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Projective modules over overrings of polynomial rings

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ABSTRACT

Let A be a commutative Noetherian ring of dimension d and let P be a projective $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$-module of rank $r \geqslant \max\{2, \dim A + 1\}$, where $f_i \in A[Y_i]$. Then

(i) The natural map $\Phi_r : \mathrm{GL}_r(R)/\mathrm{EL}_r^1(R) \to K_1(R)$ is surjective (3.8).
(ii) Assume f_i is a monic polynomial. Then Φ_{r+1} is an isomorphism (3.8).
(iii) $\mathrm{EL}^1(R \oplus P)$ acts transitively on $\mathrm{Um}(R \oplus P)$. In particular, P is cancellative (3.12).
(iv) If A is an affine algebra over a field, then P has a unimodular element (3.13).

In the case of Laurent polynomial ring (i.e. $f_i = Y_i$), (i), (ii) are due to Suslin (1977) [12], (iii) is due to Lindel (1995) [4] and (iv) is due to Bhatwadekar, Lindel and Rao (1985) [2].
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1. Introduction

All the rings are assumed to be commutative Noetherian and all the modules are finitely generated.

Let A be a ring of dimension d and let P be a projective A-module of rank n. We say that P is *cancellative* if $P \oplus A^m \xrightarrow{\sim} Q \oplus A^m$ for some projective A-module Q implies $P \xrightarrow{\sim} Q$. We say that P has a *unimodular element* if $P \xrightarrow{\sim} P' \oplus A$ for some projective A-module P'.

Assume $\operatorname{rank} P > \dim A$. Then (i) Bass [1] proved that $\mathrm{EL}^1(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$. In particular, P is cancellative and (ii) Serre [11] proved that P has a unimodular element.

Later, Plumstead [7] generalized above results by proving that if P is a projective $A[T]$-module of rank $> \dim A = \dim A[T] - 1$, then (i) P is cancellative and (ii) P has a unimodular element.

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Let P be a projective $A[X_1,\dots,X_l]$-module of rank $>\dim A$. Then (i) Ravi Rao [9] proved that P is cancellative and (ii) Bhatwadekar and Roy [3] proved that P has a unimodular element, thus generalizing Plumstead's results.

Let P be a projective $R=A[X_1,\dots,X_l,Y_1^{\pm1},\dots,Y_m^{\pm1}]$-module of rank $\geqslant \max(2,1+\dim A)$. Then (i) Lindel [4] proved that $\mathrm{EL}^1(R\oplus P)$ acts transitively on $\mathrm{Um}(R\oplus P)$. In particular, P is cancellative and (ii) Bhatwadekar, Lindel and Rao [2] proved that P has a unimodular element.

In another direction, Ravi Rao [10] generalized Plumstead's result by proving that if $R=A[T,1/g(T)]$ or $R=A[T,\frac{f_1(T)}{g(T)},\dots,\frac{f_r(T)}{g(T)}]$, where $g(T)\in A[T]$ is a non-zerodivisor and if P is a projective R-module of rank $>\dim A$, then P is cancellative. We will generalize Rao's result by proving that $\mathrm{EL}^1(R\oplus P)$ acts transitively on $\mathrm{Um}(R\oplus P)$ (3.14).

Let $R=A[X_1,\dots,X_l,Y_1,\dots,Y_m,\frac{1}{f_1\cdots f_m}]$, where $f_i\in A[Y_i]$ and let P be a projective R-module of rank $\geqslant \max\{2,\dim A+1\}$ Then we show that (i) $\mathrm{EL}^1(R\oplus P)$ acts transitively on $\mathrm{Um}(R\oplus P)$ and (ii) if A is an affine algebra over a field, then P has a unimodular element (3.12), (3.13), thus generalizing results of [4,2] where it is proved for $f_i=Y_i$.

As an application, we prove the following result (3.16): Let $\bar{k}$ be an algebraically closed field with $1/d!\in\bar{k}$ and let A be an affine $\bar{k}$-algebra of dimension d. Let $R=A[T,1/f(T)]$ or $R=A[T,\frac{f_1(T)}{f(T)},\dots,\frac{f_r(T)}{f(T)}]$, where $f(T)$ is a monic polynomial and $f(T),f_1(T),\dots,f_r(T)$ is $A[T]$-regular sequence. Then every projective R-module of rank $\geqslant d$ is cancellative. (See [5] for motivation.)

2. Preliminaries

Let A be a ring and let M be an A-module. For $m\in M$, we define $O_M(m)=\{\varphi(m)\mid \varphi\in \mathrm{Hom}_A(M,A)\}$. We say that m is *unimodular* if $O_M(m)=A$. The set of all unimodular elements of M will be denoted by $\mathrm{Um}(M)$. We denote by $\mathrm{Aut}_A(M)$, the group of all A-automorphism of M. For an ideal J of A, we denote by $\mathrm{Aut}_A(M,J)$, the kernel of the natural homomorphism $\mathrm{Aut}_A(M)\to \mathrm{Aut}_A(M/JM)$.

We denote by $\mathrm{EL}^1(A\oplus M,J)$, the subgroup of $\mathrm{Aut}_A(A\oplus M)$ generated by all the automorphisms $\Delta_{a\varphi}=\begin{pmatrix}1 & a\varphi\\ 0 & id_M\end{pmatrix}$ and $\Gamma_m=\begin{pmatrix}1 & 0\\ m & id_M\end{pmatrix}$ with $a\in J$, $\varphi\in\mathrm{Hom}_A(M,A)$ and $m\in M$.

We denote by $\mathrm{Um}^1(A\oplus M,J)$, the set of all $(a,m)\in\mathrm{Um}(A\oplus M)$ such that $a\in 1+J$ and by $\mathrm{Um}(A\oplus M,J)$, the set of all $(a,m)\in\mathrm{Um}^1(A\oplus M,J)$ with $m\in JM$. We will write $\mathrm{Um}_r^1(A,J)$ for $\mathrm{Um}^1(A\oplus A^{r-1},J)$ and $\mathrm{Um}_r(A,J)$ for $\mathrm{Um}(A\oplus A^{r-1},J)$.

We will write $\mathrm{EL}_r^1(A,J)$ for $\mathrm{EL}^1(A\oplus A^{r-1},J)$, $\mathrm{EL}_r^1(A)$ for $\mathrm{EL}_r^1(A,A)$ and $\mathrm{EL}^1(A\oplus M)$ for $\mathrm{EL}^1(A\oplus M,A)$.

Remark 2.1. (i) Let $I\subset J$ be ideals of a ring A and let P be a projective A-module. Then, it is easy to see that the natural map $\mathrm{EL}^1(A\oplus P,J)\to\mathrm{EL}^1(\frac{A}{I}\oplus\frac{P}{IP},\frac{J}{I})$ is surjective.

(ii) Let $E_r(A)$ be the group generated by elementary matrices $E_{i_0j_0}(a)=(a_{ij})$, where $i_0\neq j_0$, $a_{ii}=1$, $a_{i_0j_0}=a\in A$ and remaining $a_{ij}=0$ for $1\leqslant i,j\leqslant r$. Then using [13, Lemma 2.1], it is easy to see that $E_r(A)=\mathrm{EL}_r^1(A)$.

The following result is a consequence of a theorem of Eisenbud–Evans as stated in [7, p. 1420].

Theorem 2.2. *Let R be a ring and let P be a projective R-module of rank r. Let $(a,\alpha)\in(R\oplus P^*)$. Then there exists $\beta\in P^*$ such that* $\mathrm{ht}\, I_a\geqslant r$, *where $I=(\alpha+a\beta)(P)$. In particular, if the ideal $(\alpha(P),a)$ has height $\geqslant r$, then* $\mathrm{ht}\, I\geqslant r$. *Further, if $(\alpha(P),a)$ is an ideal of height $\geqslant r$ and I is a proper ideal of R, then* $\mathrm{ht}\, I=r$.

The following two results are due to Wiemers [13, Proposition 2.5 and Theorem 3.2].

Proposition 2.3. *Let A be a ring and let $R=A[X_1,\dots,X_n,Y_1^{\pm1},\dots,Y_m^{\pm1}]$. Let c be the element 1, X_n or Y_m-1. If $s\in A$ and $r\geqq\max\{3,\dim A+2\}$, then $\mathrm{EL}_r^1(R,sc)$ acts transitively on $\mathrm{Um}_r^1(R,sc)$.*

Theorem 2.4. *Let A be a ring and let $R = A[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$. Let P be a projective R-module of rank $r \geqq \max\{2, \dim A + 1\}$. If J denotes the ideal R, $X_n R$ or $(Y_m - 1)R$, then $\mathrm{EL}^1(R \oplus P, J)$ acts transitively on $\mathrm{Um}^1(R \oplus P, J)$.*

The following result is due to Ravi Rao [10, Lemma 2.1].

Lemma 2.5. *Let $B \subset C$ be rings of dimension d and $x \in B$ such that $B_x = C_x$. Then*

(i) *$B/(1 + xb)B = C/(1 + xb)C$ for all $b \in B$.*
(ii) *If I is an ideal of C such that $\mathrm{ht}\, I \geqslant d$ and $I + xC = C$, then there exists $b \in B$ such that $1 + xb \in I$.*
(iii) *If $c \in C$, then $c = 1 + x + x^2 h \bmod (1 + xb)$ for some $h \in B$ and for all $b \in B$.*

Definition 2.6. Let A be a ring and let M, N be A-modules. Suppose $f, g : M \xrightarrow{\sim} N$ be two isomorphisms. We say that "f is *isotopic* to g" if there exists an isomorphism $\phi(X) : M[X] \xrightarrow{\sim} N[X]$ such that $\phi(0) = f$ and $\phi(1) = g$.

Note that if $\sigma \in \mathrm{EL}^1(A \oplus M)$, then σ is isotopic to identity.

The following lemma follows from the well-known Quillen splitting lemma [8, Lemma 1] and its proof is essentially contained in [8, Theorem 1].

Lemma 2.7. *Let A be a ring and let P be a projective A-module. Let $s, t \in A$ be two comaximal elements. Let $\sigma \in \mathrm{Aut}_{A_{st}}(P_{st})$ which is isotopic to identity. Then $\sigma = \tau_s \theta_t$, where $\tau \in \mathrm{Aut}_{A_t}(P_t)$ such that $\tau = id$ modulo sA and $\theta \in \mathrm{Aut}_{A_s}(P_s)$ such that $\theta = id$ modulo tA.*

The following two results are due to Suslin [12, Corollary 5.7 and Theorem 6.3].

Theorem 2.8. *Let A be a ring and let $f \in A[X]$ be a monic polynomial. Let $\alpha \in \mathrm{GL}_r(A[X])$ be such that $\alpha_f \in \mathrm{EL}_r^1(A[X]_f)$. Then $\alpha \in \mathrm{EL}_r^1(A[X])$.*

Theorem 2.9. *Let A be a ring and $B = A[X_1, \ldots, X_l]$. Then the canonical map $\mathrm{GL}_r(B)/\mathrm{EL}_r^1(B) \to K_1(B)$ is an isomorphism for $r \geqslant \max\{3, \dim A + 2\}$. In particular, if $\alpha \in \mathrm{GL}_r(B)$ is stably elementary, then α is elementary.*

3. Main theorem

We begin this section with the following result which is easy to prove. We give the proof for the sake of completeness.

Lemma 3.1. *Let A be a ring and let P be a projective A-module. Let "bar" denote reduction modulo the nil radical of A. For an ideal J of A, if $\mathrm{EL}^1(\overline{A} \oplus \overline{P}, \overline{J})$ acts transitively on $\mathrm{Um}^1(\overline{A} \oplus \overline{P}, \overline{J})$, then $\mathrm{EL}^1(A \oplus P, J)$ acts transitively on $\mathrm{Um}^1(A \oplus P, J)$.*

Proof. Let $(a, p) \in \mathrm{Um}^1(A \oplus P, J)$. By hypothesis, there exists a $\sigma \in \mathrm{EL}^1(\overline{A} \oplus \overline{P}, \overline{J})$ such that $\sigma(\bar{a}, \bar{p}) = (1, 0)$. Using (2.1), let $\varphi \in \mathrm{EL}^1(A \oplus P, J)$ be a lift of σ such that $\varphi(a, p) = (1 + b, q)$, where $b \in N = nil(A)$ and $q \in NP$. Note that $b \in N \cap J$. Since $1 + b$ is a unit, we get $\Gamma_1 = \Gamma_{\frac{-q}{1+b}} \in \mathrm{EL}^1(A \oplus P, J)$ such that $\Gamma_1(1 + b, q) = (1 + b, 0)$. It is easy to see that there exist $p_1, \ldots, p_n \in P$ and $\alpha_1, \ldots, \alpha_n \in P^*$ such that $\alpha_1(p_1) + \cdots + \alpha_n(p_n) = 1$. Write $h = \sum_2^n \alpha_i(p_i)$. Note that $(1 + b, 0) = (1 + \sum_1^n b\alpha_i(p_i), 0)$, $\Gamma_{\frac{p_1}{1+b}}(1 + b, 0) = (1 + b, p_1)$ and $\Delta_{-b\alpha_1}(1 + b, p_1) = (1 + bh, p_1)$, where $\Delta_{-b\alpha_1} \in \mathrm{EL}^1(A \oplus P, J)$. Since $1 + bh$ is a unit, $\Gamma_{\frac{-p_1}{1+bh}}(1 + bh, p_1) = (1 + bh, 0) = (1 + \sum_2^n b\alpha_i(p_i), 0)$. Applying further transformations as above, we can take $(1 + \sum_2^n b\alpha_i(p_i), 0)$ to $(1, 0)$ by elements of $\mathrm{EL}^1(A \oplus P, J)$. □

The following lemma is similar to the Quillen's splitting lemma (2.7). We will sketch the proof. Recall that for a ring B and an element $s \in B$, $\mathrm{SL}_n^1(B,s)$ denotes the subgroup of $\mathrm{SL}_n(B)$ consisting of those elements whose first row is $(1,0,\ldots,0)$ modulo the ideal (s).

Lemma 3.2. *Let A be a ring and let u, v be two comaximal elements of A. For any $s \in A$, every $\alpha \in \mathrm{EL}_n^1(A_{uv},s)$ has a splitting $(\alpha_1)_v \circ (\alpha_2)_u$, where $\alpha_1 \in \mathrm{SL}_n^1(A_u,s)$ and $\alpha_2 \in \mathrm{EL}_n^1(A_v,s)$.*

Proof. If $\alpha \in \mathrm{EL}_n^1(A_{uv},s)$, then $\alpha = \prod_{i=1}^r \alpha_i$, where α_i is of the form $\begin{pmatrix} 1 & s\underline{v} \\ 0 & Id_M \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ \underline{w}^t & Id_M \end{pmatrix}$, where $M = A_{uv}^{n-1}$, $\underline{v}, \underline{w} \in M$.

Define $\alpha(X) \in \mathrm{EL}_n^1(A[X]_{uv},s)$ by $\alpha(X) = \prod_{i=1}^r \alpha_i(X)$, where $\alpha_i(X)$ is of the form $\begin{pmatrix} 1 & sX\underline{v} \\ 0 & Id_{M[X]} \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ X\underline{w}^t & Id_{M[X]} \end{pmatrix}$ as may by the case above.

Since $\alpha(0) = id$ and $\alpha(1) = \alpha$, α is isotopic to identity. Using proof of (2.7) [6, Lemma 2.19], we get that $\alpha(X) = (\psi_1(X))_v \circ (\psi_2(X))_u$, where $\psi_1(X) = \alpha(X) \circ \alpha(\lambda u^k X)^{-1} \in \mathrm{SL}_n^1(A_u[X],s)$ and $\psi_2(X) = \alpha(\lambda u^k X) \in \mathrm{EL}_n^1(A_v[X],s)$ with $\lambda \in A$, $k \gg 0$. Write $\psi_1(1) = \alpha_1 \in \mathrm{SL}_n^1(A_u,s)$ and $\psi_2(1) = \alpha_2 \in \mathrm{EL}_n^1(A_v,s)$, we get that $\alpha(1) = \alpha = (\alpha_1)_v \circ (\alpha_2)_u$. $\square$

Remark 3.3. We do not know whether $\alpha_1 \in \mathrm{EL}_n^1(A_u,s)$ in the above result. In particular, we can ask the following question: Let A be a ring and let u, v be two comaximal elements of A. Let $\alpha \in EL_n^1(A_{uv})$. Does α have a splitting $(\alpha_1)_v \circ (\alpha_2)_u$, where $\alpha_1 \in EL_n^1(A_u)$ and $\alpha_2 \in EL_n^1(A_v)$?

Definition 3.4. Let A be a ring of dimension d and let $l,m,n \in \mathbb{N} \cup \{0\}$. We say that a ring R is of the type $A\{d,l,m,n\}$, if R is an A-algebra generated by $X_1,\ldots,X_l$, $Y_1,\ldots,Y_m$, $T_1,\ldots,T_n$, $\frac{1}{f_1 \cdots f_m}$, $\frac{g_{11}}{h_1},\ldots,\frac{g_{1t_1}}{h_1},\ldots,\frac{g_{n1}}{h_n},\ldots,\frac{g_{nt_n}}{h_n}$, where X_i's, Y_i's and T_i's are variables over A, $f_i \in A[Y_i]$, $g_{ij} \in A[T_i]$, $h_i \in A[T_i]$ and h_i's are non-zerodivisors.

For Laurent polynomial ring (i.e. $f_i = Y_i$), the following result is due to Wiemers (2.3).

Proposition 3.5. *Let A be a ring of dimension d and let $R = A[X_1,\ldots,X_l,Y_1,\ldots,Y_m,\frac{1}{f_1 \cdots f_m}]$, where $f_i \in A[Y_i]$ (i.e. R is of the type $A\{d,l,m,0\}$). If $s \in A$ and $r \geqslant \max\{3, d+2\}$, then $\mathrm{EL}_r^1(R,s)$ acts transitively on $\mathrm{Um}_r^1(R,s)$.*

Proof. Without loss of generality, we may assume that A is reduced. The case $m = 0$ is due to Wiemers (2.3). Assume $m \geqslant 1$ and apply induction on m.

Let $(a_1,\ldots,a_r) \in \mathrm{Um}_r^1(R,s)$. Consider a multiplicative closed subset $S = 1 + f_m A[Y_m]$ of $A[Y_m]$. Then $R_S = B[X_1,\ldots,X_l,Y_1,\ldots,Y_{m-1},\frac{1}{f_1 \cdots f_{m-1}}]$, where $B = A[Y_m]_{f_m S}$ and $\dim B = \dim A$. Since R_S is of the type $B\{d,l,m-1,0\}$, by induction hypothesis on m, there exists $\sigma \in \mathrm{EL}_r^1(R_S,s)$ such that $\sigma(a_1,\ldots,a_r) = (1,0,\ldots,0)$. We can find $g \in S$ and $\sigma' \in \mathrm{EL}_r^1(R_g,s)$ such that $\sigma'(a_1,\ldots,a_r) = (1,0,\ldots,0)$.

Write $C = A[X_1,\ldots,X_l,Y_1,\ldots,Y_m,\frac{1}{f_1,\ldots,f_{m-1}}]$. Consider the following fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & R = C_{f_m} \\ \downarrow & & \downarrow \\ C_g & \longrightarrow & R_g = C_{gf_m} \end{array}$$

Since $\sigma' \in \mathrm{EL}_r^1(C_{gf_m},s)$, by (3.2), $\sigma' = (\sigma_2)_{f_m} \circ (\sigma_1)_g$, where $\sigma_2 \in \mathrm{SL}_r^1(C_g,s)$ and $\sigma_1 \in \mathrm{EL}_r^1(R,s)$. Since $(\sigma_1)_g(a_1,\ldots,a_r) = (\sigma_2)_{f_m}^{-1}(1,0,\ldots,0)$, patching $\sigma_1(a_1,\ldots,a_r) \in \mathrm{Um}_r^1(C_{f_m},s)$ and $(\sigma_2)^{-1}(1,0,\ldots,0) \in$

$\mathrm{Um}^1_r(C_g, s)$, we get a unimodular row $(c_1, \dots, c_r) \in \mathrm{Um}^1_r(C, s)$. Since C is of the type $A\{d, l+1, m-1, 0\}$, by induction hypothesis on m, there exists $\phi \in \mathrm{EL}^1_r(C, s)$ such that $\phi(c_1, \dots, c_r) = (1, 0, \dots, 0)$. Taking projection, we get $\Phi \in \mathrm{EL}^1_r(R, s)$ such that $\Phi\sigma_1(a_1, \dots, a_r) = (1, 0, \dots, 0)$. This completes the proof. □

Corollary 3.6. *Let A be a ring of dimension d and let $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$, where $f_i \in A[Y_i]$. Let c be 1 or X_l. If $s \in A$ and $r \geqslant \max\{3, d+2\}$, then $\mathrm{EL}^1_r(R, sc)$ acts transitively on $\mathrm{Um}^1_r(R, sc)$.*

Proof. Let $(a_1, \dots, a_r) \in \mathrm{Um}^1_r(R, sc)$. The case $c = 1$ is done by (3.5). Assume $c = X_l$. We can assume, after an $\mathrm{EL}^1_r(R, sX_l)$-transformation, that $a_2, \dots, a_r \in sX_lR$. Then we can find $(b_1, \dots, b_r) \in \mathrm{Um}_r(R, sX_l)$ such that the following equation holds:

$$a_1b_1 + \cdots + a_rb_r = 1. \tag{i}$$

Now consider the A-automorphism $\mu : R \to R$ defined as follows

$$X_i \mapsto X_i \quad \text{for } i = 1, \dots, l-1,$$

$$X_l \mapsto X_l(f_1 \dots f_m)^N \quad \text{for some large positive integer } N.$$

Applying μ, we can read the image of Eq. (i) in the subring $S = A[X_1, \dots, X_l, Y_1, \dots, Y_m]$. By (2.3), we obtain $\psi \in \mathrm{EL}^1_r(R, sX_l)$ such that $\psi(\mu(a_1), \dots, \mu(a_r)) = (1, 0, \dots, 0)$. Since $\mu^{-1}(X_l)$ and X_l generate the same ideal in R, applying μ^{-1}, the proof follows. □

Corollary 3.7. *Let A be a ring of dimension d and let $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$, where $f_i \in A[Y_i]$. Then $\mathrm{EL}^1_r(R)$ acts transitively on $\mathrm{Um}_r(R)$ for $r \geqslant \max\{3, d+2\}$.*

The following result is similar to [10, Theorem 5.1]. The Laurent polynomial case (i.e. $f_i = Y_i$) is due to Suslin [12].

Theorem 3.8. *Let A be a ring of dimension d and let $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$, where $f_i \in A[Y_i]$ (i.e. R is of the type $A\{d, l, m, 0\}$). Then*

(i) *The canonical map $\Phi_r : \mathrm{GL}_r(R)/\mathrm{EL}^1_r(R) \to K_1(R)$ is surjective for $r \geqslant \max\{2, d+1\}$.*
(ii) *Assume $f_i \in A[Y_i]$ is a monic polynomial for all i. Then for $r \geqslant \max\{3, d+2\}$, any stably elementary matrix in $\mathrm{GL}_r(R)$ is in $\mathrm{EL}^1_r(R)$. In particular, Φ_{d+2} is an isomorphism.*

Proof. (i) Let $[M] \in K_1(R)$. We have to show that $[M] = [N]$ in $K_1(R)$ for some $N \in \mathrm{GL}_{d+1}(R)$. Without loss of generality, we may assume that $M \in \mathrm{GL}_{d+2}(R)$. By (3.5), there exists an elementary matrix $\sigma \in \mathrm{EL}^1_{d+2}(R)$ such that $M\sigma = \begin{pmatrix} M' & a \\ 0 & 1 \end{pmatrix}$. Applying further $\sigma' \in \mathrm{EL}^1_{d+2}(R)$, we get $\sigma' M\sigma = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, where $M', N \in \mathrm{GL}_{d+1}(R)$. Hence $[M] = [N]$ in $K_1(R)$. This completes the proof of (i).

(ii) Let $M \in \mathrm{GL}_r(R)$ be a stably elementary matrix. For $m = 0$, we are done by (2.9). Assume $m \geqslant 1$.

Let $S = 1 + f_mA[Y_m]$. Then $R_S = B[X_1, \dots, X_l, Y_1, \dots, Y_{m-1}, \frac{1}{f_1 \cdots f_{m-1}}]$, where $B = A[Y_m]_{f_mS}$ and $\dim B = \dim A$. Since R_S is of the type $B\{d, l, m-1, 0\}$, by induction hypothesis on m, $M \in \mathrm{EL}^1_r(R_S)$. Hence there exists $g \in S$ such that $M \in \mathrm{EL}^1_r(R_g)$. Let $\sigma \in \mathrm{EL}^1_r(R_g)$ be such that $\sigma M = Id$.

Write $C = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_{m-1}}]$. Consider the following fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & C_{f_m} = R \\ \downarrow & & \downarrow \\ C_g & \longrightarrow & C_{gf_m} = R_g \end{array}$$

By (3.2), $\sigma = (\sigma_2)_{f_m} \circ (\sigma_1)_g$, where $\sigma_2 \in \mathrm{SL}_r(C_g)$ and $\sigma_1 \in \mathrm{EL}_r^1(C_{f_m})$. Since $(\sigma_1 M)_g = (\sigma_2)_{f_m}^{-1}$, patching $\sigma_1 M$ and $(\sigma_2)^{-1}$, we get $N \in \mathrm{GL}_r(C)$ such that $N_{f_m} = \sigma_1 M$.

Write $D = A[X_1, \dots, X_n, Y_1, \dots, Y_{m-1}, \frac{1}{f_1 \cdots f_{m-1}}]$. Then $D[Y_m] = C$ and $D[Y_m]_{f_m} = R$. Since $N \in \mathrm{GL}_r(D[Y_m])$, $f_m \in D[Y_m]$ is a monic polynomial and $N_{f_m} = \sigma_1 M$ is stably elementary, by (2.8), N is stably elementary. Since C is of the type $A\{d, l+1, m-1, 0\}$, by induction hypothesis on m, $N \in \mathrm{EL}_r^1(C)$. Since σ_1 is elementary, we get that $M \in \mathrm{EL}_r^1(R)$. This completes the proof of (ii). □

Lemma 3.9. *Let R be a ring of the type $A\{d, l, m, n\}$. Let P be a projective R-module of rank $r \geqslant \max\{2, 1+d\}$. Then there exist an $s \in A$, $p_1, \dots, p_r \in P$ and $\varphi_1, \dots, \varphi_r \in \mathrm{Hom}(P, R)$ such that the following properties holds.*

(i) *P_s is free.*
(ii) *$(\varphi_i(p_j)) = \text{diagonal}(s, s, \dots, s)$.*
(iii) *$sP \subset p_1 A + \cdots + p_r A$.*
(iv) *The image of s in A_{red} is a non-zerodivisor.*
(v) *$(0 : sA) = (0 : s^2 A)$.*

Proof. Without loss of generality, we may assume that A is reduced. Let S be the set of all non-zerodivisors in A. Since $\dim A_S = 0$ and projective R_S-module P_S has constant rank, we may assume that A_S is a field. Then it is easy to see that $A_S[T_i, \frac{g_{ij}}{h_i}] = A_S[T_i, \frac{1}{h_i}]$ (assuming $\gcd(g_{ij}, h_i) = 1$). Therefore $R_S = A_S[X_1, \dots, X_l, Y_1, \dots, Y_m, T_1, \dots, T_n, \frac{1}{f_1 \dots f_m h_1 \dots h_n}]$ is a localization of a polynomial ring over a field. Hence projective modules over R_S are stably free. Since P_S is stably free of rank $\geqslant \max\{2, 1+d\}$, by (3.5), P_S is a free R_S-module of rank r. We can find an $s \in S$ such that P_s is a free R_s-module. The remaining properties can be checked by taking a basis $p_1, \dots, p_r \in P$ of P_s, a basis $\varphi_1, \dots, \varphi_r \in \mathrm{Hom}(P, R)$ of P_s^* and replacing s by some power of s, if needed. This completes the proof. □

Lemma 3.10. *Let R be a ring of the type $A\{d, l, m, n\}$. Let P be a projective R-module of rank r. Choose $s \in A$, $p_1, \dots, p_r \in P$ and $\varphi_1, \dots, \varphi_r \in \mathrm{Hom}(P, R)$ satisfying the properties of (3.9). Let $(a, p) \in \mathrm{Um}(R \oplus P, sA)$ with $p = c_1 p_1 + \cdots + c_r p_r$, where $c_i \in sR$ for $i = 1$ to r. Assume there exists $\phi \in \mathrm{EL}_{r+1}^1(R, s)$ such that $\phi(a, c_1, \dots, c_r) = (1, 0, \dots, 0)$. Then there exists $\Phi \in \mathrm{EL}^1(R \oplus P)$ such that $\Phi(a, p) = (1, 0)$.*

Proof. Since $\phi \in \mathrm{EL}_{r+1}^1(R, s)$, $\phi = \prod_{j=1}^n \phi_j$, where $\phi_j = \Delta_{s\psi_j}$ or Γ_{v^t} with $\psi_j = (b_{1j}, \dots, b_{rj}) \in R^{r*}$ and $v = (f_1, \dots, f_r) \in R^r$.

Define a map $\Theta : \mathrm{EL}_{r+1}^1(R, s) \to \mathrm{EL}^1(R \oplus P)$ as follows

$$\Theta(\Delta_{s\psi_j}) = \begin{pmatrix} 1 & \sum_{i=1}^r b_{ij}\varphi_i \\ 0 & id_P \end{pmatrix} \quad \text{and} \quad \Theta(\Gamma_{v^t}) = \begin{pmatrix} 1 & 0 \\ \sum_{i=1}^r f_i p_i & id_P \end{pmatrix}.$$

Let $\Phi = \prod_{j=1}^n \Theta(\phi_j) \in \mathrm{EL}^1(R \oplus P)$. Then it is easy to see that $\Phi(a, p) = (1, 0)$. This completes the proof. □

Remark 3.11. From the proof of above lemma, it is clear that if $\phi \in \mathrm{EL}^1_{r+1}(R, sX_l)$ such that $\phi(a, c_1, \dots, c_r) = (1, 0, \dots, 0)$, then $\Phi \in \mathrm{EL}^1(R \oplus P, X_l)$ such that $\Phi(a, p) = (1, 0)$.

For Laurent polynomial ring (i.e. $f_i = Y_i$ and $J = R$), the following result is due to Lindel [4].

Theorem 3.12. *Let A be a ring of dimension d and let $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$, where $f_i \in A[Y_i]$ (i.e. R is of the type $A\{d, l, m, 0\}$). Let P be a projective R-module of rank $r \geqslant \max\{2, d+1\}$. If J denotes the ideal R or $X_l R$, then $\mathrm{EL}^1(R \oplus P, J)$ acts transitively on $\mathrm{Um}^1(R \oplus P, J)$.*

Proof. Without loss of generality, we may assume that A is reduced. We will use induction on d. When $d = 0$, we may assume that A is a field. Hence projective modules over R are stably free (proof of Lemma 3.9). Using (3.6), we are done.

Assume $d > 0$. By (3.9), there exist a non-zerodivisor $s \in A$, $p_1, \dots, p_r \in P$ and $\phi_1, \dots, \phi_r \in P^* = \mathrm{Hom}_R(P, R)$ satisfying the properties of (3.9). If $s \in A$ is a unit, then P is a free and the result follows from (3.6). Assume s is not a unit.

Let $(a, p) \in \mathrm{Um}^1(R \oplus P, J)$. Let "bar" denote reduction modulo the ideal s^2R. Since $\dim \overline{A} < \dim A$, by induction hypothesis, there exists $\varphi \in \mathrm{EL}^1(\overline{R} \oplus \overline{P}, \overline{J})$ such that $\varphi(\bar{a}, \bar{p}) = (1, 0)$. Using (2.1), let $\Phi \in \mathrm{EL}^1(R \oplus P, J)$ be a lift of φ and $\Phi(a, p) = (b, q)$, where $b \equiv 1 \bmod s^2 JR$ and $q \in s^2 JP$.

By (3.9), there exist $a_1, \dots, a_r \in sJR$ such that $q = a_1 p_1 + \cdots + a_r p_r$. It follows that $(b, a_1, \dots, a_r) \in \mathrm{Um}_{r+1}(R, sJ)$. By (3.6), there exists $\phi \in \mathrm{EL}^1_{r+1}(R, sJ)$ such that $\phi(b, a_1, \dots, a_r) = (1, 0, \dots, 0)$. Applying (3.11), we get $\Psi \in \mathrm{EL}^1(R \oplus P, J)$ such that $\Psi(b, q) = (1, 0)$. Therefore $\Psi\Phi(a, p) = (1, 0)$. This completes the proof. □

For Laurent polynomial ring (i.e. $f_i = Y_i$), the following result is due to Bhatwadekar, Lindel and Rao [2].

Theorem 3.13. *Let k be a field and let A be an affine k-algebra of dimension d. Let $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$, where $f_i \in A[Y_i]$ (i.e. R is of the type $A\{d, l, m, 0\}$). Then every projective R-module P of rank $\geqslant d+1$ has a unimodular element.*

Proof. We assume that A is reduced and use induction on $\dim A$. If $\dim A = 0$, then every projective module of constant rank is free (3.5), (3.9). Assume $\dim A > 0$.

By (3.9), there exists a non-zerodivisor $s \in A$ such that P_s is free R_s-module. Let "bar" denote reduction modulo the ideal sR. By induction hypothesis, $\overline{P}$ has a unimodular element, say $\bar{p}$. Clearly $(p, s) \in \mathrm{Um}(P \oplus R)$, where $p \in P$ is a lift of $\bar{p}$. By (2.2), we may assume that $\mathrm{ht}\, I \geqslant d+1$, where $I = O_P(p)$. We claim that $I_{(1+sA)} = R_{(1+sA)}$ (i.e. $p \in \mathrm{Um}(P_{1+sA})$).

Since R is a Jacobson ring, $\sqrt{I} = \bigcap \mathfrak{m}$ is the intersection of all maximal ideals of R containing I. Since $I + sR = R$, $s \notin (I \cap A)$. Let $\mathfrak{m}$ be any maximal ideal of R which contains I. Since A and R are affine k-algebras, $\mathfrak{m} \cap A$ is a maximal ideal of A. Hence $\mathfrak{m} \cap A$ contains an element of the form $1 + sa$ for some $a \in A$ (as $s \notin \mathfrak{m} \cap A$). Hence $\mathfrak{m}R_{(1+sA)} = R_{(1+sA)}$ and $I_{(1+sA)} = R_{(1+sA)}$. This proves the claim.

Let $S = 1 + sA$. Let $t \in S$ be such that $p \in \mathrm{Um}(P_t)$. Choose $p_1 \in \mathrm{Um}(P_s)$. Since R_{sS} is of the type $A_{sS}\{d-1, l, m, 0\}$, by (3.12), there exists $\varphi \in \mathrm{EL}^1(P_{sS})$ such that $\varphi(p_1) = p$. We can choose $t_1 = tt_2 \in S$ such that $\varphi \in \mathrm{EL}^1(P_{st_1})$. By (2.7), $\varphi = (\varphi_1)_s \circ (\varphi_2)_{t_1}$, where $\varphi_2 \in \mathrm{Aut}(P_s)$ and $\varphi_1 \in \mathrm{Aut}(P_{t_1})$. Consider the following fiber product diagram

$$\begin{array}{ccc} P & \longrightarrow & P_s \\ \big\downarrow & & \big\downarrow \\ P_{t_1} & \longrightarrow & P_{st_1} \end{array}$$

Since $(\varphi_2)_{t_1}(p_1) = (\varphi_1)_s^{-1}(p)$, patching $\varphi_2(p_1) \in \mathrm{Um}(P_s)$ and $\varphi_1^{-1}(p) \in \mathrm{Um}(P_t)$, we get a unimodular element in P. This proves the result. □

The following result generalizes a result of Ravi Rao [10] where it is proved that P is cancellative.

Theorem 3.14. *Let A be a ring of dimension d and let $R = A[X, \frac{f_1}{g}, \dots, \frac{f_n}{g}]$, where $g, f_i \in A[X]$ with g a non-zerodivisor. Let P be a projective R-module of rank $r \geqslant \max\{2, d+1\}$. Then $\mathrm{EL}^1(R \oplus P)$ acts transitively on $\mathrm{Um}(R \oplus P)$.*

Proof. We will assume that A is reduced and apply induction on $\dim A$. If $\dim A = 0$, then we may assume that A is a field. Hence R is a PID and P is free. By (2.3), we are done.

Assume $\dim A = d > 0$. By (3.9), we can choose a non-zerodivisor $s \in A$, $p_1, \dots, p_r \in P$ and $\phi_1, \dots, \phi_r \in P^*$ satisfying the properties of (3.9).

Let $(a, p) \in \mathrm{Um}(R \oplus P)$. Let "bar" denote reduction modulo sgR. Then $\dim \overline{R} < \dim R$ and $r \geqslant \dim \overline{R} + 1$. By Serre's result [11], $\overline{P}$ has a unimodular element, say $\bar{q}$. Then $(0, \bar{q}) \in \mathrm{Um}(\overline{R} \oplus \overline{P})$. By Bass result [1], there exists $\phi \in \mathrm{EL}^1(\overline{R} \oplus \overline{P})$ such that $\phi(\bar{a}, \bar{p}) = (0, \bar{q})$. Using (2.1), let $\Phi \in \mathrm{EL}^1(R \oplus P)$ be a lift of ϕ and $\Phi(a, p) = (b, q)$, where $b \in sgR$. By (2.2), we may assume that $\mathrm{ht}\, O_P(q) \geqslant d+1$.

Write $B = A[X]$, $x = sg$, $I = O_P(q)$ and $C = R$. Then $\dim B = \dim C$ and $B_{sg} = C_{sg}$. By (2.5(ii)), there exists $h \in A[X]$ such that $1 + sgh \in O_P(q)$. Hence there exists $\varphi \in P^*$ such that $\varphi(q) = 1 + sgh$.

By (2.5(iii)), there exists $b' \in R$ such that $b - b'(1 + sgh) = 1 + sg + s^2g^2h'$ for some $h' \in A[X]$. Since $\Delta_{-b'\varphi}(b, q) = (b - b'\varphi(q), q) = (1 + sg + s^2g^2h', q) = (b_0, q)$ and $\Gamma_{-q}(b_0, q) = (b_0, q - b_0q) = (b_0, sgq_1)$ for some $q_1 \in P$ and $b_0 \in A[X]$ with $b_0 = 1 \mod sgA[X]$.

Write $sgq_1 = c_1p_1 + \cdots + c_rp_r$ for some $c_i \in R$. Then $(b_0, c_1, \dots, c_r) \in \mathrm{Um}^1_{r+1}(R, sg)$. It is easy to see that by adding some multiples of b_0 to $c_1, \dots, c_r$, we may assume that $(b_0, c_1, \dots, c_r) \in \mathrm{Um}^1(A[X], sgA[X])$. By (2.3), there exists $\Theta \in \mathrm{EL}^1_{r+1}(A[X], s)$ such that $\Theta(b_0, c_1, \dots, c_r) = (1, 0, \dots, 0)$. Applying (3.10), there exists $\Psi \in \mathrm{EL}^1(R \oplus P)$ such that $\Psi(b_0, sgq_1) = (1, 0)$. This proves the result. □

Question 3.15. *Let R be a ring of type $A\{d, l, m, n\}$ and let P be a projective R-module of rank $\geqslant \max\{2, d+1\}$.*

(i) *Does $\mathrm{EL}^1(R \oplus P)$ act transitively on $\mathrm{Um}(R \oplus P)$? In particular, is P cancellative?*
(ii) *Does P have a unimodular element?*

Assume $n = 0$. Then (i) is (3.12) and for affine algebras over a field, (ii) is (3.13).

When either P is free or $\bar{k} = \overline{\mathbb{F}}_p$, then the following result is proved in [5].

Theorem 3.16. *Let $\bar{k}$ be an algebraically closed field with $1/d! \in \bar{k}$ and let A be an affine $\bar{k}$-algebra of dimension d. Let $f(T) \in A[T]$ be a monic polynomial and assume that either*

(i) *$R = A[T, \frac{1}{f(T)}]$ or*
(ii) *$R = A[T, \frac{f_1}{f}, \dots, \frac{f_n}{f}]$, where $f, f_1, \dots, f_n$ is $A[T]$-regular sequence.*

Then every projective R-module P of rank d is cancellative.

Proof. By (3.9), there exists a non-zerodivisor $s \in A$ satisfying the properties of (3.9). Let $(a, p) \in \mathrm{Um}(R \oplus P)$.

Let "bar" denote reduction modulo ideal s^3A. Since $\dim \overline{A} < \dim A$, by (3.12), (3.14), there exists $\phi \in \mathrm{EL}^1(\overline{R} \oplus \overline{P})$ such that $\phi(\bar{a}, \bar{p}) = (1, 0)$. Let $\Phi \in \mathrm{EL}^1(R \oplus P)$ be a lift of ϕ. Then $\Phi(a, p) = (b, q)$, where $(b, q) \in \mathrm{Um}^1(R \oplus P, s^2A)$. Now the proof follows by [5, Theorem 4.4]. □

The proof of the following result is same as of (3.16) using [5, Theorem 5.5].

Theorem 3.17. *Let k be a real closed field and let A be an affine k-algebra of dimension $d-2$. Let $f \in A[X, T]$ be a monic polynomial in T which does not belong to any real maximal ideal of $A[X, T]$. Assume that either*

(i) $R = A[X, T, 1/f]$ *or*
(ii) $R = A[X, T, f_1/f, \dots, f_n/f]$, *where* $f, f_1, \dots, f_n$ *is* $A[X, T]$*-regular sequence.*

Then every projective R*-module of rank* $d - 1$ *is cancellative.*

4. An analogue of Wiemers result

We begin this section with the following result which can be proved by the same arguments as in [13, Corollary 3.4] and using (3.12).

Theorem 4.1. *Let* A *be a ring of dimension* d *and* $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$, *where* $f_i \in A[Y_i]$. *Let* P *be a projective* R*-module of rank* $\geqslant d + 1$. *Then the natural map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_{\overline{R}}(P/X_l P)$ *with* $\overline{R} = R/X_l R$ *is surjective.*

Using the automorphism μ defined in (3.6), the following result can be proved by the same arguments as in [13, Proposition 4.1].

Proposition 4.2. *Let* A *be a ring of dimension* d, $1/d! \in A$ *and* $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$ *with* $l \geqslant 1$, $f_i \in A[Y_i]$. *Then* $\mathrm{GL}_{d+1}(R, X_l J R)$ *acts transitively on* $\mathrm{Um}_{d+1}(R, X_l J R)$, *where* J *is an ideal of* A.

When $f_i = Y_i$, the following result is due to Wiemers [13, Theorem 4.3]. The proof of this result is same as of [13, Theorem 4.3] using (4.1), (4.2).

Theorem 4.3. *Let* A *be a ring of dimension* d *with* $1/d! \in A$ *and let* $R = A[X_1, \dots, X_l, Y_1, \dots, Y_m, \frac{1}{f_1 \cdots f_m}]$ *with* $f_i \in A[Y_i]$ *for* $i = 1$ *to* m. *Let* P *be a projective* R*-module of rank* $\geqslant d$. *If* Q *is another projective* R*-module such that* $R \oplus P \cong R \oplus Q$ *and* $\overline{P} \cong \overline{Q}$, *then* $P \cong Q$, *where "bar" denotes reduction modulo the ideal* $(X_1, \dots, X_l)R$.

Using (3.16), (4.3), we get the following result.

Corollary 4.4. *Let* $\bar{k}$ *be an algebraically closed field with* $1/d! \in \bar{k}$ *and let* A *be an affine* $\bar{k}$*-algebra of dimension* d. *Let* $f(T) \in A[T]$ *be a monic polynomial and let* $R = A[X_1, \dots, X_l, T, \frac{1}{f(T)}]$. *Then every projective* R*-module of rank* $\geqslant d$ *is cancellative.*

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