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Discrete skew self-adjoint canonical system and the isotropic Heisenberg magnet model

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Abstract

A discrete analog of a skew self-adjoint canonical (Zakharov–Shabat or AKNS) system with a pseudo-exponential potential is introduced. For the corresponding Weyl function the direct and inverse problem are solved explicitly in terms of three parameter matrices. As an application explicit solutions are obtained for the discrete integrable nonlinear equation corresponding to the isotropic Heisenberg magnet model. State–space techniques from mathematical system theory play an important role in the proofs.

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0. Introduction

In this paper, we shall treat a discrete analog of the well-known skew self-adjoint canonical (Dirac type, Zakharov–Shabat or AKNS) system:

$$-iJ\frac{dY}{dx}(x,z) = zY(x,z) + V(x)Y(x,z), \quad J = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, \quad x \geqslant 0. \tag{0.1}$$

Here z is a spectral variable, Y and V are $2m \times 2m$ matrix functions on the half-line, and V is skew self-adjoint, that is, $V(x)^* = JV(x)J$ with $V(x)^*$ being the matrix adjoint of $V(x)$. To obtain the discrete analog of (0.1) let U be the unique solution of the initial value problem

$$\frac{dU}{dx}(x) = -iU(x)JV(x), \quad x \geqslant 0, \quad U(0) = I_{2m}. \tag{0.2}$$

Since $JV(x)$ is self-adjoint, we get from (0.2) that $U(x)$ is unitary for each $x \geqslant 0$. Now put $S(x) = U(x)JU(x)^*$ and $W(x,z) = U(x)Y(x,z)$. Then

$$\frac{dW}{dx}(x,z) = izS(x)W(x,z), \quad S(x) = S(x)^* = S(x)^{-1}, \quad x \geqslant 0. \tag{0.3}$$

It is now immediate that

$$W_{n+1}(\lambda) - W_n(\lambda) = -\frac{i}{\lambda}S_n W_n(\lambda), \quad S_n = S_n^* = S_n^{-1}, \quad n = 0, 1, \ldots \tag{0.4}$$

is a natural discrete analog of (0.1). This discrete analog of the continuous pseudo-canonical system is very important. In fact, when $m = 1$, then the system (0.4) turns out to be an auxiliary system for the nonlinear isotropic Heisenberg magnet (IHM) model [36] (see also the detailed discussion after Theorem 0.5 below and the historical remarks in [12]). Motivated by the IHM model we shall use the term *spin sequence* to denote any sequence of $N \times N$ matrices $\{S_n\}$ satisfying

$$S_n = S_n^* = S_n^{-1}, \quad n = 0, 1, 2, \ldots \ . \tag{0.5}$$

As for the skew self-adjoint continuous case [29] (see also [16]), one can associate with (0.4) an $m \times m$ matrix function $\varphi(\lambda)$, meromorphic on $\Im\lambda < -\delta < 0$, such that

$$\sum_{n=0}^{\infty} [\varphi(\lambda)^* \quad I_m]\, W_n(\lambda)^* W_n(\lambda) \begin{bmatrix} \varphi(\lambda) \\ I_m \end{bmatrix} < \infty, \tag{0.6}$$

where $W_n(\lambda)$, $n \geqslant 0$, is the *fundamental solution* of (0.4), i.e., the $2m \times 2m$ matrix solution $W_n(\lambda)$ of (0.4) normalized by the condition $W_0(\lambda) = I_{2m}$. One refers to φ as the *Weyl function* of (0.4). When the Weyl function is rational and strictly proper we shall recover the system (0.4) explicitly from its Weyl function. For this purpose, we need to introduce spin sequences that are the discrete analogs of the pseudo-exponential potentials from [16,17] (see also the references therein).

The spin sequences from this special class are defined in terms of three parameter matrices with the following properties. First fix an integer $N > 0$, and consider an $N \times N$ matrix α with $\det \alpha \neq 0$, an $N \times N$ matrix Σ_0 such that $\Sigma_0 = \Sigma_0^*$, and an $N \times 2m$ matrix Λ_0. These matrices should satisfy the following matrix identity:

$$\alpha\Sigma_0 - \Sigma_0\alpha^* = i\Lambda_0\Lambda_0^*. \tag{0.7}$$

Given these three matrices α, Σ_0, and Λ_0 we define for $n = 1, 2, \dots$ the $N \times 2m$ matrix Λ_n and the $N \times N$ matrix Σ_n via recursion:

$$\begin{aligned} \Lambda_{n+1} &= \Lambda_n + i\alpha^{-1}\Lambda_n J, \\ \Sigma_{n+1} &= \Sigma_n + \alpha^{-1}\Sigma_n(\alpha^*)^{-1} + \alpha^{-1}\Lambda_n J \Lambda_n^*(\alpha^*)^{-1}. \end{aligned} \tag{0.8}$$

Next assume that the matrices Σ_n, $n = 0, 1, 2, \dots,$ are non-singular. Then we say that the sequence of matrices $\{S_n\}$ defined by

$$S_n = J + \Lambda_n^*\Sigma_n^{-1}\Lambda_n - \Lambda_{n+1}^*\Sigma_{n+1}^{-1}\Lambda_{n+1}, \quad n = 0, 1, 2, \dots, \tag{0.9}$$

is the *spin sequence determined* by the parameter matrices α, Σ_0 and Λ_0. Notice that this requires the invertibility of the matrices Σ_n.

For spin sequences defined in this way our first theorem presents an formula for the fundamental solution $W_n(\lambda)$ of (0.4).

Theorem 0.1. *Let* α ($\det \alpha \neq 0$), Σ_0 ($\Sigma_0 = \Sigma_0^*$) *and* Λ_0 *satisfy* (0.7), *and assume that* $\det \Sigma_n \neq 0$ *for* $0 \leqslant n \leqslant M$, *where* Σ_n *is given by* (0.8). *For* $0 \leqslant n \leqslant M-1$ *let* S_n *be the matrices determined by* α, Σ_0 *and* Λ_0 *via* (0.9) *and* (0.8). *Then* $S_n = S_n^* = S_n^{-1}$ *for* $0 \leqslant n \leqslant M-1$, *and for* $0 \leqslant n \leqslant M$ *the fundamental solution* $W_n(\lambda)$ *of the discrete system* (0.4) *can be represented in the form*

$$W_n(\lambda) = W_{\alpha,\Lambda}(n, \lambda)\left(I_{2m} - \frac{i}{\lambda}J\right)^n W_{\alpha,\Lambda}(0, \lambda)^{-1}, \tag{0.10}$$

where $W_{\alpha,\Lambda}(n, \lambda)$ *is defined by*

$$W_{\alpha,\Lambda}(n, \lambda) = I_{2m} + i\Lambda_n^*\Sigma_n^{-1}(\lambda I_N - \alpha)^{-1}\Lambda_n. \tag{0.11}$$

When $\Sigma_0 > 0$, there exist simple conditions on α and Λ_0 to guarantee that $\det \Sigma_n \neq 0$. First, if $\Sigma_0 > 0$, then without loss of generality we can assume that $\Sigma_0 = I_N$.

Indeed, it is easy to see that the sequence of matrices $\{S_n\}$ defined by (0.9) and (0.8) does not change if we substitute α, Σ_0 and Λ_0 by $\Sigma_0^{-\frac{1}{2}}\alpha\Sigma_0^{\frac{1}{2}}$, I_N and $\Sigma_0^{-\frac{1}{2}}\Lambda_0$. So let us assume that $\Sigma_0 = I_N$. Next, we partition Λ_0 into two $N \times m$ blocks θ_1 and θ_2 as follows: $\Lambda_0 = [\theta_1 \quad \theta_2]$. This together with $\Sigma_0 = I_N$ allows us to rewrite (0.7) in the form

$$\alpha - \alpha^* = i(\theta_1\theta_1^* + \theta_2\theta_2^*). \tag{0.12}$$

Furthermore, in this case Λ_n is given by

$$\Lambda_n = [(I_N + i\alpha^{-1})^n\theta_1 \quad (I_N - i\alpha^{-1})^n\theta_2]. \tag{0.13}$$

Finally, we shall assume that the pair $\{\alpha, \theta_1\}$ is *full range* which means that

$$\mathbb{C}^N = \operatorname{span}\{\alpha^k\theta_1\mathbb{C}^m \mid k = 0, 1, 2, \ldots, N-1\}.$$

The following proposition shows that under these conditions automatically $\det \alpha \neq 0$ and $\det \Sigma_n \neq 0$ for $n = 0, 1, 2, \ldots$.

Proposition 0.2. *Let α be a square matrix of order N, and θ_1 and θ_2 be $N \times m$ matrices satisfying* (0.12). *Assume that the pair $\{\alpha, \theta_1\}$ is full range. Then all the eigenvalues of α are in the open upper half-plane $\mathbb{C}_+$, and for $n = 1, 2, \ldots$ the matrices Σ_n defined by* (0.8), *with $\Sigma_0 = I_N$ and Λ_n given by* (0.13), *are positive definite and satisfy the identity*

$$\alpha\Sigma_n - \Sigma_n\alpha^* = i\Lambda_n\Lambda_n^*. \tag{0.14}$$

Definition 0.3. A triple of matrices α, θ_1 and θ_2, with α square of order N and θ_1 and θ_2 of size $N \times m$, is called admissible if the pairs $\{\alpha, \theta_1\}$ and $\{\alpha, \theta_2\}$ are full range and the identity (0.12) holds.

We denote by the acronym FG (*finitely generated*) the class of spin sequences $\{S_n\}$ determined by the matrices α, $\Sigma_0 = I_N$ and $\Lambda_0 = [\theta_1 \quad \theta_2]$, where α, θ_1 and θ_2 form an admissible triple. In this case, we also say that these spin sequences are *determined* by the corresponding admissible triples. The next two theorems present the solutions of the direct and inverse problem in terms of the Weyl function.

Theorem 0.4. *Assume that the spin sequence $\{S_n\}_{n\geqslant 0}$ of the discrete pseudo-canonical system* (0.4) *belongs to the class FG and is determined by the admissible triple α, θ_1 and θ_2. Then the system* (0.4) *has a unique Weyl function φ, which satisfies* (0.6) *on the half-plane $\Im\lambda < -\frac{1}{2}$, a finite number of poles excluded, and this function is given* *by the formula*

$$\varphi(\lambda) = i\theta_1^*(\lambda I_N - \beta)^{-1}\theta_2, \tag{0.15}$$

where $\beta = \alpha - i\theta_2\theta_2^*$.

Notice that the function φ in (0.15) is a strictly proper $m \times m$ rational matrix function. Conversely, if φ is a strictly proper $m \times m$ rational matrix function, then it admits a representation of the form

$$\varphi(\lambda) = i\vartheta_1^*(\lambda I_n - \gamma)^{-1}\vartheta_2, \tag{0.16}$$

where γ is a square matrix and ϑ_1, ϑ_2 are matrices of size $n \times m$. We refer to the right-hand side of (0.16) as a *minimal realization* of φ if among all possible representations (0.16) of φ the order n of the matrix γ is as small as possible. This terminology is taken from mathematical system theory. We can now state the solution of the inverse problem.

Theorem 0.5. *Let φ be a strictly proper rational $m \times m$ matrix function, given by the minimal realization* (0.16). *There is a unique positive definite $n \times n$ matrix solution X of the algebraic Riccati equation*

$$\gamma X - X\gamma^* = i(X\vartheta_1\vartheta_1^*X - \vartheta_2\vartheta_2^*). \tag{0.17}$$

Using X define matrices θ_1, θ_2, and $\alpha = \beta + i\theta_2\theta_2^*$ by

$$\theta_1 = X^{\frac{1}{2}}\vartheta_1, \quad \theta_2 = X^{-\frac{1}{2}}\vartheta_2, \quad \beta = X^{-\frac{1}{2}}\gamma X^{\frac{1}{2}}. \tag{0.18}$$

Then α, θ_1, and θ_2 form an admissible triple, and the given matrix function φ is the Weyl function of a system (0.4) *of which the spin sequence $\{S_n\} \in FG$ and is uniquely determined by the admissible triple α, θ_1, and θ_2.*

Next, we describe connections with the nonlinear IHM equation. For this purpose consider the zero curvature representation [36] of the IHM model:

$$\frac{d}{dt}G_n(t,\lambda) = F_{n+1}(t,\lambda)G_n(t,\lambda) - G_n(t,\lambda)F_n(t,\lambda), \tag{0.19}$$

where

$$G_n(t,\lambda) = I_2 - \frac{i}{\lambda}S_n(t), \quad F_n(t,\lambda) = \frac{V_n^+(t)}{\lambda - i} + \frac{V_n^-(t)}{\lambda + i}, \tag{0.20}$$

$$V_r^{\pm}(t) := (1 + \overrightarrow{S}_{r-1}(t) \cdot \overrightarrow{S}_r(t))^{-1}(I_2 \pm S_r(t))(I_2 \pm S_{r-1}(t)). \tag{0.21}$$

Here the vectors $\overrightarrow{S}_r = [S_r^1 \quad S_r^2 \quad S_r^3]$ belong to $\mathbb{R}^3$, $\mathbb{R}$ is real axis, the dot $\cdot$ denotes the scalar product in $\mathbb{R}^3$, and the correspondence between the spin matrix S_r and the spin vector $\overrightarrow{S}_r$ is given by the equality

$$S_r = \begin{bmatrix} S_r^3 & S_r^1 - iS_r^2 \\ S_r^1 + iS_r^2 & -S_r^3 \end{bmatrix}. \tag{0.22}$$

In other words the IHM equation

$$\frac{d\overrightarrow{S}_n}{dt} = 2\overrightarrow{S}_n \bigwedge \left(\frac{\overrightarrow{S}_{n+1}}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1}} + \frac{\overrightarrow{S}_{n-1}}{1 + \overrightarrow{S}_{n-1} \cdot \overrightarrow{S}_n} \right), \tag{0.23}$$

where $\bigwedge$ stands for the vector product in $\mathbb{R}^3$, is equivalent to the compatibility condition (0.19) of the systems

$$W_{n+1}(t,\lambda) = G_n(t,\lambda)W_n(t,\lambda),$$
$$\frac{d}{dt}W_n(t,\lambda) = F_n(t,\lambda)W_n(t,\lambda) \quad (n \geqslant 0). \tag{0.24}$$

In (0.23) it is required that $\|\overrightarrow{S}_n\| = 1$. Now one can see easily that the representation (0.22), where $\overrightarrow{S}_n \cdot \overrightarrow{S}_n = 1$, is equivalent to the equalities (0.5) with $S_n \neq \pm I_2$. Thus the first system in (0.24) coincides with system (0.4), where $m = 2$, and $S_n \neq \pm I_2$. We use these connections to obtain explicit solutions of the IHM model.

The literature on continuous canonical systems is very rich, especially for the self-adjoint case; see, for instance, the books [10,12,14,26,35]. Self-adjoint continuous canonical systems with pseudo-exponential potentials have been introduced in [15], and for this class of potentials various direct and inverse problems have been solved; see [17] and the references therein. The subclass of strictly pseudo-exponential potentials has been treated in [3–5]. Interesting recent results on the spectral theory of self-adjoint discrete systems and various useful references on this subject can be found in [2,7,9,13,28,37]. Mainly Jacobi matrices (or block Jacobi matrices as in [2]) that are related to Toda chain problems have been studied. For the skew self-adjoint discrete case some references can be found in [11,12,18].

Theorem 0.1 is the discrete analog of Theorem 1.2 in [16]. The right-hand side of (0.11) can be viewed as the transfer function of a linear input–output system (see [6]). Transfer functions of the special form given by (0.11) were introduced in [33], and also used for the representation of the fundamental solutions of continuous canonical systems [34,35]. In Theorem 0.1 we are closer to [16] (see also [30,32]), where the dependence on the parameter x differs from the one in [34,35]. The condition on an admissible triple α, θ_1, θ_2 that the pairs $\{\alpha, \theta_1\}$ and $\{\alpha, \theta_2\}$ are full range pairs is specific for the discrete case. Nevertheless, Theorems 0.4, 0.5 and parts of their proofs are analogous to results and proofs in Section 2 of Gohberg et al. [16].

This paper consists of four sections not counting this introduction. Since elements from mathematical system theory play an important role in this paper, we present the necessary preliminaries from that area in Section 1. In Section 2, we prove Theorem 0.1 and present some auxiliary results that will be used in the application to the IHM model. Theorems 0.4, 0.5 and Proposition 0.2 are proved in Section 3. In Section 4, we construct solutions of the IHM Eq. (0.23), describe the evolution of the Weyl function and consider a simple example.

1. Preliminaries from mathematical system theory

The material from the state–space theory of rational matrix functions, that is used in this paper, has its roots in the Kalman theory of input–output systems [21], and can be found in books, see, e.g., [19,8]. In general the rational matrix functions appearing in this paper are *proper*, that is, analytic at infinity, and they are square, of size $m \times m$, say. Such a function F can be represented in the form

$$F(\lambda) = D + C(\lambda I_N - A)^{-1}B, \tag{1.1}$$

where A is a square matrix (of which the order N may be much larger than m), the matrices B and C are of sizes $N \times m$ and $m \times N$, respectively, and $D = F(\infty)$. In this paper D is often a zero matrix, and in that case F is called *strictly proper*. The representation (1.1) is called a *realization* or a *transfer matrix representation* of F, and the number $N = \mathrm{ord}(A)$, is called the *state–space dimension* of the realization. Here, $\mathrm{ord}(A)$ denotes the *order* of the matrix A.

Realizations of a fixed F are not unique. The realization (1.1) is said to be *minimal* if its state–space dimension N is minimal among all possible realizations of F. The state–space dimension of a minimal realization of F is called the *McMillan degree* of F and is denoted by $\deg F$. Notice that $\deg F = 0$ corresponds to the case when $\mathrm{ord}(A) = 0$, and this occurs if and only if $F(\lambda) \equiv D$. The realization (1.1) of F is minimal if and only if

$$\mathrm{span} \bigcup_{k=0}^{N-1} \mathrm{Im}\, A^k B = \mathbb{C}^N, \quad \bigcap_{k=0}^{N-1} \mathrm{Ker}\, CA^k = \{0\}, \quad N = \mathrm{ord}(A). \tag{1.2}$$

If for a pair of matrices $\{A, B\}$ the first equality in (1.2) holds, then $\{A, B\}$ is called *controllable* or a *full range pair*. If the second equality in (1.2) is fulfilled, then $\{C, A\}$ is said to be *observable* or a *zero kernel pair*. If a pair $\{A, B\}$ is full range, and K is an $m \times N$ matrix, where N is the order of A and m is the number of columns for B, then the pair $\{A - BK, B\}$ is also full range. An analogous result holds true for zero kernel pairs.

Minimal realizations are unique up to a basis transformation, that is, if (1.1) is a minimal realization of F and $F(\lambda) = D + \widetilde{C}(\lambda I_N - \widetilde{A})^{-1}\widetilde{B}$ is a second minimal realization of F, then there exists a unique invertible matrix S such that $\widetilde{A} = SAS^{-1}$, $\widetilde{B} = SB$, and $\widetilde{C} = CS^{-1}$. In this case S is called a *state–space similarity*.

Finally, if in (1.1) we have $D = I_m$, then $F(\lambda)$ is invertible whenever λ is not an eigenvalue of $A - BC$ and in that case

$$F(\lambda)^{-1} = I_m - C(\lambda I_N - A^{\times})^{-1}B, \quad A^{\times} = A - BC. \tag{1.3}$$

2. The fundamental solution

In this section, we prove Theorem 0.1 and present some results that will be used in Section 4 (a result on the invertibility of matrices Σ_n, in particular).

Proof of Theorem 0.1. First, we shall show that equalities (0.7) and (0.8) yield the identity (0.14) for all $n \geqslant 0$. The statement is proved by induction. Indeed, for $n = 0$ it is true by assumption. Suppose (0.14) is true for $n = r$. Then using the expression for Σ_{r+1} from (0.8) and identity (0.14) for $n = r$ we get

$$\begin{aligned} &\alpha\Sigma_{r+1} - \Sigma_{r+1}\alpha^* \\ &\quad = i\Lambda_r\Lambda_r^* + i\alpha^{-1}\Lambda_r\Lambda_r^*(\alpha^*)^{-1} + \Lambda_r J\Lambda_r^*(\alpha^*)^{-1} - \alpha^{-1}\Lambda_r J\Lambda_r^*. \end{aligned} \tag{2.1}$$

The first relation in (0.8) and formula (2.1) yield (0.14) for $r = n + 1$ and thus for all $n \geqslant 0$.

The next equality will be crucial for our proof. Namely, we shall show that for $0 \leqslant n \leqslant M - 1$ we have

$$W_{\alpha,\Lambda}(n+1,\lambda)\left(I_{2m} - \frac{i}{\lambda}J\right) = \left(I_{2m} - \frac{i}{\lambda}S_n\right)W_{\alpha,\Lambda}(n,\lambda). \tag{2.2}$$

By (0.11) formula (2.2) is equivalent to the formula

$$\begin{aligned} \frac{1}{\lambda}(S_n - J) = {} & \left(I_{2m} - \frac{i}{\lambda}S_n\right)\Lambda_n^*\Sigma_n^{-1}(\lambda I_N - \alpha)^{-1}\Lambda_n \\ & - \Lambda_{n+1}^*\Sigma_{n+1}^{-1}(\lambda I_N - \alpha)^{-1}\Lambda_{n+1}\left(I_{2m} - \frac{i}{\lambda}J\right). \end{aligned} \tag{2.3}$$

Using the Taylor expansion of $(\lambda I_N - \alpha)^{-1}$ at infinity one shows that (2.3) is in its turn equivalent to the set of equalities:

$$S_n - J = \Lambda_n^*\Sigma_n^{-1}\Lambda_n - \Lambda_{n+1}^*\Sigma_{n+1}^{-1}\Lambda_{n+1}, \tag{2.4}$$

$$\begin{aligned} &\Lambda_{n+1}^*\Sigma_{n+1}^{-1}\alpha^p\Lambda_{n+1} - i\Lambda_{n+1}^*\Sigma_{n+1}^{-1}\alpha^{p-1}\Lambda_{n+1}J \\ &\quad = \Lambda_n^*\Sigma_n^{-1}\alpha^p\Lambda_n - iS_n\Lambda_n^*\Sigma_n^{-1}\alpha^{p-1}\Lambda_n \quad (p > 0). \end{aligned} \tag{2.5}$$

Equality (2.4) is equivalent to (0.9). Taking into account the first relation in (0.8) we have $\alpha\Lambda_{n+1} - i\Lambda_{n+1}J = \alpha\Lambda_n + \alpha^{-1}\Lambda_n$. Thus the equalities in (2.5) can be rewritten in the form $K_n\alpha^{p-2}\Lambda_n = 0$, where

$$K_n = \Lambda_{n+1}^*\Sigma_{n+1}^{-1}(\alpha^2 + I_N) - \Lambda_n^*\Sigma_n^{-1}\alpha^2 + iS_n\Lambda_n^*\Sigma_n^{-1}\alpha. \tag{2.6}$$

Therefore, if we prove that $K_n = 0$, then equalities (2.5) will be proved, and so formula (2.2) will be proved too. Substitute (0.9) into (2.6), and again use the first relation in (0.8) to obtain

$$\begin{aligned} K_n = {} & \Lambda_{n+1}^*\Sigma_{n+1}^{-1}(\alpha^2 + I_N) - \Lambda_n^*\Sigma_n^{-1}\alpha^2 + iJ\Lambda_n^*\Sigma_n^{-1}\alpha \\ & + i\Lambda_n^*\Sigma_n^{-1}\Lambda_n\Lambda_n^*\Sigma_n^{-1}\alpha - i\Lambda_{n+1}^*\Sigma_{n+1}^{-1}(\Lambda_n + i\alpha^{-1}\Lambda_n J)\Lambda_n^*\Sigma_n^{-1}\alpha. \end{aligned} \tag{2.7}$$

Now, we use (0.14) to obtain $i\Lambda_n\Lambda_n^*\Sigma_n^{-1} = \alpha - \Sigma_n\alpha^*\Sigma_n^{-1}$ and substitute this relation into (2.7). After easy transformations it follows that

$$\begin{aligned} K_n = {} & \Lambda_{n+1}^*\Sigma_{n+1}^{-1}\Big(\alpha^{-1}\Sigma_n(\alpha^*)^{-1} + \Sigma_n + \alpha^{-1}\Lambda_n J\Lambda_n^*(\alpha^*)^{-1}\Big)\alpha^*\Sigma_n^{-1}\alpha \\ & + iJ\Lambda_n^*\Sigma_n^{-1}\alpha - \Lambda_n^*\alpha^*\Sigma_n^{-1}\alpha. \end{aligned} \tag{2.8}$$

In view of the second relation in (0.8) the first term on the right-hand side of (2.8) equals $\Lambda_{n+1}^*\alpha^*\Sigma_n^{-1}\alpha$ and we have

$$K_n = (\Lambda_{n+1}^* + iJ\Lambda_n^*(\alpha^*)^{-1} - \Lambda_n^*)\alpha^*\Sigma_n^{-1}\alpha. \tag{2.9}$$

By the first relation in (0.8) the equality $K_n = 0$ is now immediate, i.e., (2.2) is true.

Notice that equality (0.10) is valid for $n = 0$. Suppose that it is valid for $n = r$. Then (0.4) and (0.10) yield

$$W_{r+1}(\lambda) = \left(I_{2m} - \frac{i}{\lambda}S_r\right)W_{\alpha,\Lambda}(r,\lambda)\left(I_{2m} - \frac{i}{\lambda}J\right)^r W_{\alpha,\Lambda}(0,\lambda)^{-1}. \tag{2.10}$$

By (2.2) and (2.10) the validity of (0.10) for $n = r+1$ easily follows, i.e., (0.10) is proved by induction.

Consider now the matrices S_n given by (0.9). It is easy to see that $S_n = S_n^*$. Notice also that in view of (0.14) we have

$$W_{\alpha,\Lambda}(r,\lambda)W_{\alpha,\Lambda}(r,\overline{\lambda})^* = I_{2m} \quad (r \geqslant 0), \tag{2.11}$$

where $\overline{\lambda}$ stands for complex conjugate for λ. From (2.2) and (2.11) it follows that $(I_{2m} - i\lambda^{-1}S_n)(I_{2m} + i\lambda^{-1}S_n) = \lambda^{-2}(\lambda^2+1)I_{2m}$. Thus the equality $S_n^* = S_n^{-1}$ holds, which finishes the proof of the theorem. $\square$

The case when $\pm i \notin \sigma(\alpha)$ (σ means spectrum) is important for the study of the IHM model. Assume this condition is fulfilled, and put

$$R_n = (I_N - i\alpha^{-1})^{-n}\Sigma_n(I_N + i(\alpha^*)^{-1})^{-n}, \tag{2.12}$$

$$Q_n = (I_N + i\alpha^{-1})^{-n}\Sigma_n(I_N - i(\alpha^*)^{-1})^{-n}. \tag{2.13}$$

The following proposition will be useful for formulating the conditions of invertibility of Σ_n in a somewhat different form then those in Proposition 0.2 (see Corollary 2.2 below).

Proposition 2.1. *Let the matrices* α (det $\alpha \neq 0$), $\Sigma_0 = \Sigma_0^*$, *and* Λ_0 *satisfy* (0.7), *and let the matrices* Σ_n *be given by* (0.8). *If* $i \notin \sigma(\alpha)$, *then the sequence of matrices* $\{R_n\}$ *is well defined and non-decreasing. If* $-i \notin \sigma(\alpha)$, *then the sequence of matrices* $\{Q_n\}$ *is well defined and non-increasing.*

Proof. To prove that the sequence $\{R_n\}$ is non-decreasing it will suffice to show that

$$\Sigma_{n+1} - (I_N - i\alpha^{-1})\Sigma_n(I_N + i(\alpha^*)^{-1}) \geqslant 0. \tag{2.14}$$

For this purpose notice that

$$\begin{aligned}&\Sigma_{n+1} - (I_N - i\alpha^{-1})\Sigma_n(I_N + i(\alpha^*)^{-1})\\ &\quad = \Sigma_{n+1} - \Sigma_n - \alpha^{-1}\Sigma_n(\alpha^*)^{-1} - i\alpha^{-1}(\alpha\Sigma_n - \Sigma_n\alpha^*)(\alpha^*)^{-1}.\end{aligned}$$

Hence, in view of (0.8) and (0.14) we get

$$\begin{aligned}&\Sigma_{n+1} - (I_N - i\alpha^{-1})\Sigma_n(I_N + i(\alpha^*)^{-1})\\ &\quad = \alpha^{-1}(\Lambda_n J\Lambda_n^* + \Lambda_n\Lambda_n^*)(\alpha^*)^{-1}.\end{aligned} \tag{2.15}$$

Since $J + I_{2m} \geqslant 0$, the inequality (2.14) is immediate from (2.15).

Similarly, from (0.8) and (0.14) we get

$$\begin{aligned}&\Sigma_{n+1} - (I_N + i\alpha^{-1})\Sigma_n(I_N - i(\alpha^*)^{-1})\\ &\quad = \alpha^{-1}(\Lambda_n J\Lambda_n^* - \Lambda_n\Lambda_n^*)(\alpha^*)^{-1} \leqslant 0,\end{aligned} \tag{2.16}$$

and so the sequence of matrices $\{Q_n\}$ is non-increasing. □

According to Proposition 2.1, when $i \notin \sigma(\alpha)$ and $\Sigma_0 > 0$, we have $R_n > 0$.

Corollary 2.2. *Let the conditions of Proposition* 2.1 *hold, and assume that* $i \notin \sigma(\alpha)$ *and* $\Sigma_0 > 0$. *Then we get* $\Sigma_n > 0$ *for all* $n > 0$.

Partition the matrices $W_{\alpha,\Lambda}(r,\lambda)$ and Λ_r into two m-column blocks each:

$$W_{\alpha,\Lambda}(r,\lambda) = \Big[\big(W_{\alpha,\Lambda}(r,\lambda)\big)_1 \quad \big(W_{\alpha,\Lambda}(r,\lambda)\big)_2\Big], \quad \Lambda_r = \big[(\Lambda_r)_1 \quad (\Lambda_r)_2\big].$$

The next lemma will be used in Section 4.

Lemma 2.3. *Let the matrices* α $(0, \pm i \notin \sigma(\alpha))$, Σ_0 $(\Sigma_0 = \Sigma_0^*)$ *and* Λ_0 *satisfy* (0.7), *and let the matrices* Σ_n *be given by* (0.8). *Then for* $n \geqslant 0$ *the following relations hold*:

$$\big(W_{\alpha,\Lambda}(n,i)\big)_1 = \big(W_{\alpha,\Lambda}(n+1,-i)\big)_1 \times\Big(I_m + 2\big(W_{\alpha,\Lambda}(n,i)\big)_1^*\Lambda_n^*\Sigma_n^{-1}(\alpha^2+I_N)^{-1}(\Lambda_n)_1\Big), \tag{2.17}$$

$$\big(W_{\alpha,\Lambda}(n,-i)\big)_2 = \big(W_{\alpha,\Lambda}(n+1,i)\big)_2 \times\Big(I_m - 2\big(W_{\alpha,\Lambda}(n,-i)\big)_2^*\Lambda_n^*\Sigma_n^{-1}(\alpha^2+I_N)^{-1}(\Lambda_n)_2\Big). \tag{2.18}$$

Proof. From the proof of Theorem 0.1 we know that $K_n = 0$, where K_n is given by (2.6). In particular, we get

$$K_n\alpha^{-1}(\alpha^2+I_N)^{-1}(\Lambda_n)_1 = 0, \quad K_n\alpha^{-1}(\alpha^2+I_N)^{-1}(\Lambda_n)_2 = 0. \tag{2.19}$$

To prove (2.17) notice that $(\Lambda_{n+1})_1 = \alpha^{-1}(\alpha + iI_N)(\Lambda_n)_1$ and rewrite the first equality in (2.19) as

$$\Lambda_{n+1}^*\Sigma_{n+1}^{-1}(\alpha+iI_N)^{-1}(\Lambda_{n+1})_1 - \Lambda_n^*\Sigma_n^{-1}(\alpha-iI_N)^{-1}(\Lambda_n)_1 + i(I_{2m}+S_n)\Lambda_n^*\Sigma_n^{-1}(\alpha^2+I_N)^{-1}(\Lambda_n)_1 = 0. \tag{2.20}$$

Put $\lambda = -i$ in (2.2) and take into account (2.11) to derive

$$I_{2m} + S_n = 2\big(W_{\alpha,\Lambda}(n+1,-i)\big)_1\big(W_{\alpha,\Lambda}(n,i)\big)_1^*. \tag{2.21}$$

In view of definition (0.11) of $W_{\alpha,\Lambda}$, equality (2.17) follows from (2.20) and (2.21).

Putting $\lambda = i$ in (2.2) we get

$$I_{2m} - S_n = 2\big(W_{\alpha,\Lambda}(n+1,i)\big)_2\big(W_{\alpha,\Lambda}(n,-i)\big)_2^*. \tag{2.22}$$

Analogously to the proof of (2.17) we derive (2.18) from (2.22) and the second equality in (2.19). □

Remark 2.4. According to (2.21) and (2.22) the rank of the matrices $I_{2m} \pm S_n$ is less than or equal to m. Together with the formula (0.5) this implies that under the conditions of Lemma 2.3 we have $S_n = U_n J U_n^*$, where U_n are unitary matrices and J is defined in (0.1).

3. Weyl functions: direct and inverse problems

In this section we prove Theorems 0.4 and 0.5, and Proposition 0.2. At the end of the section a lemma on the case $i \notin \sigma(\alpha)$ is treated too.

Proof of Proposition 0.2. Suppose f is an eigenvector of α, that is, $\alpha f = cf,\ f \neq 0$. Then formula (0.12) yields the equality

$$i(\overline{c} - c) f^* f = f^*(\theta_1 \theta_1^* + \theta_2 \theta_2^*) f \geqslant 0. \tag{3.1}$$

So $c \in \overline{\mathbb{C}_+}$. Moreover, if $c = \overline{c}$, then according to (3.1) we have $\theta_1^* f = \theta_2^* f = 0$, and therefore $\alpha f = \alpha^* f$. It follows that

$$f^* \theta_1 = 0, \quad f^*(\alpha - cI_N) = 0 \quad (f \neq 0). \tag{3.2}$$

As $\{\alpha, \theta_1\}$ is a full range pair, so the pair $\{\alpha - cI_N, \theta_1\}$ is full range, which contradicts (3.2). This implies that $c \in \mathbb{C}_+$, i.e., $\sigma(\alpha) \subset \mathbb{C}_+$.

Recall that identity (0.14) was deduced in the proof of Theorem 0.1. Taking into account that $\sigma(\alpha) \subset \mathbb{C}_+$, identity (0.14) yields

$$\Sigma_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha - \lambda I_N)^{-1} \Lambda_n \Lambda_n^* (\alpha^* - \lambda I_N)^{-1} \, d\lambda. \tag{3.3}$$

Notice now that the pair $\{\alpha, (I_N + i\alpha^{-1})^n \theta_1\}$ is full range and use (0.13), (3.3) to obtain $\Sigma_n > 0$ for all $n \geqslant 0$. $\square$

Remark 3.1. In the same way as in the proof of Proposition 0.2 above the inclusion $\sigma(\alpha) \subset \mathbb{C}_+$ follows from the weaker condition that the pair $\{\alpha, \Lambda_0\}$ is full range. However, the example $N = 1$, $\alpha = i$, $\theta_1 = 0$, $\theta_2 \theta_2^* = 2$, which yields $\Sigma_n \equiv 0$ for $n > 0$, shows that we have to require that the pair $\{\alpha, \theta_1\}$ is full range in order to get $\Sigma_n > 0$. The full range condition on the pair $\{\alpha, \Lambda_0\}$ is not enough for this conclusion.

Recall now Definition 0.3 of the admissible triple. Proposition 0.2 implies, in particular, that $\det \alpha \neq 0$ for the admissible triple and the spin sequences $\{S_n\}$ determined by it are well defined for all $n \geqslant 0$. In other words the class FG is well defined.

Proof of Theorem 0.4. Let $W_{\alpha,\Lambda}(n,\lambda)$ be given by (0.11). Write $W_{\alpha,\Lambda}(0,\lambda)$ as

$$W_{\alpha,\Lambda}(0,\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}, \tag{3.4}$$

where the $m \times m$ matrix functions $b(\lambda)$ and $d(\lambda)$ are given by

$$b(\lambda) = i\theta_1^*(\lambda I_N - \alpha)^{-1}\theta_2, \quad d(\lambda) = I_m + i\theta_2^*(\lambda I_N - \alpha)^{-1}\theta_2. \tag{3.5}$$

We first prove that

$$b(\lambda)d(\lambda)^{-1} = i\theta_1^*(\lambda I_N - \beta)^{-1}\theta_2. \tag{3.6}$$

Using (1.3) in preliminaries, from $\beta = \alpha - i\theta_2\theta_2^*$ and (3.5) we obtain

$$d(\lambda)^{-1} = I_m - i\theta_2^*(\lambda I_N - \beta)^{-1}\theta_2. \tag{3.7}$$

From $\beta = \alpha - i\theta_2\theta_2^*$ and the equalities (3.5) and (3.7) we see that

$$\begin{aligned} &b(\lambda)d(\lambda)^{-1} \\ &\quad = i\theta_1^*(\lambda I_N - \alpha)^{-1}\theta_2 + i\theta_1^*(\lambda I_N - \alpha)^{-1}(\beta - \alpha)(\lambda I_N - \beta)^{-1}\theta_2. \end{aligned} \tag{3.8}$$

From (3.8) formula (3.6) follows.

Let φ be defined by (0.15), and thus by virtue of (3.6) we have

$$\varphi(\lambda) = b(\lambda)d(\lambda)^{-1}. \tag{3.9}$$

By (3.4), (3.9) and the representation (0.10) of the fundamental solution we get

$$W_n(\lambda)\begin{bmatrix} \varphi(\lambda) \\ I_m \end{bmatrix} = \left(\frac{\lambda + i}{\lambda}\right)^n W_{\alpha,\Lambda}(n,\lambda)\begin{bmatrix} 0 \\ d(\lambda)^{-1} \end{bmatrix}. \tag{3.10}$$

Notice also that (0.14) yields a more general formula than (2.11), namely

$$\begin{aligned} &W_{\alpha,\Lambda}(n,\lambda)^* W_{\alpha,\Lambda}(n,\lambda) \\ &\quad = I_{2m} - i(\lambda - \overline{\lambda})\Lambda_n^*(\overline{\lambda} I_N - \alpha^*)^{-1}\Sigma_n^{-1}(\lambda I_N - \alpha)^{-1}\Lambda_n. \end{aligned} \tag{3.11}$$

As the second term in the right-hand side of (3.11) is non-positive, it follows from formula (3.11) that

$$W_{\alpha,\Lambda}(n,\lambda)^* W_{\alpha,\Lambda}(n,\lambda) \leqslant I_{2m} \quad (\lambda \in \mathbb{C}_-). \tag{3.12}$$

Taking into account (3.10) and (3.12) we obtain (0.6), i.e., φ is a Weyl function.

It remains only to prove the uniqueness of the Weyl function. Let us first show that for some $M > 0$ and all $n \geqslant 0$ we have the inequality

$$f^*(I_N - i(\alpha^*)^{-1})^n \Sigma_n^{-1}(I_N + i\alpha^{-1})^n f \leqslant M f^* f, \quad f \in L, \tag{3.13}$$

where

$$L := \mathrm{span}_{\lambda \notin \sigma(\alpha)} \,\mathrm{Im}\,(\lambda I_N - \alpha)^{-1}\theta_1.$$

In view of (0.13) formula (3.11) yields

$$\theta_1^*(\bar{\lambda} I_N - \alpha^*)^{-1}(I_N - i(\alpha^*)^{-1})^n \Sigma_n^{-1} \times (I_N + i\alpha^{-1})^n(\lambda I_N - \alpha)^{-1}\theta_1 \leqslant \frac{i}{\bar{\lambda} - \lambda} I_m. \tag{3.14}$$

Now to get (3.13) from (3.14) we note that $\mathrm{span}_{\lambda \notin \sigma(\alpha)} \,\mathrm{Im}\,(\lambda I_N - \alpha)^{-1}\theta_1$ coincides with the same span when λ runs over an ε-neighbourhood O_ε of any $\lambda_0 \notin \sigma_\alpha$, for any sufficiently small $\varepsilon > 0$.

By (3.11) and (3.13) we can choose $M_1 > 0$ such that we have

$$[I_m \quad 0] W_{\alpha,\Lambda}(n,\lambda)^* W_{\alpha,\Lambda}(n,\lambda) \begin{bmatrix} I_m \\ 0 \end{bmatrix} \geqslant \frac{1}{2} \quad \text{for all } |\lambda| > M_1. \tag{3.15}$$

Without loss of generality we may assume that M_1 is large enough in order that $M_1 > \|\alpha\|$ and $a(\lambda)$ is invertible for $|\lambda| > M_1$. Then, taking into account (0.10) and (3.15), we obtain

$$\sum_{n=0}^{r} [I_m \quad \big(c(\lambda)a(\lambda)^{-1}\big)^*] \times W_n(\lambda)^* W_n(\lambda) \begin{bmatrix} I_m \\ c(\lambda)a(\lambda)^{-1} \end{bmatrix} > \frac{r}{2}\big(a(\lambda)^{-1}\big)^* a(\lambda)^{-1} \tag{3.16}$$

for all λ in the domain $D = \{\lambda : |\lambda| > M_1, \quad \Im\lambda < -1/2\}$. In other words, for $\lambda \in D$ we have

$$\sum_{n=0}^{\infty} f^* W_n(\lambda)^* W_n(\lambda) f = \infty \quad (f \in L_1), \tag{3.17}$$

where

$$L_1 := \mathrm{Im} \begin{bmatrix} I_m \\ c(\lambda)a(\lambda)^{-1} \end{bmatrix}.$$

Suppose now that φ and $\tilde{\varphi}$ are Weyl functions of (0.4) and that for some fixed $\lambda_0 \in D$ we have $\tilde{\varphi}(\lambda_0) \neq \varphi(\lambda_0)$. Put

$$L_2 = \operatorname{Im} \begin{bmatrix} \varphi(\lambda_0) \\ I_m \end{bmatrix} + \operatorname{Im} \begin{bmatrix} \tilde{\varphi}(\lambda_0) \\ I_m \end{bmatrix}.$$

According to the definition of the Weyl function we have

$$\sum_{n=0}^{\infty} f^* W_n(\lambda)^* W_n(\lambda) f < \infty \quad (f \in L_2). \tag{3.18}$$

As $\dim L_1 = m$ and $\dim L_2 > m$, there is a non-zero vector f such that $f \in (L_1 \cap L_2)$, which contradicts (3.17) and (3.18). □

For the proof of Theorem 0.5 we shall use the following lemma which is of independent interest.

Lemma 3.2. *A strictly proper rational $m \times m$ matrix function φ admits a minimal realization of the form*

$$\varphi(\lambda) = i\theta_1^*(\lambda I_N - \beta)^{-1}\theta_2, \tag{3.19}$$

such that $\beta - \beta^* = i(\theta_1\theta_1^* - \theta_2\theta_2^*)$.

Proof. We may assume that φ is given by the minimal realization (0.16). First let us show that Eq. (0.17) has a unique solution $X > 0$.

The minimality of the realization (0.16) means that the pair $\{\vartheta_1^*, \gamma\}$ is observable and the pair $\{\gamma, \vartheta_2\}$ is controllable. Notice that $\operatorname{Im} \vartheta \supseteq \operatorname{Im} \vartheta\vartheta^*$ and $f^*\vartheta\vartheta^* = 0$ yields $f^*\vartheta = 0$, i.e., $\operatorname{Im} \vartheta = \operatorname{Im} \vartheta\vartheta^*$. Hence the pair $\{\gamma, \vartheta_2\vartheta_2^*\}$ is controllable too. Therefore the pair $\{\vartheta_2\vartheta_2^*, i\gamma^*\}$ is observable. The pair $\{i\gamma^*, \vartheta_1\}$ is controllable and hence c-stabilizable, that is, there exists a matrix K such that $i\gamma^* + \vartheta_1 K$ has all its eigenvalues in the open left half-plane. But then we can use Theorem 16.3.3 [24] (see also [20]) to show that the Eq. (0.17) has a unique non-negative solution X and that this solution X is positive definite.

Next, let $\theta_1, \theta_2, \beta$ be defined by (0.18). From (0.17) and (0.18) we see that $\beta - \beta^* = i(\theta_1\theta_1^* - \theta_2\theta_2^*)$.

According to (0.16) and (0.18) the function φ is also given by the realization (3.19). Moreover as the realization (0.16) is minimal, the same is true for the realization (3.19). □

Proof of Theorem 0.5. Let φ be a strictly proper rational $m \times m$ matrix function. Let $\theta_1, \theta_2, \beta$ be as in Lemma 3.2, and put $\alpha = \beta + i\theta_2\theta_2^*$. Then the triple α, θ_1, and θ_2 satisfies (0.12). Furthermore, the pairs $\{\beta, \theta_2\}$ and $\{\beta^*, \theta_1\}$ are full range. Since

$\alpha = \beta + i\theta_2\theta_2^*$, it is immediate that the pair $\{\alpha, \theta_2\}$ is full range (see Section 1). By (0.12) we have $\beta^* = \alpha^* + i\theta_2\theta_2^* = \alpha - i\theta_1\theta_1^*$. Hence, as $\{\beta^*, \theta_1\}$ is a full range pair, so the pair $\{\alpha, \theta_1\}$ is full range too. Therefore the triple α, θ_1, and θ_2 is admissible. From Theorem 0.4 it follows now that the function $\{S_n\}$ determined by the admissible triple α, θ_1 and θ_2 is indeed a solution of the inverse problem.

Let us prove now the uniqueness of the solution of the inverse problem. Suppose that there is system (0.4) with another spin sequence $\{\widetilde{S}_n\}$, given by the admissible triple $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$, and with the same Weyl function φ. According to Theorem 0.4 we have another realization for φ, namely

$$\varphi(\lambda) = i\widetilde{\theta}_1^*(\lambda I_{\widetilde{N}} - \widetilde{\beta})^{-1}\widetilde{\theta}_2, \quad \widetilde{\beta} = \widetilde{\alpha} - i\widetilde{\theta}_2\widetilde{\theta}_2^*. \tag{3.20}$$

As the pairs $\{\widetilde{\alpha}, \widetilde{\theta}_1\}$ and $\{\widetilde{\alpha}, \widetilde{\theta}_2\}$ are controllable, and $\widetilde{\beta} = \widetilde{\alpha} - i\widetilde{\theta}_2\widetilde{\theta}_2^*$, $\widetilde{\beta}^* = \widetilde{a} - i\widetilde{\theta}_1\widetilde{\theta}_1^*$, it follows that the pairs $\{\widetilde{\beta}, \widetilde{\theta}_2\}$ and $\{\widetilde{\beta}^*, \widetilde{\theta}_1\}$ are also controllable. Thus the realization (3.20) is minimal and $\widetilde{N} = N$. Moreover, there is (see Section 1) a state–space similarity transforming the realization (0.15) into (3.20), that is, there exists an invertible $N \times N$ matrix S such that

$$\widetilde{\beta} = S\beta S^{-1}, \quad \widetilde{\theta}_2 = S\theta_2, \quad \widetilde{\theta}_1^* = \theta_1^* S^{-1}. \tag{3.21}$$

Recall that $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ is an admissible triple, and therefore we have

$$\widetilde{\beta} - \widetilde{\beta}^* = i(\widetilde{\theta}_1\widetilde{\theta}_1^* - \widetilde{\theta}_2\widetilde{\theta}_2^*). \tag{3.22}$$

From (3.21) and (3.22) it follows that $Z = S^{-1}(S^*)^{-1}$ satisfies

$$\beta Z - Z\beta^* = i(Z\theta_1\theta_1^* Z - \theta_2\theta_2^*), \quad Z > 0. \tag{3.23}$$

Completely similar to the uniqueness of $X > 0$ in (0.17) one obtains the uniqueness of the solution $Z > 0$ of (3.23). By comparing the identity $\beta - \beta^* = i(\theta_1\theta_1^* - \theta_2\theta_2^*)$ with (3.23) we see that $Z = I_N$ and thus S is unitary. In view of this we have $\widetilde{\alpha} = S\alpha S^*$, $\widetilde{\theta}_2 = S\theta_2$, and $\widetilde{\theta}_1 = S\theta_1$. This unitary equivalence transformation does not change the spin sequence, i.e., $\{S_n\} = \{\widetilde{S}_n\}$. □

The case when $i \notin \sigma(\alpha)$ is important in the next section. We shall use the acronym $\widetilde{\mathrm{FG}}$ to denote the class of spin sequences $\{S_n\}$ determined by the triples α, θ_1, θ_2, with α an $N \times N$ non-singular matrix and θ_1, θ_2 of size $N \times m$, satisfying the identity (0.12) and the additional special condition $i \notin \sigma(\alpha)$. Notice (see the beginning of the proof of Proposition 0.2) that (0.12) implies that $\sigma(\alpha) \subset \overline{\mathbb{C}_+}$. The next lemma shows that without loss of generality we can also require that $\{\alpha, \theta_1\}$ and $\{\alpha, \theta_2\}$ are full range pairs, i.e., $\widetilde{\mathrm{FG}} \subseteq \mathrm{FG}$.

Lemma 3.3. *Assume that the spin sequence* $\{S_n\}$ *belongs* $\widetilde{\widetilde{\mathrm{FG}}}$. *Then it can be determined by a triple* α, θ_1, θ_2 *such that* α *is non-singular,* (0.12) *holds,* $i \notin \sigma(\alpha)$, *and the pairs* $\{\alpha, \theta_1\}$ *and* $\{\alpha, \theta_2\}$ *are full range.*

Proof. Let N denote the minimal order of α $(0, i \notin \sigma(\alpha))$ in the set of triples that satisfy (0.12) and determine the given spin sequence $\{S_n\}$. Suppose the $N \times N$ matrix $\widehat{\alpha}$ and the $N \times m$ matrices $\widehat{\theta}_1$ and $\widehat{\theta}_2$ form such a triple but the pair $\{\widehat{\alpha}, \widehat{\theta}_2\}$ is not full range. Put

$$\widehat{L}_0 := \mathrm{span} \bigcup_{k=0}^{\infty} \mathrm{Im}\, \widehat{\alpha}^k \widehat{\theta}_2, \quad N_0 := \dim \widehat{L}_0$$

and choose a unitary matrix q that maps $\widehat{L}_0$ onto the $L_0 := \mathrm{Im}\,[I_{N_0} \ \ 0]^*$. Then we have

$$\alpha := q\widehat{\alpha}q^* = \begin{bmatrix} \widetilde{\alpha} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix}, \quad \theta_1 := q\widehat{\theta}_1 = \begin{bmatrix} \widetilde{\theta}_1 \\ \kappa \end{bmatrix}, \quad \theta_2 := q\widehat{\theta}_2 = \begin{bmatrix} \widetilde{\theta}_2 \\ 0 \end{bmatrix}, \tag{3.24}$$

where the $N_0 \times N_0$ matrix $\widetilde{\alpha}$ $(0, i \notin \sigma(\widetilde{\alpha}))$ and the $N_0 \times m$ matrices $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ form a triple which satisfies (0.12) and determines $\{S_n\}$. To show this we need to make some preparations. As $\widehat{L}_0$ is an invariant subspace of $\widehat{\alpha}$, so L_0 is an invariant subspace of α, and thus α has the block triangular form given in (3.24). Moreover in view of the inclusion $\mathrm{Im}\,\widehat{\theta}_2 \subseteq \widehat{L}_0$, we have $\mathrm{Im}\,\theta_2 \subseteq L_0$, i.e., θ_2 has the block form given in (3.24). Taking into account that q is unitary, $0, i \notin \sigma(\widehat{\alpha})$ and that $\widehat{\alpha}$, $\widehat{\theta}_1$, $\widehat{\theta}_2$ satisfy (0.12) and determine $\{S_n\}$, we see that $0, i \notin \sigma(\alpha)$ and that α, θ_1, θ_2 satisfy (0.12) and determine $\{S_n\}$ too. So in view of (3.24) we have $0, i \notin \sigma(\widetilde{\alpha})$ and the triple $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ satisfies (0.12) as well. Now use the matrices Q_n defined in (2.13) to rewrite $\Lambda_n^* \Sigma_n^{-1} \Lambda_n$ as a 2×2 block matrix with blocks of size $m \times m$. This yields

$$\Lambda_n^* \Sigma_n^{-1} \Lambda_n = \{(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{kj}\}_{k,j=1}^2$$

with blocks

$$\begin{aligned}
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{11} &= \theta_1^* Q_n^{-1} \theta_1, \\
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{12} &= \theta_1^* Q_n^{-1} (I_N + i\alpha^{-1})^{-n} (I_N - i\alpha^{-1})^n \theta_2, \\
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{21} &= \theta_2^* (I_N + i(\alpha^*)^{-1})^n (I_N - i(\alpha^*)^{-1})^{-n} Q_n^{-1} \theta_1, \\
(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{22} &= \theta_2^* (I_N + i(\alpha^*)^{-1})^n (I_N - i(\alpha^*)^{-1})^{-n} Q_n^{-1} \\
&\quad \times (I_N + i\alpha^{-1})^{-n} (I_N - i\alpha^{-1})^n \theta_2.
\end{aligned} \tag{3.25}$$

Partition $Q_{n+1} - Q_n$ into four blocks: $Q_{n+1} - Q_n = \{\chi_{kj}\}_{k,j=1}^2$, where χ_{11} is an $N_0 \times N_0$ block. In view of (0.13), (2.16), and (3.24) we obtain

$$\chi_{21} = 0, \quad \chi_{12} = 0, \quad \chi_{22} = 0,$$

$$\chi_{11} = -2\widetilde{\alpha}^{-1}(I_{N_0} + i\widetilde{\alpha}^{-1})^{-n-1}(I_{N_0} - i\widetilde{\alpha}^{-1})^n \widetilde{\theta}_2 \widetilde{\theta}_2^* (I_{N_0} + i(\widetilde{\alpha}^*)^{-1})^n \times (I_{N_0} - i(\widetilde{\alpha}^*)^{-1})^{-n-1}(\widetilde{\alpha}^*)^{-1}.$$

Denote by $\widetilde{\Lambda}_n$, $\widetilde{\Sigma}_n$, $\widetilde{Q}_n$, $\widetilde{S}_n$, etc. the matrices generated by the triple $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$. One can see that $\chi_{11} = \widetilde{Q}_{n+1} - \widetilde{Q}_n$. Taking into account $Q_0 = I_N$ we obtain that the matrices Q_n are block diagonal: $Q_n = \mathrm{diag}\{\widetilde{Q}_n, I_{N-N_0}\}$. Now according to (3.24), (3.25) it follows that

$$\Lambda_n^* \Sigma_n^{-1} \Lambda_n - \widetilde{\Lambda}_n^* \widetilde{\Sigma}_n^{-1} \widetilde{\Lambda}_n = \begin{bmatrix} \kappa^*\kappa & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.26}$$

By (0.9) and (3.26) we get $S_n = \widetilde{S}_n$, i.e., the triple $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ determines the spin sequence $\{S_n\}$. This contradicts the assumption that N is minimal and therefore the pair $\{\widehat{\alpha}, \widehat{\theta}_2\}$ should be full range.

In the same way we shall show that the pair $\{\widehat{\alpha}, \widehat{\theta}_1\}$ is full range too. Indeed, suppose $\{\widehat{\alpha}, \widehat{\theta}_1\}$ is not full range. Put now

$$\widehat{L}_0 := \mathrm{span} \bigcup_{k=0}^{\infty} \mathrm{Im}\, \widehat{\alpha}^k \widehat{\theta}_1, \quad N_0 := \dim \widehat{L}_0$$

and choose a unitary matrix q that maps $\widehat{L}_0$ onto the $L_0 := \mathrm{Im}\,[I_{N_0} \quad 0]^*$. Then similar to the previous case we obtain

$$\alpha := q\widehat{\alpha}q^* = \begin{bmatrix} \widetilde{\alpha} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix}, \quad \theta_1 := q\widehat{\theta}_1 = \begin{bmatrix} \widetilde{\theta}_1 \\ 0 \end{bmatrix}, \quad \theta_2 := q\widehat{\theta}_2 = \begin{bmatrix} \widetilde{\theta}_2 \\ \kappa \end{bmatrix}, \tag{3.27}$$

where the $N_0 \times N_0$ matrix $\widetilde{\alpha}$ $(0, i \notin \sigma(\widetilde{\alpha}))$ and th e $N_0 \times m$ matrices $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ form a triple which satisfies (0.12). To show that the triple $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ determines S_n we shall use the fact that α, θ_1, θ_2 determines S_n and rewrite $\Lambda_n^*\Sigma_n^{-1}\Lambda_n$ as a 2×2 block matrix with blocks of size $m \times m$ as follows:

$$\Lambda_n^* \Sigma_n^{-1} \Lambda_n = \{(\Lambda_n^* \Sigma_n^{-1} \Lambda_n)_{kj}\}_{k,j=1}^2$$

with

$$\begin{aligned}
(\Lambda_n^*\Sigma_n^{-1}\Lambda_n)_{22} &= \theta_2^* R_n^{-1} \theta_2, \\
(\Lambda_n^*\Sigma_n^{-1}\Lambda_n)_{21} &= \theta_2^* R_n^{-1} (I_N - i\alpha^{-1})^{-n}(I_N + i\alpha^{-1})^n \theta_1, \\
(\Lambda_n^*\Sigma_n^{-1}\Lambda_n)_{12} &= \theta_1^* (I_N - i(\alpha^*)^{-1})^n (I_N + i(\alpha^*)^{-1})^{-n} R_n^{-1} \theta_2, \\
(\Lambda_n^*\Sigma_n^{-1}\Lambda_n)_{11} &= \theta_1^* (I_N - i(\alpha^*)^{-1})^n (I_N + i(\alpha^*)^{-1})^{-n} R_n^{-1} \\
&\quad \times (I_N - i\alpha^{-1})^{-n}(I_N + i\alpha^{-1})^n \theta_1.
\end{aligned} \tag{3.28}$$

Here the matrices R_n are given by (2.12). Partition now $R_{n+1} - R_n$ into four blocks: $R_{n+1} - R_n = \{\chi_{kj}\}_{k,j=1}^2$, where χ_{11} is an $N_0 \times N_0$ block. In view of (2.15) and (3.27) we obtain

$$\chi_{21} = 0, \quad \chi_{12} = 0, \quad \chi_{22} = 0,$$
$$\chi_{11} = 2\widetilde{\alpha}^{-1}(I_{N_0} - i\widetilde{\alpha}^{-1})^{-n-1}(I_{N_0} + i\widetilde{\alpha}^{-1})^n\widetilde{\theta}_1\widetilde{\theta}_1^*(I_{N_0} - i(\widetilde{\alpha}^*)^{-1})^n$$
$$\times(I_{N_0} + i(\widetilde{\alpha}^*)^{-1})^{-n-1}(\widetilde{\alpha}^*)^{-1}.$$

Denote by $\widetilde{\Lambda}_n$, $\widetilde{\Sigma}_n$, $\widetilde{R}_n$, $\widetilde{S}_n$, etc. the matrices generated by the triple $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$. One can see that $\chi_{11} = \widetilde{R}_{n+1} - \widetilde{R}_n$. Taking into account $R_0 = I_N$ we obtain that the matrices R_n are block diagonal: $R_n = \operatorname{diag}\{\widetilde{R}_n, I_{N-N_0}\}$. Now according to (3.27), (3.28) it follows that

$$\Lambda_n^*\Sigma_n^{-1}\Lambda_n - \widetilde{\Lambda}_n^*\widetilde{\Sigma}_n^{-1}\widetilde{\Lambda}_n = \begin{bmatrix} 0 & 0 \\ 0 & \kappa^*\kappa \end{bmatrix}. \tag{3.29}$$

By (0.9) and (3.29) we get $S_n = \widetilde{S}_n$, i.e., the triple $\widetilde{\alpha}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ determines the spin sequence $\{S_n\}$. So the pair $\{\widehat{\alpha}, \widehat{\theta}_1\}$ should also be full range. □

Finally, from the proof of Theorem 0.4 we have the following corollary.

Corollary 3.4. *Let the parameter matrices* α $(0, i \notin \sigma(\alpha))$, $\Sigma_0(0) > 0$ *and* $\Lambda_0(0)$ *satisfy the identity* (0.7). *Then the Weyl function* φ *of the system determined by these matrices is given by the formula*

$$\varphi(\lambda) = i\theta_1^*\Sigma_0^{-1}(\lambda I_N - \widetilde{\beta})^{-1}\theta_2, \quad \widetilde{\beta} = \alpha - i\theta_2\theta_2^*\Sigma_0^{-1}.$$

This corollary is proved by transforming the matrices α, $\Sigma_0(0)$ and $\Lambda_0(0)$ into the equivalent set $\Sigma_0^{-\frac{1}{2}}\alpha\Sigma_0^{\frac{1}{2}}$, I_N, $\Sigma_0^{-\frac{1}{2}}\Lambda_0$.

4. Isotropic Heisenberg magnet

Explicit solutions of the discrete integrable nonlinear equations form an interesting and actively studied domain (see Refs. [1,11,12,22,23,25,27,37]). To study the IHM model we insert an additional variable t in our notations: $\Lambda_n(t)$, $\Sigma_n(t)$, $S_n(t)$, $W_{\alpha,\Lambda}(n, t, \lambda)$, $\varphi(t, \lambda)$ and so on. Notice that the order N and the parameter matrix α do not depend on t, and the dependence on t of the other matrix functions is defined by the equations:

$$\frac{d\Lambda_0}{dt} = -2\big((\alpha - iI_N)^{-1}\Lambda_0 P_+ + (\alpha + iI_N)^{-1}\Lambda_0 P_-\big),$$
$$P_\pm = \tfrac{1}{2}(I_{2m} \pm J) \tag{4.1}$$

and

$$\frac{d\Sigma_0}{dt} = -\Big((\alpha - iI_N)^{-1}\Sigma_0(t) + (\alpha + iI_N)^{-1}\Sigma_0(t) + \Sigma_0(t)(\alpha^* + iI_N)^{-1} + \Sigma_0(t)(\alpha^* - iI_N)^{-1} + 2(\alpha^2 + I_N)^{-1}\big(\alpha\Lambda_0(t)J\Lambda_0(t)^* + \Lambda_0(t)J\Lambda_0(t)^*\alpha^*\big)((\alpha^*)^2 + I_N)^{-1}\Big). \quad (4.2)$$

We assume that the parameter matrices α, $\Sigma_0(0)$ and $\Lambda_0(0)$ satisfy the identity

$$\alpha\Sigma_0(t) - \Sigma_0(t)\alpha^* = i\Lambda_0(t)\Lambda_0(t)^* \quad (4.3)$$

at $t = 0$. Then according to (4.1), (4.2) the identity (4.3) holds for all t. (This result can be obtained by differentiating both sides of (4.3).)

Theorem 4.1. *Assume the parameter matrices* α $(0,\ i \notin \sigma(\alpha))$, $\Sigma_0(0) > 0$ *and* $\Lambda_0(0)$ *satisfy the identity*

$$\alpha\Sigma_0(0) - \Sigma_0(0)\alpha^* = i\Lambda_0(0)\Lambda_0(0)^*.$$

Define $\Sigma_0(t)$ *and* $\Lambda_0(t)$ *by Eqs.* (4.1) *and* (4.2). *Then* $\Sigma_0(t) > 0$ *on some interval* $-\varepsilon < t < \varepsilon$, *and the sequence* $\{S_n(t)\}$ *given by* (0.9) *and* (0.8) *belongs to* $\widetilde{\mathrm{FG}}$ *for each t from this interval. Moreover,* $\{S_n(t)\}$ $(-\varepsilon < t < \varepsilon)$ *satisfies the IHM equations* (0.22), (0.23).

Proof. As α does not depend on t and (4.3) is true, it is immediate that $\{S_n(t)\} \subset \widetilde{\mathrm{FG}}$.

Similar to the cases treated in [31,32] we shall successively obtain the derivatives $\frac{d}{dt}\Lambda_n$, $\frac{d}{dt}\Sigma_n$, $\frac{d}{dt}\big(\Lambda_n^*\Sigma_n^{-1}\big)$, and $\frac{d}{dt}W_{\alpha,\Lambda}(n,t,\lambda)$, and use the expressions for these derivatives to derive the zero curvature equation (0.19), which is equivalent to (0.22), (0.23); see [12]. In view of (0.13) and (4.1) we have

$$\Lambda_n(t) = \big[(I_N + i\alpha^{-1})^n e^{-2t(\alpha - iI_N)^{-1}}\theta_1 \quad (I_N - i\alpha^{-1})^n e^{-2t(\alpha + iI_N)^{-1}}\theta_2\big], \quad (4.4)$$

where $\theta_p := (\Lambda_0(0))_p$ $(p = 1, 2)$. Hence it follows that

$$\frac{d\Lambda_n}{dt} = -2\big((\alpha - iI_N)^{-1}\Lambda_n P_+ + (\alpha + iI_N)^{-1}\Lambda_n P_-\big). \quad (4.5)$$

Now we shall show by induction that

$$\frac{d\Sigma_n}{dt} = -\Big((\alpha - iI_N)^{-1}\Sigma_n(t) + (\alpha + iI_N)^{-1}\Sigma_n(t) + \Sigma_n(t)(\alpha^* + iI_N)^{-1} + \Sigma_n(t)(\alpha^* - iI_N)^{-1} + 2(\alpha^2 + I_N)^{-1}\big(\alpha\Lambda_n(t)J\Lambda_n(t)^* + \Lambda_n(t)J\Lambda_n(t)^*\alpha^*\big)((\alpha^*)^2 + I_N)^{-1}\Big). \quad (4.6)$$

By (4.2) formula (4.6) is true for $n = 0$. Suppose it is true for $n = r$. Then, taking into account the second relation in (0.8) we obtain

$$\begin{aligned}\frac{d\Sigma_{r+1}}{dt} = -\Big(&(\alpha - iI_N)^{-1}\Sigma_{r+1}(t) - (\alpha - iI_N)^{-1}\alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}\\ &+ (\alpha + iI_N)^{-1}\Sigma_{r+1}(t) - (\alpha + iI_N)^{-1}\alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}\\ &+ \Sigma_{r+1}(t)(\alpha^* + iI_N)^{-1} - \alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}(\alpha^* + iI_N)^{-1}\\ &+ \Sigma_{r+1}(t)(\alpha^* - iI_N)^{-1} - \alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}(\alpha^* - iI_N)^{-1}\\ &+ 2(\alpha^2 + I_N)^{-1}\big((\alpha + \alpha^{-1})\Lambda_r(t)J\Lambda_r(t)^* + \Lambda_r(t)J\Lambda_r(t)^*\\ &\times(\alpha^* + (\alpha^*)^{-1})\big)((\alpha^*)^2 + I_N)^{-1}\Big) + \frac{d}{dt}\big(\alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}\big). \end{aligned} \tag{4.7}$$

In view of (4.5) we easily calculate that

$$\begin{aligned} C_1(t) = &\frac{d}{dt}\big(\alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}\big)\\ &+ (\alpha - iI_N)^{-1}\alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1} + (\alpha + iI_N)^{-1}\alpha^{-1}\Lambda_r(t)J\\ &\times\Lambda_r(t)^*(\alpha^*)^{-1} + \alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}(\alpha^* + iI_N)^{-1} + \alpha^{-1}\\ &\times\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}(\alpha^* - iI_N)^{-1}\\ = &-(\alpha - iI_N)^{-1}\alpha^{-1}\Lambda_r(t)\\ &\times\Lambda_r(t)^*(\alpha^*)^{-1} + (\alpha + iI_N)^{-1}\alpha^{-1}\Lambda_r(t)\Lambda_r(t)^*(\alpha^*)^{-1} - \alpha^{-1}\Lambda_r(t)\\ &\times\Lambda_r(t)^*(\alpha^*)^{-1}(\alpha^* + iI_N)^{-1} + \alpha^{-1}\Lambda_r(t)\Lambda_r(t)^*(\alpha^*)^{-1}(\alpha^* - iI_N)^{-1}. \end{aligned} \tag{4.8}$$

Notice that $(\alpha - iI_N)^{-1} - (\alpha + iI_N)^{-1} = 2i(\alpha^2 + I_N)^{-1}$. Therefore we can rewrite (4.8) as

$$\begin{aligned} C_1(t) = 2i\big(&\alpha^{-1}\Lambda_r(t)\Lambda_r(t)^*(\alpha^*)^{-1}((\alpha^*)^2 + I_N)^{-1}\\ &- (\alpha^2 + I_N)^{-1}\alpha^{-1}\Lambda_r(t)\Lambda_r(t)^*(\alpha^*)^{-1}\big). \end{aligned} \tag{4.9}$$

Notice also that

$$\begin{aligned} C_2(t) := 2(\alpha^2 + I_N)^{-1}\big(&(\alpha + \alpha^{-1})\Lambda_r(t)J\Lambda_r(t)^* + \Lambda_r(t)J\Lambda_r(t)^*\\ &\times(\alpha^* + (\alpha^*)^{-1})\big)((\alpha^*)^2 + I_N)^{-1} = 2\big(\alpha^{-1}\Lambda_r(t)J\Lambda_r(t)^*\\ &\times((\alpha^*)^2 + I_N)^{-1} + (\alpha^2 + I_N)^{-1}\Lambda_r(t)J\Lambda_r(t)^*(\alpha^*)^{-1}\big). \end{aligned} \tag{4.10}$$

Using the first relation in (0.8) and equalities (4.9), (4.10) we get

$$
\begin{aligned}
C_2(t) - C_1(t) = {} & 2\big(\alpha^{-1}\Lambda_r(t)J\Lambda_{r+1}(t)^*((\alpha^*)^2 + I_N)^{-1} \\
& + (\alpha^2 + I_N)^{-1}\Lambda_{r+1}(t)J\Lambda_r(t)^*(\alpha^*)^{-1}\big) = 2(\alpha^2 + I_N)^{-1} \\
& \times\big(\alpha\Lambda_{r+1}(t)J\Lambda_{r+1}(t)^* + \Lambda_{r+1}(t)J\Lambda_{r+1}(t)^*\alpha^*\big) \\
& \times((\alpha^*)^2 + I_N)^{-1}.
\end{aligned} \tag{4.11}
$$

From (4.7) and (4.11) it follows that (4.6) is valid for $n = r + 1$ and so for all $n > 0$.

Taking into account (4.5) and (4.6) we can obtain the equation

$$
\begin{aligned}
\frac{d}{dt}\big(\Lambda_n(t)^*\Sigma_n(t)^{-1}\big) = {} & H_n^+(t)\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha - iI_N)^{-1} \\
& + H_n^-(t)\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha + iI_N)^{-1},
\end{aligned} \tag{4.12}
$$

where

$$
H_n^+(t) = 2W_{\alpha,\Lambda}(n, t, i)P_+W_{\alpha,\Lambda}(n, t, -i)^*, \tag{4.13}
$$

$$
H_n^-(t) = 2W_{\alpha,\Lambda}(n, t, -i)P_-W_{\alpha,\Lambda}(n, t, i)^*. \tag{4.14}
$$

Indeed, by (4.5) we have

$$
\begin{aligned}
\frac{d}{dt}\big(\Lambda_n(t)^*\Sigma_n(t)^{-1}\big) = {} & -\Big(2P_+\Lambda_n(t)^*(\alpha^* + iI_N)^{-1} \\
& + 2P_-\Lambda_n(t)^*(\alpha^* - iI_N)^{-1} + \Sigma_n(t)^{-1}\frac{d\Sigma_n}{dt}(t)\Big)\Sigma_n(t)^{-1}.
\end{aligned} \tag{4.15}
$$

Identity (0.14) yields

$$
\begin{aligned}
((\alpha^*) \pm iI_N)^{-1}\Sigma_n(t)^{-1} = {} & \Sigma_n(t)^{-1}(\alpha \pm iI_N)^{-1} + i((\alpha^*) \pm iI_N)^{-1}\Sigma_n(t)^{-1} \\
& \times\Lambda_n(t)\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha \pm iI_N)^{-1}.
\end{aligned} \tag{4.16}
$$

Finally notice that

$$
2(\alpha^2 + I_N)^{-1}\alpha = (\alpha - iI_N)^{-1} + (\alpha + iI_N)^{-1}. \tag{4.17}
$$

By using (4.6), (4.16) and (4.17), and after some calculations, we rewrite (4.15) in the form (4.12), where

$$H_n^+(t) = I_{2m} + J - i\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha - iI_N)^{-1}\Lambda_n(t)J \\ + iJ\Lambda_n(t)^*(\alpha^* - iI_N)^{-1}\Sigma_n(t)^{-1}\Lambda_n(t) + \Lambda_n(t)^*\Sigma_n(t)^{-1} \\ \times(\alpha - iI_N)^{-1}\Lambda_n(t)J\Lambda_n(t)^*(\alpha^* - iI_N)^{-1}\Sigma_n(t)^{-1}\Lambda_n(t), \quad (4.18)$$

$$H_n^-(t) = I_{2m} - J + i\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha + iI_N)^{-1}\Lambda_n(t)J \\ - iJ\Lambda_n(t)^*(\alpha^* + iI_N)^{-1}\Sigma_n(t)^{-1}\Lambda_n(t) - \Lambda_n(t)^*\Sigma_n(t)^{-1} \\ \times(\alpha + iI_N)^{-1}\Lambda_n(t)J\Lambda_n(t)^*(\alpha^* + iI_N)^{-1}\Sigma_n(t)^{-1}\Lambda_n(t). \quad (4.19)$$

From (0.11) and (4.18) it follows that

$$H_n^+(t) = I_{2m} + W_{\alpha,\Lambda}(n, t, i)JW_{\alpha,\Lambda}(n, t, -i)^*$$

and so, taking into account (2.11), we derive (4.13). Equality (4.14) follows from (4.19) in a similar way.

Recall now that $m = 2$ and (2.11) holds. Then according to (2.17), (2.21) and (4.13) we get

$$H_n^+(t) = c_n^+(t)(I_2 + S_n(t))(I_2 + S_{n-1}(t)) \quad (4.20)$$

and according to (2.18), (2.22) and (4.14) we get

$$H_n^-(t) = c_n^-(t)(I_2 - S_n(t))(I_2 - S_{n-1}(t)), \quad (4.21)$$

where $c_n^\pm(t)$ are scalar functions. In view of (4.13) and (4.14) we obtain also $\mathrm{Tr}\, H_n^\pm$, where Tr denotes the trace. Indeed,

$$\mathrm{Tr}\, H_n^\pm(t) \equiv 2. \quad (4.22)$$

By Remark 2.4 and formula (0.22) we derive

$$\mathrm{Tr}\,(I_2 \pm S_n(t))(I_2 \pm S_{n-1}(t)) = \mathrm{Tr}\,(I_2 + S_n(t)S_{n-1}(t)) \\ = 2(1 + \vec{S}_{n-1}(t) \cdot \vec{S}_n(t)). \quad (4.23)$$

From (4.20)–(4.23) it follows that $1 + \vec{S}_{n-1}(t) \cdot \vec{S}_n(t) \neq 0$. Now formulas (0.21), (4.23) and (4.22) yield that

$$\mathrm{Tr}\, V_n^\pm(t) \equiv 2 \equiv \mathrm{Tr}\, H_n^\pm(t). \quad (4.24)$$

Taking into account (0.21), (4.20) and (4.21) we see that $V_n^{\pm} = \widehat{c}^{\pm} H_n^{\pm}$, and so (4.24) yields equalities $V_n^{\pm} \equiv H_n^{\pm}$. Hence we have

$$\frac{d}{dt}\left(\Lambda_n(t)^* \Sigma_n(t)^{-1}\right) = V_n^+(t)\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha - iI_N)^{-1} + V_n^-(t)\Lambda_n(t)^*\Sigma_n(t)^{-1}(\alpha + iI_N)^{-1}. \tag{4.25}$$

From $V_n^{\pm} \equiv H_n^{\pm}$, (2.11) and definitions (4.13) and (4.14) we also get

$$V_n^{\pm}(t) W_{\alpha,\Lambda}(n, t, \pm i) = 2W_{\alpha,\Lambda}(n, t, \pm i)P_{\pm}. \tag{4.26}$$

Let us differentiate now $W_{\alpha,\Lambda}$. For this purpose notice that

$$(\alpha \pm iI_N)^{-1}(\alpha - \lambda I_N)^{-1} = \frac{(\alpha - \lambda I_N)^{-1} - (\alpha \pm iI_N)^{-1}}{\lambda \pm i}. \tag{4.27}$$

Using (4.5), (4.25) and (4.27) we derive

$$\begin{aligned}\frac{d}{dt} W_{\alpha,\Lambda}(n, t, \lambda) &= V_n^+(t)\frac{W_{\alpha,\Lambda}(n, t, \lambda) - W_{\alpha,\Lambda}(n, t, i)}{\lambda - i} + V_n^-(t) \\ &\quad \times \frac{W_{\alpha,\Lambda}(n, t, \lambda) - W_{\alpha,\Lambda}(n, t, -i)}{\lambda + i} - 2\,\frac{W_{\alpha,\Lambda}(n, t, \lambda) - W_{\alpha,\Lambda}(n, t, i)}{\lambda - i} P_+ \\ &\quad - 2\frac{W_{\alpha,\Lambda}(n, t, \lambda) - W_{\alpha,\Lambda}(n, t, -i)}{\lambda + i} P_-. \end{aligned} \tag{4.28}$$

In view of (4.26) we can rewrite (4.28) as

$$\frac{d}{dt} W_{\alpha,\Lambda}(n, t, \lambda) = F_n(t, \lambda)W_{\alpha,\Lambda}(n, t, \lambda) - W_{\alpha,\Lambda}(n, t, \lambda)\widehat{F}(t, \lambda), \tag{4.29}$$

where F_n is given by the second relation in (0.20) and

$$\widehat{F} = 2((\lambda - i)^{-1}P_+ + (\lambda + i)^{-1}P_-).$$

Thus, in view of Theorem 0.1 and formula (4.29) the non-degenerate matrix functions $\{\widehat{W}_n\}$ given by

$$\widehat{W}_n(t, \lambda) = \lambda^{-n} W_{\alpha,\Lambda}(n, t, \lambda)\begin{bmatrix} (\lambda - i)^n e^{2t(\lambda - i)^{-1}} & 0 \\ 0 & (\lambda + i)^n e^{2t(\lambda + i)^{-1}} \end{bmatrix}$$

satisfy Eq. (0.24), i.e., the compatibility condition (0.19) is valid. As Eq. (0.19) is equivalent to (0.23), the theorem is proved. □

Theorem 4.1 together with Corollary 3.4 yields the following result.

Corollary 4.2. *Under the conditions of Theorem* 4.1 *the evolution of the Weyl function* φ *of the system* $W_{n+1}(t,\lambda) = G_n(t,\lambda)W_n(t,\lambda)$ *is given by the formula*

$$\varphi(t,\lambda) = i\theta_1^* e^{-2t(\alpha^*+iI_N)^{-1}}\Sigma_0(t)^{-1}(\lambda I_N - \widetilde{\beta}(t))^{-1}e^{-2t(\alpha+iI_N)^{-1}}\theta_2, \tag{4.30}$$

$$\widetilde{\beta}(t) = \alpha - ie^{-2t(\alpha+iI_N)^{-1}}\theta_2\theta_2^* e^{-2t(\alpha^*-iI_N)^{-1}}\Sigma_0(t)^{-1}.$$

As an illustration let us consider a simple example.

Example 4.3. Put $m = n = 1$ and $\alpha = ih$ $(h > 0,\ h \neq 1)$, and choose scalars θ_1, θ_2 such that $|\theta_1|^2 + |\theta_2|^2 = 2h$. Then α, θ_1, θ_2 form an admissible triple and α, $\Sigma_0(0) = 1$, $\Lambda_0(0) = [\theta_1 \quad \theta_2]$ satisfy the conditions of Theorem 4.1 and Corollary 4.2. Therefore by (4.4) we have

$$\Lambda_n(t) = h^{-n}\left[(h+1)^n\theta_1 \exp\left\{\frac{2it}{h-1}\right\} \qquad (h-1)^n\theta_2 \exp\left\{\frac{2it}{h+1}\right\}\right]. \tag{4.31}$$

From (0.14) and (4.31) it follows that

$$\Sigma_n(t) \equiv \frac{c_n(h)}{2h^{2n+1}}, \quad c_n(h) := (h+1)^{2n}|\theta_1|^2 + (h-1)^{2n}|\theta_2|^2. \tag{4.32}$$

According to (0.9), (4.31), and (4.32) we get now

$$(S_n(t))_{11} = 1 - \frac{8h^2|\theta_1\theta_2|^2(h^2-1)^{2n}}{c_n(h)c_{n+1}(h)}, \quad (S_n(t))_{22} = -(S_n(t))_{11},$$

$$(S_n(t))_{12} = (S_n(t))_{21}^* = \frac{4h\overline{\theta_1}\theta_2}{c_n(h)c_{n+1}(h)} \exp\left\{\frac{4it}{1-h^2}\right\}$$
$$\times (h^2-1)^n\left((h+1)^{2n+1}|\theta_1|^2 - (h-1)^{2n+1}|\theta_2|^2\right).$$

Finally Corollary 4.2 yields

$$\varphi(t,\lambda) = \exp\left\{\frac{4it}{1-h^2}\right\}\frac{i\overline{\theta_1}\theta_2}{\lambda + i(|\theta_2|^2 - h)}.$$

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