ is non-degenerate then $(\mathcal{X}, \omega)$ is a symplectic DM stack, but this condition is not important for the following definition.

We say that an action $\alpha : G \times \mathcal{X} \to \mathcal{X}$ is Hamiltonian if there exists an equivariant map $\mu : \mathcal{X} \to \mathfrak{g}^*$ such that

$$\iota_{d\alpha(e)}\omega = d\langle \varepsilon, \mu \rangle$$

for any $\varepsilon \in \mathfrak{g}$. We refer to $\mu$ as a moment map.
In a $G$-presentation $X_1 \rightrightarrows X_0 \rightarrow \mathcal{X}$ there are two points of view on differential forms. If one thinks of $\omega$ and $\mu$ as forms on $X_0$ satisfying pullback conditions on $X_1$ then Eq. (3.1) reads
\begin{equation}
\iota_{\mu_0} \omega = d\langle \varepsilon, \mu \rangle
\end{equation}
where $u_{\varepsilon}$ is the $\text{Vect}(X_0)$-component of the vector field $d\gamma(\varepsilon)$. Essentially, in a $G$-presentation we have a Hamiltonian action on $X_0$. Note that, while $u_{\varepsilon}$ is defined only up to addition of sections of the algebroid $A$, this ambiguity does not matter in Eq. (3.2) because $\omega$ vanishes on $A$.

If one thinks of differential forms, vector fields, actions, etc., as compatible pairs of the corresponding objects on $X_0$ and $X_1$, then a Hamiltonian $G$-action on a symplectic DM stack is a presentation $X_1 \rightrightarrows X_0 \rightarrow \mathcal{X}$ together with a pair of free Hamiltonian $G$-actions on $X_1$ and $X_0$ (with respect to two possibly degenerate 2-forms) such that the groupoid structure maps intertwine these actions. This is our preferred point of view.

3.12. **Proposition.** Let $(a : G \times \mathcal{X} \rightarrow \mathcal{X}, \mu : \mathcal{X} \rightarrow g^*)$ be a Hamiltonian action of a Lie group $G$ on a symplectic DM stack $(\mathcal{X}, \omega)$. Then the action is locally free iff the moment map is regular everywhere (i.e., any value of $\mu$ is regular).

**Proof.** In a $G$-presentation $X_1 \rightrightarrows X_0 \rightarrow \mathcal{X}$ both conditions of the theorem are equivalent to the condition that $B_0$ and $A$ are transverse in $TX_0$, where $A$ is the Lie algebroid of $X_1 \rightrightarrows X_0$ and $B_0$ is as in Lemma 3.8. □

We are now in the position to state and prove the main result of the section: the DM version of the symplectic quotient construction.

3.13. **Theorem** (Symplectic reduction). Let $(a : G \times \mathcal{X} \rightarrow \mathcal{X}, \mu : \mathcal{X} \rightarrow g^*)$ be a Hamiltonian action of a compact Lie group $G$ on a symplectic DM stack $(\mathcal{X}, \omega)$. Suppose $\xi \in g^*$ is a regular value of $\mu$ which is fixed by the coadjoint action of $G$. Then

$$\mathcal{X}/\xi G := \mu^{-1}(\xi)/G$$

is a DM stack, and $\omega|_{\mu^{-1}(\xi)}$ descends to a symplectic form on $\mathcal{X}/\xi G$.

**Proof.** By Lemma 3.10 $\mu^{-1}(\xi)$ is a DM stack. Since the coadjoint action of $G$ fixes $\xi$ by assumption, $G$ acts on $\mu^{-1}(\xi)$. An argument similar to the proof of Proposition 3.12 shows that the action of $G$ on $\mu^{-1}(\xi)$ is locally free. Hence, by Lemma 3.8, $\mu^{-1}(\xi)/G$ is a DM stack.

Let $X_1 \rightrightarrows X_0 \rightarrow \mathcal{X}$ be a $G$-presentation of $\mathcal{X}$. Then $\omega = (\omega_1, \omega_0)$, $\mu = (\mu_1, \mu_0)$, and $\mu_1^{-1}(\xi)/G \cong \mu_0^{-1}(\xi)/G \rightarrow \mu^{-1}(\xi)/G$ is a presentation of $\mu^{-1}(\xi)/G$, where the groupoid maps are induced by those of $X_1 \rightrightarrows X_0$. Lemma 3.8 implies that the Lie algebroid $A$ of $X_1 \rightrightarrows X_0$ descends to a subbundle $\hat{A}$ of $T(\mu_0^{-1}(\xi)/G)$ and $\hat{A}$ is the Lie algebroid of $\mu_1^{-1}(\xi)/G \cong \mu_0^{-1}(\xi)/G$. Let $B_1 \subset TX_1$ and $B_0 \subset TX_0$ be as in Lemma 3.8. The moment map equation (3.1) implies that $\ker \omega_0|_{\mu_1^{-1}(\xi)} = A|_{\mu_1^{-1}(0)} + B_0|_{\mu_0^{-1}(0)}$. Hence $\omega_0$ descends to a closed 2-form $\hat{\omega}_0$ on $\mu_0^{-1}(0)/G$ with the kernel $\ker \hat{\omega}_0 = \hat{A}$. Since the groupoid maps of $\mu_1^{-1}(\xi)/G \cong \mu_0^{-1}(\xi)/G \rightarrow \mu^{-1}(\xi)/G$ are induced by those of $X_1 \rightrightarrows X_0$ we have $t^*\hat{\omega}_0 = s^*\hat{\omega}_0$ and so $\hat{\omega}_0$ defines a symplectic form on $\mu^{-1}(\xi)/G$. □
3.14. Remark. It is easy to modify the above discussion to describe reduction on a level \( \xi \) which is regular but not coadjoint-invariant. One either replaces \( G \)-quotient by the quotient with respect to the stabilizer of \( \xi \) or uses the usual multiplication by the coadjoint orbit trick. Nor is the restriction that the group \( G \) is compact very important. The same result holds for proper actions of non-compact Lie groups. We concentrate on the simple case to avoid complicating the notation and to emphasize the stacky features of the reduction.

4. Group extension and actions on quotient stacks

A typical example of a DM stack is the quotient \([M/H]\) of a manifold \(M\) by a proper locally free action of a Lie group \(H\). This stack is represented by the Lie groupoid \(H \times M \rightrightarrows M\), where the source map is the projection and the target map is the action. The corresponding Lie algebroid (as a subbundle of \(TM\)) is spanned by vector fields generating the action of \(h\), the Lie algebra of \(H\).

4.1. Actions on a quotient stack. Suppose we have an exact sequence

\[
1 \to H \to K \to G \to 1
\]

of Lie groups and an action of \(K\) on a manifold \(M\). Then the quotient \(G = K/H\) acts on the topological quotient \(M/H\). It is then reasonable to expect that the extension of \(G\) by \(H\) also defines an action of \(G\) on the stack quotient \([M/H]\). To define this action we describe a \(G\)-atlas of \([M/H]\).

Consider a \(K\)-action groupoid \(K \times (M \times G) \rightrightarrows M \times G\) associated to the following \(K\)-action on \((M \times G)\):

\[
k \cdot (m, g) = (k \cdot m, g[k]^{-1}),
\]

where \([k]\) is the image of \(k\) in \(G = K/H\). Note that \(G\) acts on \(M \times G\) by left translations on \(G\):

\[
g \cdot (m, g') = (m, gg').
\]

This \(G\)-action commutes with the \(K\)-action and hence \(K \times (M \times G) \rightrightarrows M \times G\) is a \(G\)-atlas of the stack \([(M \times G)/K]\) with a \(G\)-action. Proposition 4.2 below shows that, in fact, \([(M \times G)/K]\) is isomorphic to \([M/H]\), where \(H\) acts on \(M\) by the restriction of the \(K\)-action. Thus we obtain a \(G\)-action on \([M/H]\) together with a \(G\)-atlas from an action of an extension \(K\) on \(M\).

4.2. Proposition. The action groupoids \(K \times (M \times G) \rightrightarrows (M \times G)\) and \(H \times M \rightrightarrows M\) are Morita-equivalent, thus define isomorphic stacks.

Proof. Consider the manifold \(M \times K\). It has a free \(K\)-action given by

\[
k \cdot (m, k') = (km, k'k^{-1})
\]

and a commuting free \(H\)-action given by

\[
h \cdot (m, k) = (m, hk).
\]
The map
\[ \pi_1 : M \times K \to M, \quad (m, k) \mapsto km \]
is a principal \( K \)-bundle and is \( H \)-equivariant. The map
\[ \pi_2 : M \times K \to M \times G, \quad (m, k) \mapsto (m, [k]) \]
is a principal \( H \)-bundle and is \( K \)-equivariant, where \( K \)-action on \( M \times G \) is given by (4.1). Thus
\[ M \xleftarrow{\pi_1} M \times K \xrightarrow{\pi_2} M \times G \]
is a biprincipal bibundle between the action groupoids \( H \times M \rightrightarrows M \) and \( K \times (M \times G) \rightrightarrows M \times G \). Thus the two action groupoids are Morita-equivalent by definition. 

4.3. Hamiltonian actions on symplectic quotients. We adapt the above constructions to symplectic geometry. Consider a Hamiltonian action of a compact Lie group \( H \) on a symplectic manifold \((M, \omega_M)\) with the moment map \( \mu_H : M \to h^* \). Let \( \xi_H \) be a regular value of \( \mu_H \). Assume \( \xi_H \in h^* \) is fixed by the coadjoint action of \( H \). As we remarked above, this is not a very restrictive assumption since we can always replace \( H \) in the subsequent discussion by the stabilizer of \( \xi_H \). Let \( Z = \mu_H^{-1}(\xi_H) \). Then \( Z \) is an \( H \)-invariant manifold and \([Z/H] \) is a symplectic DM stack, the symplectic quotient \( M//\xi_H H : = [Z/H] \).

Next we construct a Hamiltonian action of another Lie group \( G \) on the symplectic stack \([Z/H] \). As in the previous section we consider a Lie group extension
\[ 1 \to H \to K \to G \to 1 \]
with the corresponding exact sequences of Lie algebras and dual spaces
\[ 0 \to h \xrightarrow{j} \mathfrak{k} \to g \to 0, \]
\[ 0 \leftarrow h^* \xleftarrow{i^*} \mathfrak{k}^* \leftarrow g^* \leftarrow 0. \]

Suppose there is a Hamiltonian action of \( K \) on \((M, \omega_M)\) with the moment map \( \mu_K \) such that the restriction of this action to \( H \) is the original \( H \)-action and the restriction of \( \mu_K \) is \( \mu_H \) (more precisely \( \mu_H = i^* \circ \mu_K \)).

The symplectic analogue of the manifold \( M \times G \) from 4.1 is the symplectic manifold \( M \times T^*G \). We denote by \( \lambda_G^l, \lambda_G^r : T^*G \to g^* \) the projections corresponding to the left and the right trivializations of \( T^*G = g^* \times G \) respectively. The canonical symplectic form on \( T^*G \) is given by \( \omega_{T^*G} = d\langle \lambda_G^l, \theta^l_G \rangle \), where \( \theta^l_G \) is the left-invariant Maurer–Cartan form on \( G \). With respect to this symplectic form \(-\lambda_G^l \) and \( \lambda_G^r \) are the moment maps of the right and the left translations respectively.

Consider the canonical lift of the action (4.1) of \( K \) on \( M \times G \) to \( M \times T^*G \). The lifted action is Hamiltonian with respect to the moment map
\[ v = \mu_K - \lambda_G^l : M \times T^*G \to \mathfrak{k}^*, \]
where we think of \( g^* \) as a subspace of \( \mathfrak{k}^* \).
Fix a lift $\xi_K \in \mathfrak{t}^*$ of $\xi_H \in \mathfrak{h}^*$ and consider the symplectic quotient of $M \times T^*G$ at the level $\xi_K \in \mathfrak{t}^*$. We assume $\xi_K \in \mathfrak{t}^*$ is invariant under the coadjoint action of $K$, in particular $Z = \mu_K^{-1}(\xi_K)$ is $K$-invariant. Again this assumption is not essential but makes statements simpler. The level set $v^{-1}(\xi_K)$ can be described as follows (we use left trivialization of $T^*G$):

$$v^{-1}(\xi_K) = \{(m, \mu_K(m) - \xi_K, g) \in M \times \mathfrak{g}^* \times G = M \times T^*G \mid \mu_H(m) = \xi_H\} \simeq Z \times G.$$ 

Hence we get the expected $G$-atlas $Z \times G$ for the DM stack $[Z/H]$. Additionally the description of this atlas as the moment level set $v^{-1}(\xi_K)$ provides it with the closed 2-form

$$\omega_{Z \times G} = \omega_{M \times T^*G}|_{v^{-1}(\xi_K)} = \pi_Z^*\omega_{M}|_{Z} + d(\mu_K \circ \pi_Z - \xi_H, \theta_G^1),$$

where $\pi_Z : Z \times G \to Z$ is the projection. The form $\omega_{Z \times G}$ is $K$-invariant and degenerate precisely along $\mathfrak{t}$-action distribution, hence defines a symplectic form on the action groupoid $K \times (Z \times G) \to Z \times G$. Moreover the left action of $G$ on $T^*G$ induces a Hamiltonian action on $Z \times G = v^{-1}(\xi_K) \subset M \times T^*G$ with the Hamiltonian $\eta = \lambda_G^r$ or, explicitly,

$$\eta(z, g) = \text{Ad}^*(g)(\mu_K(z) - \xi_K) : Z \times G \to \mathfrak{g}^*.$$ 

Note that different choices of the lift $\xi_K$ of $\xi_H$ correspond to shifts of $\eta$ by a $(G$-invariant) constant. Also $\eta^{-1}(0) = \mu_K^{-1}(\xi_K)$, hence the symplectic reduction of the groupoid $K \times (Z \times G) \to Z \times G$ with respect to the $G$-action is symplectomorphic to $K \times \mu_K^{-1}(\xi_K) \simeq \mu_K^{-1}(\xi_K)$, representing the symplectic quotient of $M$ with respect to $K$. This is the reduction in stages theorem.

Finally, repeating the proof of Proposition 4.2 with two commuting free Hamiltonian actions of $K$ and $H$ on $M \times T^*K$ in place of the actions on $M \times K$ one can easily show that the symplectic action groupoids $K \times (M \times G) \to (M \times G)$ and $H \times M \to M$ define isomorphic symplectic stacks. The modifications required in the proof are similar to the above discussion of the relation between $M \times T^*G$ and $Z \times G$ and are left to the reader.

Putting everything together we have the following theorem.

**4.4. Theorem.** Let $1 \to H \to K \to G \to 1$ be a sequence of compact Lie groups and $i^* : \mathfrak{t}^* \to \mathfrak{h}^*$ the corresponding canonical projection. Suppose there is a Hamiltonian action of $K$ on a symplectic manifold $(M, \omega_M)$ with a moment map $\mu_K : M \to \mathfrak{t}^*$. Suppose $\xi_K \in \mathfrak{t}^*$ is $K$-invariant and suppose $\xi_K := i^*(\xi_K)$ is a regular value of the $H$-moment map $\mu_H := i^* \circ \mu_K$. Then

- (4.4.i) $[\mu_H^{-1}(\xi_H)/H]$ is a symplectic DM stack with the symplectic form induced by the restriction $\omega_M|_{\mu_H^{-1}(\xi_H)}$;
- (4.4.ii) There is a Hamiltonian action of $G$ on $[\mu_H^{-1}(\xi_H)/H]$ induced by the action of $K$ on $M$;
- (4.4.iii) Assume $\xi_K$ is a regular value of $\mu_K$. Then the symplectic reduction $[\mu_H^{-1}(\xi_H)/H]/\partial G$ at 0 of $[\mu_H^{-1}(\xi_H)/H]$ with respect to the $G$-action defined in (4.4.ii) is a symplectic DM stack isomorphic to $[\mu_K^{-1}(\xi_K)/K]$:

$$[\mu_H^{-1}(\xi_H)/H]/\partial G \simeq [\mu_K^{-1}(\xi_K)/K].$$
5. Symplectic toric DM stacks

5.1. Definition. A symplectic G-toric DM stack is a symplectic DM stack $\mathcal{X}$ with a Hamiltonian action of a compact torus $G$, such that

- $\dim \mathcal{X} = 2\dim G$ and
- the action of $G$ on the coarse moduli space of $\mathcal{X}$ is effective.

This is a natural definition of a toric object in the context of symplectic DM stacks.

5.2. Finite extension of a torus. As an example of an application of Theorem 4.4 we construct a symplectic toric DM stack as symplectic quotients of $\mathbb{C}^N$ (a generalization of Delzant’s construction [4] of symplectic toric manifolds). We start with an extension

$$1 \to \Gamma \to \hat{T}_N \to T_N \to 1$$

of the $N$-dimensional compact torus $T_N := \mathbb{R}^N/\mathbb{Z}^N$ by a finite group $\Gamma$. Then for any closed subgroup $A$ of $T_N$ we have an extension $\hat{A}$ by $\Gamma$. The standard action of $T_N$ on the symplectic space $(\mathbb{C}^N, \omega_{\mathbb{C}^N} = \sqrt{-1} \sum_j dz_j \wedge d\bar{z}_j)$ gives rise to an (ineffective) Hamiltonian action of $\hat{T}_N$ on $(\mathbb{C}^N, \omega_{\mathbb{C}^N})$ with the “same” moment map $\mu: \mathbb{C}^N \to (\mathbb{R}^N)^*$. We would like to apply Theorem 4.4 which requires a suitable ($\hat{T}_N$-invariant) choice of the moment map level $\xi$. The following lemma ensures that any level of the moment map works fine in our situation.

5.3. Lemma. Let $1 \to \Gamma \to \hat{T} \to T \to 1$ be an extension of a connected abelian Lie group $T$ by a finite group $\Gamma$. Then the coadjoint representation of $\hat{T}$ is trivial.

Proof. Denote the connected component of the identity of $\hat{T}$ by $\hat{T}_0$. Since $\hat{T}_0$ is a connected cover of an abelian Lie group, it is an abelian Lie group and its adjoint action is trivial. Hence it is enough to show that the adjoint action of $\Gamma$ is trivial. Since $\Gamma$ is finite, for any $\gamma \in \Gamma$, $X \in \hat{T}$, and $t \in \mathbb{R}$, we have

$$\exp(tX) \gamma \exp(-tX) = \exp(0X) \gamma \exp(-0X) = \gamma.$$ 

Hence

$$\gamma \exp(tX) \gamma^{-1} = \exp(tX)$$

for all $t \in \mathbb{R}$. Taking derivatives of both sides with respect to $t$ at $t = 0$ we get $Ad(\gamma)X = X$. \qed

5.4. Theorem. Let $1 \to \Gamma \to \hat{T}_N \to T_N \to 1$ be an extension of the standard $N$-dimensional torus $T_N = \mathbb{R}^N/\mathbb{Z}^N$ by a finite group $\Gamma$, $A < T_N$ a closed subgroup and $\hat{A} < \hat{T}_N$ the corresponding subgroup of $\hat{T}_N$. Let $a \in a^*$ be a regular value of the A-moment map $\mu_{\hat{A}}: \mathbb{C}^N \to a^*$. Then the stack quotient

$$\mathbb{C}^N//\hat{A} = \left[ \mu_{\hat{A}}^{-1}(a)/\hat{A} \right]$$

is a symplectic toric $G$-stack, where $G = \hat{T}_N/\hat{A} = T_N/A$. 

5.5. Remark. The same stack can be obtained as the reduction of the symplectic DM stack $\mathbb{C}^N \times B\Gamma$ with respect to the diagonal Hamiltonian action of $A$, where the action of $A$ on $B\Gamma$ is defined by the extension $1 \to \Gamma \to \hat{A} \to A \to 1$:

$$\mathbb{C}^N \sslash_a \hat{A} \cong (\mathbb{C}^N \times B\Gamma) \sslash_a A.$$ 

This is the point of view taken in the Introduction.

Proof of Theorem 5.4. By Theorem 4.4 with $K = \hat{T}^N$, $H = \hat{A}$, $G = \hat{T}^N / \hat{A} = T^N / A$ and $(M, \omega) = (\mathbb{C}^N, \omega_{\mathbb{C}^N})$ the symplectic quotient

$$\mathbb{C}^N \sslash_a \hat{A} := \left[ \frac{\mu_\hat{A}^{-1}(a)}{\hat{A}} \right]$$

is a symplectic DM stack with a Hamiltonian action of the torus $G$ for any regular value $a \in a^*$ of the moment map $\mu_\hat{A}$. We have $\dim \mathbb{C}^N \sslash_a \hat{A} = 2N - \dim A - \dim A = 2\dim G$. So to prove that the symplectic DM stack $\mathbb{C}^N \sslash_a \hat{A}$ is $G$-toric it remains to check that the action of $G$ on its coarse moduli space $\mu_\hat{A}^{-1}(a)/\hat{A}$ is effective.

We first argue that $Z \cap (\mathbb{C}^N)^* \neq \emptyset$, where $Z = \mu_H^{-1}(a)$. The moment map $\mu_K : \mathbb{C}^N \to (\mathbb{R}^N)^*$ for the action of $K = \hat{T}^N$ on $\mathbb{C}^N$ is given by $\mu_K(z_1, \ldots, z_N) = \sum |z_j|^2 e_j^*$, where $\{e_j^*\}$ is the basis of $(\mathbb{R}^N)^*$ dual to the standard basis $\{e_j\}$ of $\mathbb{R}^N = \mathfrak{k}$ (which is also a basis of $\mathbb{Z}^N$). The image $\mu_K(\mathbb{C}^N)$ is the orthant

$$\left(\mathbb{R}^N\right)_+^* := \left\{ \eta \in \left(\mathbb{R}^N\right)^* \mid \langle \eta, e_j \rangle \geq 0, \ 1 \leq j \leq N \right\}$$

with $\mu_K$ mapping $(\mathbb{C}^N)^*$ to the interior of the orthant. Since $\mu_H$ is the restriction of $\mu_K$, we have

$$Z \cap (\mathbb{C}^N)^* \neq \emptyset \iff V_a \cap \text{interior}(\mathbb{R}^N)^*_+ \neq \emptyset, \quad (5.1)$$

where the affine subspace $V_a \subset (\mathbb{R}^N)^* = \mathfrak{k}^*$ is the preimage of $a \in \mathfrak{h}^*$. If $\tilde{a} \in V_a$ then $V_a = \tilde{a} + \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ denotes the annihilator of $\mathfrak{h}$ in $(\mathbb{R}^N)^*$.

The faces of the orthant $(\mathbb{R}^N)^*_+$ are images under $\mu_K$ of coordinate subspaces of the form

$$\mathcal{S} = \mathcal{S}(i_1, \ldots, i_n) := \{z_{i_1} = 0, \ldots, z_{i_n} = 0\}$$

for some subset $\{i_1, \ldots, i_n\} \subset \{1, \ldots, N\}$. The subspace $\mathcal{S}$ above is precisely the fixed point set of the subtorus

$$\mathcal{S} = \{ (\lambda_1, \ldots, \lambda_N) \in T^N \mid \lambda_i = 1 \text{ for } i \notin \{i_1, \ldots, i_n\} \}$$

with Lie algebra $\mathfrak{s}$. Since the action of $H$ on $Z$ is locally free, $Z \cap \mathcal{S} \neq \emptyset$ implies that $\mathfrak{s} \cap \mathfrak{h} = 0$. Hence $\mathfrak{s}^\perp + \mathfrak{h}^\perp = (\mathbb{R}^N)^*$. We conclude that the affine plane $V_a$ intersects the faces of the orthant $(\mathbb{R}^N)^*_+$ transversely. Therefore $V_a$ contains points in the interior of $(\mathbb{R}^N)^*_+$, hence $Z \cap (\mathbb{C}^N)^* \neq \emptyset$, and there is a point $z \in Z$ on which $T^N$ acts freely.
An element $g \in G$ acts trivially on the coarse moduli space of the stack $[Z/H] = [(Z \times G)/\hat{T^n}]$ if for any $(z, g') \in Z \times G$ there is $\hat{x} \in \hat{T^N}$ such that
\[
\hat{x} \cdot (z, g') = g \cdot (z, g'),
\]
that is,
\[
(\hat{x}z, g'[\hat{x}]^{-1}) = (z, gg'),
\]
(5.2)
where $[\hat{x}] \in G = \hat{T^n}/\hat{A}$ is the class of $\hat{x} \in \hat{T^n}$. Now take $g' = e_G$, the identity of $G$, and $z \in Z$ an element on which $T^N$ acts freely. Then (5.2) implies
\[
\hat{x} \in \Gamma, \quad g = [\hat{x}] = e_G
\]
and, since $\Gamma \subset \hat{A}$, we have $g = e_G$. Hence the action of $G$ on the coarse moduli space $[Z/\hat{A}]$ is effective, and $\mathbb{C}^N//_{\hat{A}}$ is a symplectic toric DM stack. \qed

References