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On toric varieties which are almost set-theoretic complete intersections

Margherita Barile

Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, 70125 Bari, Italy

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Abstract

We describe a class of affine toric varieties V that are set-theoretically minimally defined by codim$V + 1$ binomial equations over fields of any characteristic.

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0. Introduction

Let K be an algebraically closed field. Let V be an affine variety, $V \subset K^N$, where N is the smallest possible. The arithmetical rank (ara) of V is defined as the least number of equations that are needed to define V set-theoretically as a subvariety of K^N. In general we have that ara $V \geq$ codim V. If equality holds, V is called a *set-theoretic complete intersection*. If ara $V \leq$ codim $V + 1$, V is called an *almost set-theoretic complete intersection*. The problem of determining when a toric variety is (almost) set-theoretic complete intersection is open in general; it includes the still unsettled case of monomial curves, which was intensively studied over the last three decades (see, e.g., [6, 10–12,14–16]). In [3] it was shown that, whenever V is a simplicial toric variety with a full parametrization, then ara V = codim V in all positive characteristics, whereas

E-mail address: barile@dm.uniba.it.

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ara $V \le$ codim $V + 1$ in characteristic zero. A different class of toric varieties with the same property has recently been described in [1]. In this paper we present a class of toric varieties V of arbitrary codimension greater than 2, such that ara $V =$ codim $V + 1$ in all characteristics: each variety V is defined, over any field, by the vanishing of the same set of codim $V + 1$ binomials, even though these do not generate the defining ideal $I(V) \subset K[x_1, \ldots, x_N]$. It seems that no example of this kind has ever appeared in the recent literature.

1. Preliminaries

Let K be an algebraically closed field. Let $n \ge 3$ be an integer and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{Z}^n$. Set $N = 2n$ and consider the following subset of $\mathbb{N}^N$:

$$T = \{d_1\mathbf{e}_1, \ldots, d_{n-1}\mathbf{e}_{n-1}, \mathbf{e}_n, f_1\mathbf{e}_1 + g_1\mathbf{e}_n, \ldots,\\ f_{n-1}\mathbf{e}_{n-1} + g_{n-1}\mathbf{e}_n, h_1\mathbf{e}_1 + \cdots + h_{n-1}\mathbf{e}_{n-1}\},$$

where $d_1, \ldots, d_{n-1}, f_1, \ldots, f_{n-1}, g_1, \ldots, g_{n-1}, h_1, \ldots, h_{n-1}$ are all positive integers such that

$$\gcd(d_i, f_i) = 1 \quad \text{for all } i = 1, \ldots, n-1, \tag{1}$$

$$\gcd(d_i, h_i) = 1 \quad \text{for all } i = 1, \ldots, n-1, \tag{2}$$

and

$$\gcd(d_i, d_j) = 1 \quad \text{for all } i, j = 1, \ldots, n-1, i \ne j. \tag{3}$$

Suppose, moreover, that there are two distinct (positive) primes p and q such that p divides d_i and q divides d_j for some indices i and j. With T we can associate the affine variety $V \subset K^N$ admitting the following parametrization.

$$V : \begin{cases} x_1 = u_1^{d_1} \\ \quad\vdots \\ x_{n-1} = u_{n-1}^{d_{n-1}} \\ x_n = u_n \\ y_1 = u_1^{f_1} u_n^{g_1} \\ \quad\vdots \\ y_{n-1} = u_{n-1}^{f_{n-1}} u_n^{g_{n-1}} \\ y_n = u_1^{h_1} \cdots u_{n-1}^{h_{n-1}} \end{cases}$$

We have that codim $V = n$. The polynomials in the defining ideal $I(V)$ of V are the linear combinations of binomials

$$B_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n} = x_1^{\alpha_1^+} \cdots x_n^{\alpha_n^+} y_1^{\beta_1^+} \cdots y_n^{\beta_n^+} - x_1^{\alpha_1^-} \cdots x_n^{\alpha_n^-} y_1^{\beta_1^-} \cdots y_n^{\beta_n^-}$$

where $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \in \mathbb{Z}^N \setminus \{\mathbf{0}\}$ is such that

$$\begin{aligned}&\alpha_1 d_1\mathbf{e}_1+\cdots+\alpha_{n-1}d_{n-1}\mathbf{e}_{n-1}+\alpha_n\mathbf{e}_n\\&\quad+\beta_1(f_1\mathbf{e}_1+g_1\mathbf{e}_n)+\cdots+\beta_{n-1}(f_{n-1}\mathbf{e}_{n-1}+g_{n-1}\mathbf{e}_n)\\&\quad+\beta_n(h_1\mathbf{e}_1+\cdots+h_{n-1}\mathbf{e}_{n-1})=\mathbf{0}\end{aligned}\tag{*}$$

and $\alpha_i^+=\max\{\alpha_i,0\}$, $\alpha_i^-=\max\{-\alpha_i,0\}$, $\beta_i^+=\max\{\beta_i,0\}$ and $\beta_i^-=\max\{-\beta_i,0\}$. There is a one-to-one correspondence between the set of binomials in $I(V)$ and the set of relations (*) between the elements of T. Our aim is to prove the following:

Theorem 1. *ara* $V=n+1$.

This will be done by proving the two inequalities separately, in Sections 2 and 4 respectively.

2. The defining equations

In this section we explicitly exhibit $n+1$ binomial equations which define V set-theoretically over any field K.

From (1) it follows that there are $\alpha_1,\ldots,\alpha_{n-1},\beta_1\ \ldots,\beta_{n-1}\in\mathbb{Z}$ such that

$$h_i=\alpha_i d_i+\beta_i f_i\quad\text{for all } i=1,\ldots,n-1.\tag{4}$$

Then

$$\sum_{i=1}^{n-1}\alpha_i d_i\mathbf{e}_i-\left(\sum_{i=1}^{n-1}\beta_i g_i\right)\mathbf{e}_n+\sum_{i=1}^{n-1}\beta_i(f_i\mathbf{e}_i+g_i\mathbf{e}_n)-\sum_{i=1}^{n-1}h_i\mathbf{e}_i=0.$$

Hence the binomial

$$G=B_{(\alpha_1,\ldots,\alpha_{n-1},-\sum_{i=1}^{n-1}\beta_i g_i,\beta_1,\ldots,\beta_{n-1},-1)}$$

belongs to $I(V)$. Let

$$F=y_n^{d_1\cdots d_{n-1}}-x_1^{h_1 d_2\cdots d_{n-1}}\cdots x_i^{d_1\cdots d_{i-1}h_i d_{i+1}\cdots d_n}\cdots x_{n-1}^{d_1\cdots d_{n-2}h_{n-1}},$$

and

$$F_i=y_i^{d_i}-x_i^{f_i}x_n^{d_i g_i}\quad\text{for all } i=1,\ldots,n-1.$$

An easy computation yields that $F,F_1,\ldots,F_{n-1}\in I(V)$.

Proposition 1. *Variety V is set-theoretically defined by*

$$F_1=\cdots=F_{n-1}=F=G=0.$$

Proof. We only have to prove that every $\mathbf{w}\in K^N$ fulfilling the given equations belongs to V. So let $\mathbf{w}=(\bar{x}_1,\ldots,\bar{x}_n,\bar{y}_1,\ldots,\bar{y}_n)\in K^N$ be such that

$$F_1(\mathbf{w})=\cdots=F_{n-1}(\mathbf{w})=0\tag{5}$$

$$F(\mathbf{w})=0\tag{6}$$

$$G(\mathbf{w})=0.\tag{7}$$

Let $u_1, \ldots, u_n \in K$ be such that $\bar{x}_i = u_i^{d_i}$ for all $i = 1, \ldots, n-1$, and set $u_n = \bar{x}_n$. We show that for a suitable choice of $u_1, \ldots, u_{n-1}$, we have

$$\bar{y}_i = u_i^{f_i} u_n^{g_i} \quad \text{for all } i = 1, \ldots, n-1, \tag{8}$$

and

$$\bar{y}_n = u_1^{h_1} \cdots u_{n-1}^{h_{n-1}}. \tag{9}$$

We first show that, up to replacing some u_i with another d_ith root of $\bar{x}_i$, (8) is fulfilled. From (6) we deduce that $\bar{y}_n^{d_1 \cdots d_{n-1}} = u_1^{h_1 d_1 \cdots d_{n-1}} \cdots u_{n-1}^{h_{n-1} d_1 \cdots d_{n-1}}$, so that

$$\bar{y}_n = u_1^{h_1} \cdots u_{n-1}^{h_{n-1}} \eta, \tag{9$'$}$$

for some $\eta \in K$ such that $\eta^{d_1 \cdots d_{n-1}} = 1$. For all $i = 1, \ldots, n-1$ let $\eta_i \in K$ be a primitive d_ith root of unity. Then (3) implies that $\eta' = \eta_1 \cdots \eta_{n-1}$ is a primitive $d_1 \cdots d_{n-1}$th root of unity. Hence, for a suitable positive integer s, we have $\eta'^s = \eta$. Choose $u_i \eta_i^s$ instead of u_i, for all $i = 1, \ldots, n-1$. Then (9)$'$ turns into (9). Condition (8) is also fulfilled if $\bar{x}_n = 0$. So suppose that $\bar{x}_n \neq 0$. We prove that, up to a modification of $u_1, \ldots, u_{n-1}$ which preserves (9), condition (8) is satisfied. From (5) we derive that, for all $i = 1, \ldots, n-1$, $\bar{y}_i^{d_i} = u_i^{d_i f_i} u_n^{d_i g_i}$, so that

$$\bar{y}_i = u_i^{f_i} u_n^{g_i} \theta_i, \tag{8$'$}$$

where $\theta_i \in K$ is some d_ith root of unity. From (1) it follows that $\theta_i = \omega_i^{f_i}$ where $\omega_i \in K$ is some d_ith root of unity. We replace u_i by $u_i \omega_i$ for all $i = 1, \ldots, n-1$. Then (8)$'$ turns into (8). Finally, from (7) it follows that $\bar{y}_n \bar{x}_n^{\beta_1 g_1 + \cdots + \beta_{n-1} g_{n-1}} = \bar{x}_1^{\alpha_1} \cdots \bar{x}_{n-1}^{\alpha_{n-1}} \bar{y}_1^{\beta_1} \cdots \bar{y}_{n-1}^{\beta_{n-1}}$, whence

$$\bar{y}_n u_n^{\beta_1 g_1 + \cdots + \beta_{n-1} g_{n-1}} = u_1^{\alpha_1 d_1 + \beta_1 f_1} \cdots u_{n-1}^{\alpha_{n-1} d_{n-1} + \beta_{n-1} f_{n-1}} u_n^{\beta_1 g_1 + \cdots \beta_{n-1} g_{n-1}},$$

which, since $u_n \neq 0$, and in view of (4), implies (9). This completes the proof. □

We have just proven that:

Corollary 1. *ara* $V \leq n + 1$.

In general, even if V is set-theoretically defined by $n + 1$ binomial equations, $n + 1$ binomials are not sufficient to generate the defining ideal of $I(V)$. This is shown in the next example.

Example 1. Let $n = 3$, and take $d_1 = 2$, $d_2 = 3$, $f_1 = 3$, $f_2 = 5$, $g_1 = g_2 = 1$, $h_1 = 3$, $h_2 = 5$. The corresponding toric variety is

$$V: \begin{cases} x_1 = u_1^2 \\ x_2 = u_2^3 \\ x_3 = u_3 \\ y_1 = u_1^3 u_3 \\ y_2 = u_2^5 u_3 \\ y_3 = u_1^3 u_2^5 \end{cases}$$

A computation by CoCoA [9] yields that the defining ideal $I(V)$ is minimally generated by the following eight binomials:

$$y_1^2 - x_1^3x_3^2, \qquad y_2^3 - x_2^5x_3^3, \qquad y_3^6 - x_1^9x_2^{10}, \qquad x_3^2y_3 - y_1y_2,$$
$$y_1y_3 - x_1^3y_2, \qquad y_2y_3^2 - x_1^3x_2^5x_3, \qquad y_2^2y_3 - x_2^5x_3y_1, \qquad x_3y_3^3 - x_1^3x_2^5y_1.$$

Since $f_i = h_i$ for $i = 1, 2$, in (4) we can take $\alpha_i = 0$ and $\beta_i = 1$ for $i = 1, 2$. Then the first four binomials in the above list are F_1, F_2, F and G respectively. According to Proposition 1 they suffice to define V set-theoretically.

3. Some cohomological results

In the next section we shall complete the proof of Theorem 1 using cohomological tools. In this section we provide the necessary preliminary lemmas on étale cohomology. We refer to [13] for the basic notions. Let $K^* = K \setminus \{0\}$.

Lemma 1. *Let n be a positive integer, and let r be an integer prime to* char K. *Let $d_1, \ldots, d_n$ be positive integers such that, for some index i, d_i and r are not coprime, and consider the morphism of schemes*

$$\delta : (K^*)^n \to (K^*)^n$$
$$(u_1, \ldots, u_n) \mapsto (u_1^{d_1}, \ldots, u_n^{d_n}).$$

Then the map

$$\kappa_n : H_c^{n+1}((K^*)^n) \to H_c^{n+1}((K^*)^n)$$

induced in cohomology with compact support is not injective.

Proof. In what follows H_{et} and H_c will denote étale cohomology and cohomology with compact support with respect to $\mathbb{Z}/r\mathbb{Z}$; for the sake of simplicity we shall omit the coefficient group in this and in the next proofs. We proceed by induction on $n \geq 1$. For $n = 1$ we have the morphism

$$\delta : K^* \to K^*$$
$$u_1 \mapsto u_1^{d_1},$$

and we know that exponentiation to d_1 in K induces multiplication by d_1 in cohomology with compact support. Thus δ gives rise to the following commutative diagram with exact rows in cohomology with compact support:

$$\begin{array}{ccccccc}
0 & & \mathbb{Z}/r\mathbb{Z} & & \mathbb{Z}/r\mathbb{Z} & & 0 \\
\| & & \wr| & \simeq & \wr| & & \| \\
H_c^1(\{0\}) & \to & H_c^2(K^*) & \to & H_c^2(K) & \to & H_c^2(\{0\}) \\
 & & \downarrow \kappa_1 & & \downarrow \cdot d_1 & & \\
H_c^1(\{0\}) & \to & H_c^2(K^*) & \to & H_c^2(K) & \to & H_c^2(\{0\}) \\
\| & & \wr| & \simeq & \wr| & & \| \\
0 & & \mathbb{Z}/r\mathbb{Z} & & \mathbb{Z}/r\mathbb{Z} & & 0
\end{array}$$

Since by assumption d_1 and r are not coprime, multiplication by d_1 in $\mathbb{Z}/r\mathbb{Z}$ is not injective. It follows that κ_1 is not injective. Now let $n > 1$ and suppose the claim true for all smaller n. Without loss of generality we may assume that d_1 is not prime to r. Note that $\{0\}\times(K^*)^{n-1}$ is a closed subset of $K \times (K^*)^{n-1}$ and $(K \times (K^*)^{n-1}) \setminus (\{0\} \times (K^*)^{n-1}) = (K^*)^n$. After identifying $\{0\} \times (K^*)^{n-1}$ with $(K^*)^{n-1}$ we have an exact sequence of cohomology with compact support

$$H_c^n(K \times (K^*)^{n-1}) \to H_c^n((K^*)^{n-1}) \to H_c^{n+1}((K^*)^n) \to H_c^{n+1}(K \times (K^*)^{n-1}). \tag{10}$$

According to the Künneth formula, for all indices i we have

$$H_c^i(K \times (K^*)^{n-1}) \simeq \bigoplus_{p+q_1+\cdots+q_{n-1}=i} H_c^p(K) \otimes H_c^{q_1}(K^*) \otimes \cdots \otimes H_c^{q_{n-1}}(K^*).$$

Since $H_c^p(K) = 0$ for $p \neq 2$ and $H_c^q(K^*) = 0$ for $q \neq 1, 2$, it follows that

$$H_c^n(K \times (K^*)^{n-1}) = 0.$$

Let

$$\delta' : (K^*)^{n-1} \to (K^*)^{n-1}$$
$$(u_1, \ldots, u_{n-1}) \mapsto (u_1^{d_1}, \ldots, u_{n-1}^{d_{n-1}})$$

be the map obtained from δ by restriction, and let

$$\kappa_{n-1} : H_c^n((K^*)^{n-1}) \to H_c^n((K^*)^{n-1})$$

be the homomorphism it induces in cohomology with compact support. Then (10) gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \to & H_c^n((K^*)^{n-1}) & \to & H_c^{n+1}((K^*)^n) \\ & & \downarrow \kappa_{n-1} & & \downarrow \kappa_n \\ 0 & \to & H_c^n((K^*)^{n-1}) & \to & H_c^{n+1}((K^*)^n). \end{array}$$

Since, by induction, κ_{n-1} is not injective, it follows that κ_n is not injective. This completes the proof. □

The following result is due to Newstead and is quoted from [7].

Lemma 2. *Let $W \subset \tilde{W}$ be affine varieties. Let $d = \dim \tilde{W} \setminus W$. If there are s equations $F_1, \ldots, F_s$ such that $W = \tilde{W} \cap V(F_1, \ldots, F_s)$, then*

$$H_{et}^{d+i}(\tilde{W} \setminus W, \mathbb{Z}/r\mathbb{Z}) = 0 \qquad \text{for all } i \geq s$$

and for all $r \in \mathbb{Z}$ which are prime to char K.

From this we derive the following cohomological criterion on the number of equations defining an affine variety set-theoretically.

Corollary 2. *Let V be a subvariety of the affine space K^N. Let $s < 2N$ be a positive integer such that*

$$H_c^s(V, \mathbb{Z}/r\mathbb{Z}) \neq 0$$

for some positive integer r prime to char K. *Then ara $V \geq N - s$.*

Proof. As in the preceding proofs, we shall omit the coefficient group $\mathbb{Z}/r\mathbb{Z}$. We have an exact sequence

$$H_c^s(K^N) \to H_c^s(V) \to H_c^{s+1}(K^N \setminus V),$$

where $H_c^s(K^N) = 0$. If $H_c^s(V) \neq 0$, it follows that $H_c^{s+1}(K^N \setminus V) \neq 0$. Applying Poincaré Duality (see [13, Corollary 11.2, p. 276]) yields

$$H_{\text{et}}^{2N-s-1}(K^N \setminus V) \simeq H_c^{s+1}(K^N \setminus V) \neq 0,$$

so that, by Lemma 2, V is not defined set-theoretically by $N - s - 1$ equations, i.e., ara $V \geq N - s$, as claimed. This completes the proof. □

4. The cohomological lower bound

We are now ready to complete the proof of Theorem 1 for the toric variety V introduced above. We show

Proposition 2. *ara $V \geq n + 1$.*

Proof. Let $r \in \{p, q\}$. Suppose that char $K \neq r$. It suffices to show that the claim holds under this assumption: since $p \neq q$, the characteristic of any ground field is different from p or different from q. All the cohomology groups considered in this proof are referred to the coefficient group $\mathbb{Z}/r\mathbb{Z}$. Let

$$\phi : K^n \to V$$
$$(u_1, \ldots, u_n) \mapsto (u_1^{d_1}, \ldots, u_{n-1}^{d_{n-1}}, u_n, u_1^{f_1} u_n^{g_1}, \ldots, u_{n-1}^{f_{n-1}} u_n^{g_{n-1}}, u_1^{h_1} \cdots u_{n-1}^{h_{n-1}})$$

and let $W \subset K^n$ be the subvariety defined by $u_n = u_1 \cdots u_{n-1} = 0$. The restriction

$$\phi' : K^n \setminus W \to V \setminus \phi(W)$$

is a bijection. Surjectivity is obvious, we only have to prove injectivity. Let $\mathbf{u} = (\bar{u}_1, \ldots, \bar{u}_{n-1}, \bar{u}_n) \in K^n \setminus W$. First assume that $\bar{u}_n \neq 0$. Let $i \in \{1, \ldots, n-1\}$. We prove that $\bar{u}_i$ is uniquely determined by $\phi(\mathbf{u})$. This is evidently true if $\bar{u}_i = 0$. So suppose that $\bar{u}_i \neq 0$. By (1) there are $a_i, b_i \in \mathbf{Z}$ such that $a_i d_i + b_i f_i = 1$. Hence

$$\bar{u}_i = \bar{u}_i^{a_i d_i + b_i f_i} = (\bar{u}_i^{d_i})^{a_i} \frac{(\bar{u}_i^{f_i} \bar{u}_n^{g_i})^{b_i}}{\bar{u}_n^{g_i b_i}}.$$

Now assume that $\bar{u}_n = 0$, so that $\bar{u}_1, \ldots, \bar{u}_{n-1} \neq 0$. Let $\mathbf{v} = (\bar{v}_1, \ldots, \bar{v}_{n-1}, 0) \in K^n$ be such that $\phi'(\mathbf{v}) = \phi'(\mathbf{u})$. Then, for all $i = 1, \ldots, n-1$, $\bar{v}_i^{d_i} = \bar{u}_i^{d_i}$ so that

$$\bar{v}_i = \eta_i \bar{u}_i \tag{11}$$

for some d_ith root of unity η_i. We also have that

$$\bar{v}_1^{h_1} \cdots \bar{v}_{n-1}^{h_{n-1}} = \bar{u}_1^{h_1} \cdots \bar{u}_{n-1}^{h_{n-1}}. \tag{12}$$

Since $\bar{u}_i \neq 0$ for all $i = 1, \ldots, n-1$, from (11) and (12) it follows that

$$\eta_1^{h_1} \cdots \eta_{n-1}^{h_{n-1}} = 1.$$

Hence

$$\omega_1 = \eta_1^{h_1} = \eta_2^{-h_2} \cdots \eta_{n-1}^{-h_{n-1}} \tag{13}$$

is both a d_1th root and an ℓ_1th root of unity, where $\ell_1 = \text{lcm}\ (d_2, \ldots, d_{n-1})$. Since, by (3), $\gcd(\ell_1, d_1) = 1$, it follows that $\omega_1 = 1$. Hence, in view of (13), η_1 is both a d_1th and a h_1th root of unity. By (2) it follows that $\eta_1 = 1$. Similarly, one shows that $\eta_i = 1$ for all $i = 2, \ldots, n-1$. Hence, by (11), $\mathbf{v} = \mathbf{u}$, which completes the proof that ϕ' is injective. Since it is finite, it is proper. Therefore, according to [8], Lemma 3.1, it induces an isomorphism in cohomology

$$\phi_i' : H_c^i(V \setminus \phi(W)) \simeq H_c^i(K^n \setminus W),$$

for all indices i. Now consider the restriction map

$$\begin{aligned} &\phi'' : W \to \phi(W) \\ &(u_1, \ldots, u_{n-1}, 0) \mapsto (u_1^{d_1}, \ldots, u_{n-1}^{d_{n-1}}, 0, \ldots, 0) \end{aligned}$$

and the maps it induces in cohomology

$$\phi_i'' : H_c^i(\phi(W)) \to H_c^i(W).$$

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H_c^{n-1}(V) & \to & H_c^{n-1}(\phi(W)) & \overset{\alpha}{\to} & H_c^n(V \setminus \phi(W)) & & \\ & & \downarrow \phi_{n-1}'' & & \wr\!\downarrow \phi_n' & & \\ H_c^{n-1}(K^n) & \to & H_c^{n-1}(W) & \underset{\simeq}{\to} & H_c^n(K^n \setminus W) & \to & H_c^n(K^n) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

Our next aim is to show that ϕ_{n-1}'' is not injective. It will follow that $\phi_n'\alpha$ is not injective, and, consequently, α is not injective, so that

$$H_c^{n-1}(V) \neq 0.$$

In view of Corollary 2, this will imply the claim. Let X be the subvariety of K^{n-1} defined by $u_1 \cdots u_{n-1} = 0$. Then, up to omitting zero coordinates, ϕ'' can be viewed as the map:

$$\begin{aligned} &\phi'' : X \to X \\ &(u_1, \ldots, u_{n-1}) \mapsto (u_1^{d_1}, \ldots, u_{n-1}^{d_{n-1}}). \end{aligned}$$

We have to show that the induced map

$$\phi''_{n-1} : H^{n-1}_{\mathrm{c}}(X) \to H^{n-1}_{\mathrm{c}}(X)$$

is not injective. Note that $K^{n-1} \setminus X = (K^*)^{n-1}$. Thus we have the following commutative diagram with exact rows, where κ_{n-1} is the map defined in Lemma 1:

$$\begin{array}{ccccccc}
0 & & & & & & 0 \\
\| & & & \simeq & & & \| \\
H^{n-1}_{\mathrm{c}}(K^{n-1}) & \to & H^{n-1}_{\mathrm{c}}(X) & \to & H^{n}_{\mathrm{c}}((K^*)^{n-1}) & \to & H^{n}_{\mathrm{c}}(K^{n-1}) \\
 & & \downarrow \phi''_{n-1} & & \downarrow \kappa_{n-1} & & \\
H^{n-1}_{\mathrm{c}}(K^{n-1}) & \to & H^{n-1}_{\mathrm{c}}(X) & \to & H^{n}_{\mathrm{c}}((K^*)^{n-1}) & \to & H^{n}_{\mathrm{c}}(K^{n-1}) \\
\| & & & \simeq & & & \| \\
0 & & & & & & 0
\end{array}$$

By Lemma 1, since r is a prime factor of some of the exponents d_i, if char $K \neq r$, κ_{n-1} is not injective, hence nor is ϕ''_{n-1}, as was to be shown. This completes the proof. □

5. Final remarks

For the toric varieties V considered in this paper, the minimal number of defining equations coincides with the minimal number of *binomial* defining equations, i.e., ara V = bar $V = n + 1$, where "bar" denotes the so-called *binomial arithmetical rank*, a notion introduced by Thoma [17]. In general, ara $V <$ bar V: an example is given by the projective monomial curves studied by Robbiano and Valla [15], which are defined by two equations (and thus are set-theoretic complete intersections), but one of these equations is, in general, non-binomial. As it was shown in [4], over fields of characteristic zero we have ara V = bar V = codim$V$ for a toric variety V only in a very special case, namely if and only if the defining ideal $I(V)$ is a complete intersection (i.e., if and only if it is generated by codim V equations). The criterion changes completely in characteristic p: there the above equality holds for a much larger class of toric varieties, whose attached affine semigroup $\mathbb{N}T$ is *completely p-glued*, a combinatorial property described in [4]. This difference explains why there are several examples of toric varieties whose arithmetical rank strictly depends on the characteristic of the ground field. One can find classes for which ara V = bar V = codim$V$ in one positive characteristic, whereas ara $V >$ codim$V$ in all remaining characteristics: this is the case for the simplicial toric varieties of codimension 2 described in [5], and for certain Veronese varieties, of arbitrarily high codimension, considered in [2]. In this respect, the main result of this paper presents a new situation: the arithmetical rank is constant in all characteristics. Moreover the defining equations are the same over all fields. This is not true for monomial curves in the three-dimensional projective space: in every positive characteristic they are binomial set-theoretic complete intersections, but the two defining binomial equations change from one characteristic to the other (see [14]), whereas in characteristic zero their arithmetical ranks are unknown in general.

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