On the zeros of complex Van Vleck polynomials

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A B S T R A C T

Polynomial solutions to the generalized Lamé equation, the Stieltjes polynomials, and the associated Van Vleck polynomials, have been studied extensively in the case of real number parameters. In the complex case, relatively little is known. Numerical investigations of the location of the zeros of the Stieltjes and Van Vleck polynomials in special cases reveal intriguing patterns in the complex case, suggestive of a deeper structure. In this article we report on these investigations, with the main result being a proof of a theorem confirming that the zeros of the Van Vleck polynomials lie on special line segments in the case of the complex generalized Lamé equation having three free parameters. Furthermore, as a result of this proposition, we are able to obtain in this case a strengthening of a classical result of Heine on the number of possible Van Vleck polynomials associated with a given Stieltjes polynomial.

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1. Introduction

Let \( \alpha_1, \ldots, \alpha_n \) be distinct complex numbers, and let \( \rho_1, \ldots, \rho_n \) be positive numbers. The generalized Lamé equation is the second-order ODE given by

\[
\prod_{j=1}^{n} (z - \alpha_j) \phi''(z) + 2 \sum_{j=1}^{n} \rho_j \prod_{i \neq j} (z - \alpha_i) \phi'(z) = V(z) \phi(z).
\] (1)

According to a result in [10], there exist at most \( \frac{n(n+1)}{2} \) polynomials \( V \) of degree \( n - 2 \) for which (1) has a polynomial solution \( \phi \) of degree \( k \). These polynomial solutions are often called Stieltjes or Heine–Stieltjes polynomials, and the corresponding polynomials \( V \) are known as Van Vleck polynomials.

On the basis of Stieltjes’ work, the zeros of the Stieltjes polynomials can be nicely interpreted in terms of the equilibrium positions of an electrostatic system with logarithmic potential [8,18,7,4]. Consider the field generated by \( n \) charges \( \rho_1, \ldots, \rho_n \) fixed at the positions \( \alpha_1, \ldots, \alpha_n \) in \( \mathbb{C} \), and \( k \) positive unit charges allowed to move freely in \( \mathbb{C} \), where the charges repel each other according to the law of logarithmic potential. This means that the charges are not point charges, but are distributed along infinite straight wires perpendicular to the plane \( \mathbb{C} \). Then the electrostatic potential of the system is given by

\[
W(z_1, \ldots, z_k) := - \log \left[ \prod_{j=1}^{n} \prod_{i=1}^{k} |z_i - \alpha_j|^\rho_j \prod_{i \neq j} |z_i - z_j| \right].
\] (2)

The equilibrium positions then consist of the points \( z_1, \ldots, z_k \in \mathbb{C}^k \) for which \( \nabla W(z_1, \ldots, z_k) = 0 \). By computing explicitly \( \nabla W(z_1, \ldots, z_k) = 0 \) using the expression (2) for \( W \), we deduce that the equilibrium positions satisfy Niven’s equation [19]:

\[
\sum_{j=1}^{n} \frac{\rho_j}{z_i - \alpha_j} + \sum_{i \neq j} \frac{1}{z_i - z_j} = 0 \quad (i = 1, \ldots, k).
\] (3)
For an equilibrium position of \( k \) charges, \((z_1, \ldots, z_k)\), we may construct the polynomial

\[
\phi(z) = \prod_{i=1}^{k} (z - z_i).
\]

Recalling a basic fact about polynomials,

\[
\phi''(z_i) = 2\phi'(z_i) \sum_{l \neq i} \frac{1}{z_l - z_i}.
\]

it follows that the polynomials (4) solve the Eq. (1) and are therefore Stieltjes polynomials. In particular, this leads to a one-to-one correspondence between the set of Stieltjes polynomials and the set of equilibrium positions of the electrostatic system described above [8].

To interpret the Van Vleck zeros, notice that if \( \phi(z) \) is a Stieltjes polynomial, and \( v \) is a zero of the corresponding Van Vleck polynomial, then

\[
\phi''(v) + \sum_{j=1}^{n} \frac{2\rho_j}{v - \alpha_j} \phi'(v) = 0.
\]

Thus, either \( v = \alpha_j \) and \( \phi'(v) = 0 \), or \( \phi'(v) \neq 0 \) and

\[
\sum_{j=1}^{n} \frac{2\rho_j}{v - \alpha_j} + \sum_{j=1}^{k-1} \frac{1}{v - z_j} = 0,
\]

where \( z_1, \ldots, z_{k-1} \) are the zeros of \( \phi'(z) \). We see that if we fix charges \( 2\rho_1, \ldots, 2\rho_n \) at \( \alpha_1, \ldots, \alpha_n \) and unit charges at \( z_1, \ldots, z_{k-1} \), then the Van Vleck zeros represent equilibria of a unit charge allowed to move freely in \( \mathbb{C} \). (See [13, cf. Theorem 3.1] for this interpretation of solutions to the above equation.)

In addition to the description of the electrostatic system with logarithmic potential, the Lamé equation (1) arises in other contexts, for example the quantum asymmetric top [1] and the Gaudin spin chains [9], as well as the classical case considered by Lamé in the 1830’s of solutions to the Laplace equation on an ellipsoid [19, Chapter XXIII].

For \( \alpha_1, \ldots, \alpha_n \) real, Stieltjes [17] showed that the locations of the zeros of the Stieltjes polynomials are completely characterized by their distribution in the subintervals \((\alpha_1, \alpha_2), \ldots, (\alpha_{n-1}, \alpha_n)\). Similar results for the zeros of the Van Vleck polynomials were also obtained in [16]. Obviously, no such result holds when the \( \alpha_i \)’s are complex numbers, and as one can expect, the situation is more complicated than in the real case. The first result in the complex case was obtained in [15]; he showed that the zeros of any Stieltjes polynomial lie inside the convex hull of the set \( \{\alpha_1, \ldots, \alpha_n\} \). Following a similar argument, Marden [12] extended Polya’s result to also include the zeros of Van Vleck polynomials. More recently, a refinement of these results for special configurations of the \( \alpha_i \)’s has also been obtained by Zaheer and Alam [20,21].

In this paper, we are interested in the location of the zeros of the Van Vleck polynomials in the case where the charges \( \rho_i \) are located on the vertices \( \alpha_i \) of cyclic polygons in the complex plane, where numerical investigations reveal substantial patterns formed by the zeros. In Section 2, we establish some rigorous results on the zero loci of the Van Vleck polynomials when the \( \alpha_i \) form the vertices of any equilateral triangle, namely, that the zeros lie on particular portions of the angle bisectors. In Section 3, we discuss non-equilateral triangles, higher order polygons, and the technical difficulties in generalizing the proof of our theorem. In particular, we present some numerical results supporting the conjecture that although there are patterns that the zeros adhere to, the result obtained in Section 2 is unique to case of the equilateral triangle. The final section contains some concluding remarks and conjectures.

2. The equilateral case

Most studies of the generalized Lamé equation (1) consider the case when the charges \( \rho_i \) are all positive. We will consider in this section, though, charges that can take on all real values. In the subsection that follows we will focus on positive charges, for which our main result applies. As a simple corollary we derive a remarkable result regarding the case when the fixed charges are zero, namely that the free charges are still distributed in the convex hull of the “zero charges”. In the succeeding subsection we consider negative fixed charges.

2.1. Nonnegative charges \( \rho_i \geq 0 \)

Specify a charge strength \( \rho > 0 \) and 2 locations \( \alpha_1, \alpha_2 \) in the complex plane, and place two charges at these locations. Up to reflection across the line passing through \( \alpha_1 \) and \( \alpha_2 \), a location \( \alpha_3 \) for a third charge is determined so that the charges lie at the vertices of an equilateral triangle. Consider further the line segments, henceforth called the bissectrices, connecting the vertices to the triangle incentre. Since the Lamé equation is invariant under complex affine transformations, we can assume without loss of generality that the \( \alpha_i \)’s are the third roots of unity, i.e.

\[
\alpha_{i+1} = e^{\frac{2\pi i}{3}} \quad \text{for} \; j = 0, 1, 2.
\]
Fig. 1. An illustration of how the zeros of the Van Vleck polynomials distribute over the bisectrices of the triangle according to Theorem 2.1. From left to right: \( q = 0, 1, 2 \).

Fig. 2. Zeros of the Van Vleck polynomials (larger dots) and zeros of the corresponding Stieltjes polynomials for \( k = 30 \) and \( \rho_j = 1 \).

Under these conditions, the generalized Lamé equation (1) takes the simpler form

\[
(z^3 - 1) \phi''(z) + 6\rho z^2 \phi'(z) = \mu (z - \nu) \phi(z).
\]  

For every \( k \in \mathbb{N} \), we denote by \( N(k) \) the number of zeros of all distinct Van Vleck polynomials \( V(z) = \mu (z - \nu) \) associated with the Stieltjes polynomials of degree \( k \). Note that Heine’s result amounts to \( N(k) \leq k + 1 \) under the current assumptions.

**Theorem 2.1.** Let \( \alpha_1, \alpha_2, \alpha_3 \) and \( \rho \) be as above, and for any \( k \in \mathbb{N} \), let \( k + 1 = 3p + q, q = 0, 1 \) or \( 2 \). Then, there exist \( p \) real, distinct \( r \in (0, 1) \) such that \( 3p \) of the zeros of the associated Van Vleck polynomials are nonzero and have the form (cf. Figs. 1 and 2)

\[ re^{2\pi i j} \quad \text{for } j = 0, 1, 2, \]  

and the remaining zeros of the Van Vleck polynomials are zero. Thus:

(i) If \( q = 0 \) or \( q = 1 \), then \( N(k) = k + 1 \).

(ii) If \( q = 2 \), then \( N(k) = k \).

As a simple consequence of Theorem 2.1, we have the following result, which is an improvement of Heine’s result in our particular setting.

**Corollary 2.1.** With \( \alpha_1, \alpha_2, \alpha_3 \) and \( \rho \) as above, the Lamé equation (1) has exactly \( k + 1 \) polynomial solutions of degree \( k \) if \( k \equiv 0 \pmod{3} \) or if \( k \equiv 2 \pmod{3} \), but has only \( k \) polynomial solutions if \( k \equiv 1 \pmod{3} \).

**Proof of Theorem 2.1.** Let \( \phi \) be a polynomial solution of degree \( k \) for the Lamé equation (5) that we write as

\[ \phi(z) = \sum_{j=0}^{k} a_j z^j. \]
Substitution of this expression for \( \phi \) into (5) yields the condition \( \mu = k(k-1+6\rho) \) and the eigenvalue problem \( B \mathbf{a} = \mu \mathbf{a} \), where \( B \) is the \((k+1) \times (k+1)\) matrix

\[
B = \begin{pmatrix}
0 & 0 & b_{13} \\
b_{21} & 0 & 0 & b_{24} \\
b_{32} & 0 & 0 & b_{35} \\
b_{43} & 0 & 0 & b_{46} \\
\vdots & \vdots & \vdots & \vdots \\
b_{k-1,k-2} & 0 & 0 & b_{k-1,k+1} \\
b_{k,k-1} & 0 & 0 & \cdot \\
b_{k+1,k} & 0 & \cdot & \cdot \\
\end{pmatrix},
\]

(7)

whose nonzero entries are given by

\[
\begin{align*}
b_{j,j+2} &= j(j+1), \\
b_{j,j-1} &= k(k-1) - (j-2)(j-3) + 6\rho(k+2-j).
\end{align*}
\]

Since \( b_{j,j+2}, b_{j,j-1} > 0 \), every eigenvector \( \mathbf{a} = (a_0, \ldots, a_k) \) of \( B \) must satisfy \( a_k \neq 0 \). Hence \( N(k) \) is equal to the number of linearly independent eigenvectors of \( B \). Since the sum of every column of \( B \) is bounded by \( \mu \), Gershgorin's Circle Theorem implies that \( |\nu| \leq 1 \). (Note that in this case we obtain a new proof of Marden's result [12].) Consequently, it is sufficient to show that the spectrum of \( B \) consists of \( q \) zeros and \( 3p \) numbers of the form

\[
\lambda = e^{2\pi i t} \quad (t = 0, 1, 2),
\]

for \( p \) distinct \( r > 0 \), and that when \( q = 2 \) the eigenspace associated with the zero eigenvalue has dimension 1.

As a consequence of the fact that \( b_{j,j+2}, b_{j,j-1} > 0 \), \( B \) is irreducible with period 3 and is therefore cogredient to a matrix in superdiagonal block form [14]. We can construct this matrix as follows. Let \( \sigma \) be the permutation of the integers from 1 to \( k+1 \) defined by

\[
\sigma = \begin{cases}
(k+1, k-2, \ldots, 3, k-3, \ldots, 2, k-1, k-4, \ldots, 1), & q = 0 \\
(k+1, k-2, \ldots, 1, k-3, \ldots, 3, k-1, k-4, \ldots, 2), & q = 1 \\
(k+1, k-2, \ldots, 2, k-3, \ldots, 1, k-1, k-4, \ldots, 3), & q = 2.
\end{cases}
\]

Let \( P \) be the permutation matrix with the value 1 in row \( \sigma_j \) of column \( j \). Then one can verify by a direct calculation that

\[
C := PBP^T
\]

is in the following superdiagonal block form:

\[
C = \begin{pmatrix}
0 & C_1 & 0 \\
0 & 0 & C_2 \\
C_3 & 0 & 0
\end{pmatrix},
\]

where \( C_1 \) and \( C_2 \) are lower bidiagonal matrices, and \( C_3 \) is an upper bidiagonal matrix.

The three matrices \( C_1, C_2, \) and \( C_3 \) have strictly positive entries on their two diagonals and are therefore totally nonnegative\(^1\) (TN) [5,6]. When \( k+1 \) is a multiple of 3, the matrices \( C_1, C_2, \) and \( C_3 \) are all \( p \times p \) square matrices. The sizes of the matrices in the other cases are as follows:

\[
q = 1: \quad C_1 \quad C_2 \quad C_3 \quad \frac{(p+1) \times p}{p \times p} \quad \frac{p \times (p+1)}{p \times p} \\
q = 2: \quad \frac{(p+1) \times (p+1)}{p \times p} \quad \frac{(p+1) \times p}{p \times (p+1)}.
\]

(8)

The third power of \( C \) is

\[
C^3 = \begin{pmatrix}
C_1 C_2 C_3 & 0 & 0 \\
0 & C_2 C_3 C_1 & 0 \\
0 & 0 & C_3 C_1 C_2
\end{pmatrix}.
\]

It then follows that the nonzero spectra of \( C_1 C_2 C_3, C_2 C_3 C_1, \) and \( C_3 C_1 C_2 \) are equal and the eigenvalues of \( C \) consist of \( q \) zeros and the third roots of the eigenvalues of the product \( C_3 C_1 C_2 \) [14]. From the considerations above, the product \( C_3 C_1 C_2 \) is a square matrix of order \( p \). It remains to show that the eigenvalues of \( C_3 C_1 C_2 \) are distinct, real and positive. For this it is sufficient to show that \( C_3 C_1 C_2 \) is oscillatory\(^2\) [14].

A matrix \( A \) is oscillatory if and only if it (a) is TN, (b) is nonsingular, and (c) satisfies \( a_{i,i+1}, a_{i+1,i} > 0 \). Condition (c) follows from the fact that \( C_1, C_2 \) and \( C_3 \) are upper and lower bidiagonal with strictly positive entries on the two diagonals. We will

\(^1\)A matrix is totally nonnegative (positive) if all minors of all sizes are nonnegative (positive).

\(^2\)A matrix \( A \) is oscillatory if there exists a positive integer \( m \) such that \( A^m \) is totally positive. Oscillatory matrices have the remarkable property that their eigenvalues are distinct, real and positive.
verify conditions (a) and (b) by showing that \( C_2 C_3 \) is TN and has positive determinant. The total nonnegativity of \( C_2 C_3 \) is an immediate consequence of the Cauchy–Binet Identity [11] and the fact that the matrices \( C_i \) are TN. The statement of the identity requires the introduction of some notation. Let \( A \) be an \( m \times m \) real matrix, and let \( \alpha, \beta \) be nonempty ordered subsets of \( \{1, \ldots, m\} \), both consisting of strictly increasing integers. Then \( A[\alpha|\beta] \) denotes the submatrix of \( A \) lying in rows indexed by \( \alpha \) and columns indexed by \( \beta \). The Cauchy–Binet Identity asserts that for any pair of nonempty ordered subsets \( \alpha, \beta \subseteq \{1, 2, \ldots, p\} \) of the same cardinality,

\[
\det((C_3 C_2)[\alpha|\beta]) = \sum_{\gamma_1, \gamma_2} \det(C_2[\alpha|\gamma_1]) \det(C_1[\gamma_1|\gamma_2]) \det(C_2[\gamma_2|\beta]).
\]

Here, the double sum is taken over all ordered subsets \( \gamma_1, \gamma_2 \) of \( \{1, \ldots, p+1\} \) with cardinality \( |\alpha| = |\beta| \), where the submatrices are defined. Since \( C_1, C_2, C_3 \) are TN, all the terms in the above sum (9) are nonnegative for all \( \alpha \) and \( \beta \), so \( C_2 C_3 \) is TN.

Furthermore, for \( \gamma_0 = \{1, \ldots, p\} \),

\[
\det(C_3 C_2) = \det(C_2[\gamma_0|\gamma_0]) \det(C_1[\gamma_0|\gamma_0]) \det(C_2[\gamma_0|\gamma_0])
+ \sum_{\gamma_1, \gamma_2 \neq \gamma_0} \det(C_3[\gamma_0|\gamma_1]) \det(C_1[\gamma_1|\gamma_2]) \det(C_2[\gamma_2|\gamma_0]).
\]

The leading principal submatrices \( C_i[\gamma_0|\gamma_0] \) are bidiagonal with strictly positive terms on their diagonals, so the first term in (10) is positive, and the remaining terms are nonnegative. Hence, \( \det(C_3 C_2) > 0 \). The three conditions are thus satisfied, and we conclude that the eigenvalues \( \omega_1, \omega_2, \ldots, \omega_q \) of \( C_2 C_3 \) are distinct, real and positive, and thus the spectrum of \( B \) consists of \( q \) zeros together with the numbers

\[
\omega_j^{1/3} e^{\frac{2\pi i}{3} l}, \quad j = 1, \ldots, p, \quad l = 0, 1, 2.
\]

This proves part (i) of the theorem. For part (ii), \( q = 2 \), so \( 0 \) is an eigenvalue of \( B \) of multiplicity 2. To complete the proof of the theorem, it therefore suffices to show that the eigenspace associated with the zero eigenvalue of \( B \) has dimension 1. But this is an immediate consequence of the fact that \( B \) has period 3. Indeed, if

\[ Ba = 0 \]

with \( a = (a_0, \ldots, a_k) \), then we have that

\[
\begin{align*}
a_0 &= a_3 = \cdots = a_{k-1} = 0, \\
a_2 &= a_5 = \cdots = a_{k-2} = 0, \\
|b_{j+2j+4a_0 + 3}| &= 0 \quad \text{for } j = 1, 4, \ldots, k-3.
\end{align*}
\]

From these equations, we deduce that \( B \) has a single independent eigenvector associated with the eigenvalue \( \lambda = 0 \). \( \square \)

The preceding theorem and proof reveal important fundamental differences between the case when the \( \alpha_i \)'s are real and that when they are complex. The first is that when the \( \alpha_i \)'s are real, \( N(k) = k + 1 \) [17], but as we saw, when the \( \alpha_i \)'s are complex, it is possible for \( N(k) < k + 1 \). Secondly, when the \( \alpha_i \)'s are real no zero of a Van Vleck polynomial is a zero of the corresponding Stieltjes polynomial [16], but when the \( \alpha_i \)'s are the vertices of an equilateral triangle in the complex plane and \( k + 1 \equiv 2 \pmod{3} \), (12) implies the existence of a Stieltjes polynomial \( \phi(z) \) and corresponding Van Vleck polynomial \( V(z) \) such that \( \phi(0) = V(0) = 0 \).

We are able to make some further comments on the distribution of Van Vleck zeros as \( \rho \) is varied from 0 to \( \infty \). Strictly speaking, \( \rho = 0 \) does not correspond to a Lamé equation. However, this can be interpreted as the limiting case \( \rho \to 0 \) when the fixed charges are reduced to zero. Remarkably, in this case there still exist equilibrium positions of the free charges inside the triangle formed by the “zero charges”. See Fig. 3, upper left. Note that when \( \rho = 0 \) the sum of every column of the matrix \( B/\mu \) is equal to 1, so the largest eigenvalues have modulus 1. And, as \( \rho \to \infty \), \( B/\mu \) tends to a lower triangular matrix with zeros on its diagonal; hence all eigenvalues tend to zero. We thus have the following corollary.

**Corollary 2.2.** Let the hypotheses of Theorem 2.1 hold. Then, if \( \rho = 0 \) and \( k \geq 2 \), three Van Vleck zeros lie on the vertices of the triangle and the remaining zeros lie inside the triangle. As \( \rho \to \infty \), the zeros concentrate at the center of the triangle.

The top panels of Fig. 3 illustrate the above corollary.

We conclude this section with a brief description of the case when the \( \rho_i \)'s are not equal. In this case, the formulation of the Van Vleck zeros as eigenvalues of a matrix no longer has the symmetric form (7). In particular, the diagonal and superdiagonal terms are no longer zero, and the preceding result does not hold. Fig. 3 shows the distribution of Van Vleck and Stieltjes zeros for various values of the \( \rho_i \)'s.
Fig. 3. Effect of $\rho$ on zeros of the Van Vleck polynomial. Upper left: $\rho_j = 0$; upper right: $\rho_j = 100$; lower left: $\rho_1 = 50, \rho_2 = \rho_3 = 1$; lower right: $\rho_1 = 50, \rho_2 = 25, \rho_3 = 1$. When the $\rho_j$’s are not equal, the zeros of the Van Vleck polynomials do not lie on the bisectrices (shown as lines in the lower figures).

2. Negative charges $\rho_j < 0$

In the proof of Theorem 2.1 we showed that the nonzero eigenvalues of $B \ (7)$ are of the form $r \ exp(2\pi i/3)$ for $r > 0$. The only fact that was necessary to prove this was that the diagonal elements $b_{j-1}$ and $b_{j+2}$ were strictly positive. But this holds also for certain negative values of $\rho$ as well, namely those $\rho$ that satisfy $\rho > -(k-1)/6$. As long as this inequality is satisfied, the assertions of Theorem 2.1 hold, with the exception that the Van Vleck and Stieltjes zeros do not all lie inside the triangle. When $\rho = -(k-1)/6, \mu = 0$ so there is no polynomial solution of (1) of degree $k$. As $\rho \to -(k-1)/6$, the set of Van Vleck zeros becomes unbounded.

Further light can be shed on this case by noting that under the transformation $z \to -z, (5)$ becomes

$$(z^3 + 1)\phi''(z) + 6\rho z^2 \phi'(z) = \mu(z + \nu)\phi(z).$$

We can, as before, formulate the problem of finding the Stieltjes and Van Vleck polynomials as an eigenvalue problem $Ba = \mu \nu a$, as in (7). In this case $B$ has the same form, but with

$$b_{i,j+2} = j(j+1),$$
$$b_{i,j-1} = -k(k-1) + (j-3) - 6\rho(k+2-j).$$

As long as $\rho < -(k-1)/3, b_{j+2}$ and $b_{j-1}$ are strictly positive. Thus, if $\rho < -(k-1)/3$, and as before letting $k+1 = 3p+q$, there are $p$ distinct $r < 0$ such that the Van Vleck zeros consist of a zero if $q \neq 0$ along with $3p$ numbers of the form

$$re^{2\pi i j} \text{ for } j = 0, 1, 2.$$

Geršgorin’s theorem also gives us that the Van Vleck zeros are in the unit circle in this case. We see that the cases $-(k-1)/6 < \rho < 0$ and $\rho < -(k-1)/3$ are qualitatively different. These two cases are illustrated in Fig. 4, and we summarize them in the following corollary.

**Corollary 2.3.** Let $\alpha_1, \alpha_2$ and $\alpha_3$ be the vertices of an equilateral triangle. Let $k+1 = 3p+q, q = 0, 1, 2$. Then the zeros of the Van Vleck polynomials consist of a zero at the incenter $\alpha = (\alpha_1 + \alpha_2 + \alpha_3)/3$ if $q \neq 0$, along with $3p$ numbers on the bisectrices of the form
3. Numerical investigations of the non-equilateral triangle and regular polygons

3.1. The non-equilateral triangle

In the case when the \( \alpha_j \)'s do not lie on the vertices of an equilateral triangle, the results described above do not hold. We make some general remarks based on numerical calculations. Fig. 5 shows cases when the \( \alpha_j \)'s are the vertices of isosceles triangles. Even when \( \rho_j = \rho \) the Van Vleck zeros do not lie on the bisectors. However, certain regularities exist. For example, let \( \alpha_{1,2} = \pm \gamma, \alpha_3 = i \) so that the \( \alpha_j \)'s form an isosceles triangle whose base is “stretched” by the factor \( \gamma \). Then the Van Vleck zeros with real part scaled by \( \gamma \) tend to a curve (upper panels in Fig. 5). These scaled zeros are indistinguishable for \( \gamma \geq 20 \). When the \( \rho_j \)'s are distinct all symmetry in the location of the Van Vleck zeros is broken.

3.2. Regular polygons

The result of Section 2 for \( n = 3 \) does not generalize to \( n > 3 \). In the case when the \( \alpha_j \)'s are the vertices of a regular polygon and \( \rho_j = \rho \), the Lamé equation (1) can be written as

\[
(z^n - 1) \phi''(z) + 2n\rho z^{n-1} \phi'(z) = V(z)\phi(z),
\]

where the Van Vleck polynomials are the polynomials \( V(z) \) of degree \( n - 2 \) such that there exist polynomial solutions of (13). We showed that when \( n = 3 \), the zeros of the Van Vleck polynomials lie on the bisectors of the triangle formed by the \( \alpha_j \)'s. A generalization of this result would state that the zeros of the Van Vleck polynomials are of the form \( r \exp ((2\pi i/n)j) \). For the \( n = 4 \) case, for example, we would expect the zeros of the Van Vleck polynomials to lie on the real and imaginary axes. However, this is not the case, as is demonstrated by the following counter-example. A Van Vleck polynomial and associated
for the zeros of the present paper. Let $\rho_5$ of equal charges shows

\[ 5 \]

Fig. 6 can be constructed by (6)

\[ 2 \]

and might suggest otherwise, we also believe this to be true on the basis of numerical evidence and a when

\[ 2 \]

Theorem 2.1 Figs. 3

\[ 14 \]

\[ \rho \]

triangles. Lower right: the distances $r$ be the Van Vleck zeros of $\alpha$

\[ \alpha \]

that in the case of three charges all on the real line, the zeros of Van Vleck polynomials corresponding to Stieltjes polynomials all lie on the bisectrices, even for non-equilateral triangles. 

preliminary examination of the asymptotic distribution of the eigenvalues. In fact, it appears that asymptotically, the zeros

\[ \alpha \]

and not on the magnitude $\rho$ line, as in the classical case considered by Lamé. In the real case the asymptotic density depends only on the location $\alpha$

\[ \alpha \]

approaches infinity. Borcea and Shapiro [14] we show the Van Vleck zeros lie on the bisectrices of the triangle. The location and density of these zeros on the bisectrices is still

4. Concluding remarks and conjectures

In the highly symmetric case considered in Section 2 of equal charges $\rho$ located on the vertices of an equilateral triangle

the Van Vleck zeros lie on the bisectrices of the triangle. The location and density of these zeros on the bisectrices is still unknown, although for finite $k$ their locations clearly depend on $\rho$ (Fig. 3).

It would be interesting to calculate the density of Van Vleck zeros in the limit as the degree of the Stieltjes polynomials approaches infinity. Borcea and Shapiro [2] have calculated such a density for Van Vleck zeros when the $\alpha_j$'s lie on the real line, as in the classical case considered by Lamé. In the real case the asymptotic density depends only on the location $\alpha_j$, and not on the magnitude $\rho_j$, of the charges. This led the authors to conjecture that the same is true in the complex case. Although Figs. 3 and 5 might suggest otherwise, we also believe this to be true on the basis of numerical evidence and a preliminary examination of the asymptotic distribution of the eigenvalues. In fact, it appears that asymptotically, the zeros all lie on the bisectrices, even for non-equilateral triangles.

A clue to the asymptotic distribution of Van Vleck zeros is indicated by the following. In a forthcoming paper [3] we show that in the case of three charges all on the real line, the zeros of Van Vleck polynomials corresponding to Stieltjes polynomials of successive degrees interlace. That is, between every two Van Vleck zeros corresponding to Stieltjes polynomials of degree $k$, there is a Van Vleck zero corresponding to a Stieltjes polynomial of degree $k + 1$, and vice versa. Numerical evidence suggests an analogous result for the symmetric case considered in Section 2 of the present paper. Let

\[ v_{n,j}^{(k+1)} = r_{n}^{(p)} e^{2\pi j i}, \quad j = 0, 1, 2, n = 1, 2, \ldots, p, \]

be the Van Vleck zeros of (6) when $k + 1 = 3p$, corresponding to Stieltjes polynomials of degree $k + 1$. We conjecture that the distances $r_{n}^{(p)}$ of the Van Vleck zeros of order $k + 1$ from the incenter interlace with those of order $k + 4$. That is, if we
label them in increasing order, then

$$0 < r_1^{(p+1)} < r_1^{(p)} < r_2^{(p+1)} < \cdots < r_p^{(p)} < r_{p+1}^{(p+1)} < 1.$$  

This suggests that the Van Vleck zeros may be asymptotically dense on the bisectrices. We plan to address this in a future paper.

Finally, we remark on the modest improvement of Heine’s result in Corollary 2.1. For equal $\rho_j$ and the $\alpha_j$’s placed at the vertices of an equilateral triangle, $N(k)$ is either $k$ or $k + 1$. We believe that this result is true for all complex $\alpha_j$ and $\rho_j > 0$.

Obtaining lower bounds for $N(k)$ for more than three complex $\alpha_j$’s is also an interesting problem.

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References


