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Stochastic Differential Equations in Hilbert Space

RUTH F. CURTAIN

School of Aeronautics, Astronautics, and Engineering Sciences, Purdue University, Lafayette, Indiana

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AND

PETER L. FALB*

Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island

Received June 17, 1970

Existence and uniqueness theorems for stochastic evolution equations are developed in a Hilbert space context. The results are based on a blending of the theorems for evolution equations with stochastic integration for Hilbert space valued random processes. The results are applied to stochastically forced parabolic partial differential equations such as have arisen in heat transfer problems.

1. INTRODUCTION

We develop existence and uniqueness theorems for stochastic abstract evolution equations in a Hilbert space context. Our results are based on a blending of the existence and uniqueness theorems for solutions of abstract evolution equations ([8–10, 12, 13]) with the theory of stochastic integration for Hilbert space valued random processes [1, 4, 6, 14]. More precisely, by giving appropriate definitions of a Wiener process and stochastic integration in a Hilbert space context, we are able to treat evolution equations of the form $u(t) = u_0 + \int_{T_1}^{t} A(s)\, u(s)\, ds + \int_{T_1}^{t} \Phi(s)\, dw$, where $A(s)$ is a closed (possibly unbounded) linear operator, Φ is a suitable transformation valued stochastic process, and $w(t)$ is a Hilbert space valued Wiener process.

We let $(\Omega, \mathscr{P}, \mu)$ be a probability space with $\mathscr{P}$ as Borel field and μ as

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measure, and we assume that μ is complete. We also let H denote a separable Hilbert space throughout the sequel. We suppose that the reader is somewhat familiar with the theory of Banach space valued random variables (see, for example [6]); however, for convenience, we include a brief appendix containing the definitions and results relevant to the paper.

We give the basic definitions of an H-valued Wiener process and the corresponding stochastic integral in Section 2 together with a number of their properties. Then we state and prove the existence and uniqueness theorems in Section 3. The results are applied to stochastically forced parabolic partial differential equations in Section 4.

2. Wiener Processes and Stochastic Integrals

We define Wiener processes and develop the stochastic integral in a Hilbert space context in this section.

Definition 2.1. Let $w(t)$ be an H-valued random process on $T = [T_1, T_2]$. Then $w(t)$ is a Wiener process if

(i) $E\{w(t) - w(s)\} = 0$ for all s, t in T;

(ii) $w(t)$ is continuous in t wp1;[1]

(iii) $E\{[w(t) - w(s)] \circ [w(t) - w(s)]\}$ [2] $= (t - s)W$ for all s, t in T, where W is a compact, positive, bounded trace class operator mapping H into itself;

(iv) $E\{\| w(t) - w(s)\|^2\} < \infty$ for all s, t in T; and,

(v) $w(t_2) - w(t_1)$ and $w(s_2) - w(s_1)$ are independent for all s_1, s_2, t_1, t_2 in T with $s_1 < s_2 \leqslant t_1 < t_2$.

We note that the operator W has countably many eigenvalues $\{\lambda_i\}$, that $\lambda_i \geqslant 0$ for all i, that $\mathrm{Tr}(W) = \sum_{i=0}^{\infty} \lambda_i$, and that there is a complete orthonormal basis $\{e_i\}$ of H for which $We_i = \lambda e_i$.

Several variants of Definition 2.1 can be obtained by replacing (v) by either of the weaker conditions:

(v)′ $\langle w(t_2) - w(t_1), h_1\rangle$ and $\langle w(s_2) - w(s_1), h_2\rangle$ are independent for all s_1, s_2, t_1, t_2 in T with $s_1 < s_2 \leqslant t_1 < t_2$ and all h_1, h_2 in H;

(v)″ $\langle w(t_2) - w(t_1), e_i\rangle$ and $\langle w(s_2) - w(s_1), e_i\rangle$ are independent for all s_1, s_2, t_1, t_2 in T with $s_1 < s_2 \leqslant t_1 < t_2$.

[1] "*wp*1" is shorthand for "with probability one."

[2] If h_1 and h_2 are elements of H, then $h_1 \circ h_2$ is the element of $\mathscr{L}(H, H)$ given by $(h_1 \circ h_2)h = h_1 \langle h, h_2\rangle$ (cf., [4]).

These variants of the definition lead to essentially identical results (see [2]).

If $w(t)$ is an H-valued Wiener process, then there are complex random processes $\{\beta_i(t)\}$ on T such that

$$w(t) = \sum_{i=0}^{\infty} \beta_i(t) e_i \tag{2.2}$$

almost everywhere in (t, ω), where $\{e_i\}$ is an orthonormal basis of H consisting of eigenvectors of W. Moreover, $\mathrm{Re}\{\beta_i(t)\}$ and $\mathrm{Im}\{\beta_i(t)\}$ are real Wiener processes. From (2.2), it is easy to see that

$$E\{\langle w(t) - w(s), w(t) - w(s)\rangle\} = \mathrm{Tr}(W) \mid t - s \mid \tag{2.3}$$

for t, s in T. We now have

PROPOSITION 2.4. *If $w(t)$ is an H-valued Wiener process, then there is a family $\{\mathscr{F}_t, t \in T\}$ of σ-algebras such that*

(i) $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{P}$ *for* $s < t$;

(ii) $w(t)$ *is measurable relative to* $\mathscr{F}_t$ *for all* t *in* T;

(iii) $w(t) - w(s)$ *is independent of* $\mathscr{F}_s$ *for* $s < t$;

(iv) $[w(t) - w(s)] \circ [w(t) - w(s)]$ *is independent of* $\mathscr{F}_s$ *for* $s < t$.

Proof. Take, for example, $\mathscr{F}_t$ to be the σ-algebra generated by the sets $w(s)^{-1}(\mathscr{O})$, $s \in T$, $s \leqslant t$, $\mathscr{O}$ a Borel set in H. Properties (i)–(iii) are obvious and (iv) is an immediate consequence of the fact that the mapping ψ of $H \oplus H$ into $\mathscr{L}(H, H)$ given by $\psi(h_1, h_2) = h_1 0 h_2$ is continuous [4, Propositions 2.2 and 2.4].

COROLLARY 2.5. $E\{w(t) - w(s) \mid \mathscr{F}_s\} = 0$ wp1 *for* $s < t$,

$$E\{[w(t) - w(s)] \circ [w(t) - w(s)] \mid \mathscr{F}_s\} = (t - s)W \text{ wp1} \qquad \textit{for} \quad s < t,$$

and $\langle [w(t) - w(s)] \circ [w(t) - w(s)]h_1, h_2\rangle$ *is independent of* $\mathscr{F}_s$ *for* $s < t$ *and all* h_1, h_2 *in* H.

If $w(t)$ is an H-valued Wiener process, then, for convenience, we fix a family $\{\mathscr{F}_t\}$ satisfying the conditions of Proposition 2.4 and associate it with $w(t)$. We then have

DEFINITION 2.6. Let K be a Hilbert space. Then $\mathscr{M}(H, K) = \{\Phi(\cdot, \cdot) : \Phi$ is an $\mathscr{L}(H, K)$-valued stochastic process on $T \times \Omega$ such that $\Phi(t)$ is measurable relative to $\mathscr{F}_t$ for all t in $T\}$, $\mathscr{M}_0(H, K) = \{\Phi \in \mathscr{M}(H, K) : \Phi$ is a t

step function on $T\}$, $\mathscr{M}_1(H, K) = \{\Phi \in \mathscr{M}(H, K) : \int_T E\|\Phi(t)\|^2\,dt < \infty\}$, and $\mathscr{M}_2(H, K) = \{\Phi \in \mathscr{M}(H, K) : \int_T \|\Phi(t)\|^2\,dt < \infty$ wp1$\}$.

If Φ is an element of $\mathscr{M}_2(H, K)$, then the K-valued stochastic integral, $\int_T \Phi(t, \omega)\,dw$, can be defined in a similar way to that used in the scalar case by Skorokhod [7]. More precisely, if Φ is an element of $\mathscr{M}_0(H, K) \cap \mathscr{M}_1(H, K)$, then $\int_T \Phi(t, \omega)\,dw$ is given by a finite sum of the form $\sum \Phi(t_j, \omega)\{w(t_{j-1}) - w(t_j)\}$. If Φ is any element of $\mathscr{M}_1(H, K)$, then there is a sequence $\{\Phi_n\}$ of elements of $\mathscr{M}_0(H, K) \cap \mathscr{M}_1(H, K)$ such that $\Phi_n \to \Phi$ almost everywhere on $T \times \Omega$ and

$$\lim_{n\to\infty} \int_T E\{\|\Phi - \Phi_n\|^2\}\,dt = 0. \tag{2.7}$$

Moreover, $\{\int_T \Phi_n(t, \omega)\,dw\}$ has a unique limit in $L_2(\Omega, K)$. This limit is the stochastic integral $\int_T \Phi(t, \omega)\,dw$. Now, if Φ is an element of $\mathscr{M}_2(H, K)$, then there is a sequence $\{\Phi_n\}$ of elements $\mathscr{M}_1(H, K)$ such that $\{\Phi_n\}$ converges to Φ almost everywhere on $T \times \Omega$ and $\{\int_T \Phi_n(t, \omega)\,dw\}$ converges in probability to a K-valued random variable. This random variable is the *stochastic integral*, $\int_T \Phi(t, \omega)\,dw$.

PROPOSITION 2.8. *If Φ is an element of $\mathscr{M}_1(H, K)$, then $E\{\int_T \Phi(t, \omega)\,dw\} = 0$ and*

$$E\left\{\left\| \int_T \Phi(t, \omega)\,dw \right\|^2\right\} \leqslant \mathrm{Tr}(W) \int_T E\{\|\Phi(t)\|^2\}\,dt. \tag{2.9}$$

Proof. See [14].

We now introduce the stochastic differential.

DEFINITION 2.10. Let $u(t)$, $t \in T$, be the K-valued stochastic process given by

$$u(t) - u(T_1) = \int_{T_1}^t q(s, \omega)\,ds + \int_{T_1}^t \Phi(s, \omega)\,dw, \tag{2.11}$$

where Φ is an element of $\mathscr{M}_2(H, K)$ and $q(s, \omega)$ is a K-valued stochastic process with $\int_T \|q(s, \omega)\|\,ds < \infty$ wp1 which is measurable relative to $\mathscr{F}_t$ for all t in T. Then u is said to have the stochastic differential $q\,dt + \Phi\,dw$ and we write $du = q\,dt + \Phi\,dw$.

The following proposition gives this definition some "substance".

PROPOSITION 2.12. *If Φ is an element of $\mathscr{M}_2(H, K)$, then the indefinite stochastic integral $u(t) = \int_{T_1}^t \Phi(s, \omega)\,dw$ is continuous in t on T* wp1.

Proof. We first show that if Φ is in $\mathscr{M}_0(H, K) \cap \mathscr{M}_1(H, K)$, then $\{u(t), \mathscr{F}_t\}$ is a K-valued martingale. Since Φ is a t step function,

$$u(t) = u(s) + \sum_{s \leqslant t_{j-1} < t} \Phi(t_j)[w(t_{j+1}) - w(t_j)] \qquad \text{for} \quad s < t.$$

It follows that $E\{u(t) \mid \mathscr{F}_s\} = u(s)$ for $s < t$ since $u(s)$ is measurable relative to $\mathscr{F}_s$, $\Phi(t_j)[w(t_{j+1}) - w(t_j)]$ is measurable relative to $\mathscr{F}_{t_j}$, and $E\{w(t_{j+1}) - w(t_j) \mid \mathscr{F}_{t_j}\} = 0$.

Now, if Φ is an element of $\mathscr{M}_1(H, K)$, then there is a sequence $\{\Phi_n\}$ of elements of $\mathscr{M}_0(H, K) \cap \mathscr{M}_1(H, K)$ such that Φ_n converges to Φ almost everywhere on $T \times \Omega$ and $\lim_{n\to\infty} \int_T E\{\| \Phi - \Phi_n \|^2\}\, dt = 0$. By virtue of a convergence property of conditional expectations [6], it follows that $\{u(t), \mathscr{F}_t\}$ is a K-valued martingale in this case also.

Finally, if Φ is an element of $\mathscr{M}_2(H, K)$ and $\{\Phi_n\}$ is a sequence of elements of $\mathscr{M}_1(H, K)$ used to define $\int_{T_1}^t \Phi(s, \omega)\, dw$, then $\{\| u(t) - u_n(t)\|, \mathscr{F}_t\}$ is a real semimartingale where $u_n(t) = \int_{T_1}^t \Phi_n(s, \omega)\, dw$ (see [6]). This semimartingale may be viewed as a separable real semimartingale [3].

The proof that $u(t)$ is continuous in t wp1 then follows exactly along the lines of the proof for the scalar case given by Skorokhod [7].[3]

We observe that if u has a stochastic differential, then the real stochastic process $\| u(t)\|$ may be viewed as a separable real process since $u(t)$ is continuous wp1. This observation will be useful in the sequel and enables us to avoid the question of generalizing the notion of separability for a random process to the Hilbert space context.

Several basic properties of the stochastic integral are given in the following propositions whose straightforward proofs are omitted [2].

PROPOSITION 2.13. *If Φ is an element of $\mathscr{M}_2(H, K)$, then $\int_T \Phi\, dw = \sum_{i=0}^{\infty} \int_T \Phi(t, \omega) e_i\, d\beta_i$ wp1, where $w(t)$ is given by 2.2.*

PROPOSITION 2.14. *Let $v_i = \int_T u_i(t)\, d\beta_i$ where u_i is an element of $\mathscr{M}_1(C, K)$. If $\sum_{i=0}^{\infty} \lambda_i \int_T E\{\| u_i \|^2\}\, dt < \infty$, then $v = \sum_{i=0}^{\infty} v_i$ is a well-defined K-valued random variable and $E\{\| v \|^2\} \leqslant \sum_{i=0}^{\infty} \lambda_i \int_T E\{\| u_i \|^2\}\, dt$.*

PROPOSITION 2.15. *If Φ, Φ_n are elements of $\mathscr{M}_2(H, K)$ such that* (i) *Φ_n converges* (*strongly*) *to Φ almost everywhere on $T \times \Omega$, and* (ii) *there is an $\alpha(t)$ in $L_2(T)$ such that $\| \Phi_n(t)\| \leqslant \alpha(t)$ almost everywhere on $T \times \Omega$, then*

$$\lim_{n\to\infty} \int \Phi_n\, dw = \int_T \Phi\, dw \tag{2.16}$$

in probability.

[3] The details can be found in [2].

PROPOSITION 2.17. *If Φ is an element of $\mathcal{M}_2(H, K)$ and B is a (nonrandom) element of $\mathcal{L}(K, G)$, G a Hilbert space, then $B(\int_T \Phi\, dw) = \int_T B\Phi\, dw$* wp1.

In the sequel, we also require the double stochastic integral. This can be defined quite naturally by analogy with the scalar case [7]. In particular, we have

DEFINITION 2.18. Let $\mathcal{N}(H, K) = \{\Psi(\cdot, \cdot, \cdot) : \Psi$ is an $\mathcal{L}(H, K)$-valued stochastic process on $T \times T \times \Omega$ such that $\Psi(t, \cdot)$ and $\Psi(\cdot, t)$ are measurable relative to $\mathcal{F}_t$ for all t in $T\}$,

$$\mathcal{N}_1(H, K) = \left\{\Psi \in \mathcal{N}(H, K) : \int_T \int_T E\{\|\Psi\|^2\}\, ds\, dt < \infty\right\}$$

and

$$\mathcal{N}_2(H, K) = \left\{\Psi \in \mathcal{N}(H, K) : \int_T \int_T \|\Psi\|^2\, ds\, dt < \infty \text{ wp1}\right\}.$$

If Ψ is in $\mathcal{N}_2(H, K)$, then $y_1(\omega) = \int_T (\int_T \Psi(s, t, \omega)\, ds)\, dw(t)$ and $y_2(\omega) = \int_T (\int_T \Psi(s, t, \omega)\, dw(t))\, ds$ are well-defined K-valued random variables with $y_1 = y_2$ wp1. Thus, the stochastic double integral is defined as y_1 or y_2 (see [2] for details). Moreover, we have

PROPOSITION 2.19. *If Ψ is an element of $\mathcal{N}_2(H, K)$ and $c(\cdot)$ is a continuous map of T into K, then*

$$\int_T \int_T \Psi(s, t)\, dw(t)\, ds = \sum_{i=0}^{\infty} \int_T \int_T \Psi(s, t) e_i\, d\beta_i\, ds \qquad \text{wp1} \qquad (2.20)$$

and

$$\int_T \sum_{i=0}^{\infty} \int_{T_1}^{s} \langle \Psi(s, t) e_i, c(s)\rangle\, d\beta_i\, ds$$

$$= \sum_{i=0}^{\infty} \int_T \int_{T_1}^{s} \langle \Psi(s, t) e_i, c(s)\rangle\, d\beta_i\, ds \qquad \text{wp1}$$

where $w(t)$ is given by 2.2.

We conclude this section with two technical lemmas which are used in the proof of the main theorem.

LEMMA 2.22. *Let A be a closed linear map of K into K and let $\mathcal{D}(A)$ be the domain of A. If $q(\cdot)$ is an element of $\mathcal{M}_2(C, K)$ such that $q(s) \in \mathcal{D}(A)$ for all s in T and $Aq(\cdot) \in \mathcal{M}_2(C, K)$, then $\int_T q\, db \in \mathcal{D}(A)$* wp1 *and $A \int_T q\, db = \int_T Aq\, db$* wp1 *where b is a scalar Wiener process.*

Proof. The proof is a simple modification of the proof of Theorem 20, p. 153, of [15] and is, therefore, omitted.

LEMMA 2.23. *Let A be a closed linear map of K into K and let $w(t)$ be an H-valued Wiener process with representation* 2.2. *Suppose that* (i) $\Phi \in \mathscr{M}_2(H, K)$ *such that* $\Phi(t)e_i \in \mathscr{D}(A)$ wp1 *for all i and t and* $A\Phi(\cdot)e_i \in \mathscr{M}_2(C, K)$ *for all i, and* (ii) $\sum_{i=0}^{\infty} \lambda_i \int_T \| A\Phi(t)e_i \|^2 \, dt < \infty$ wp1. *Then* $\int_T \Phi \, dw \in \mathscr{D}(A)$ wp1, $A \int_T \Phi \, dw = \sum_{i=0}^{\infty} \int_T A\Phi e_i \, d\beta_i$ wp1, *and* $A \int_{T_1}^{t} \Phi(s) \, dw$ *is continuous in t* wp1.

Proof. A complete proof is given in [2]. Here, to indicate the ideas involved, we treat the special case where $\Phi \in \mathscr{M}_1(H, K)$, $A\Phi e_i \in \mathscr{M}_1(C, K)$, and $\sum_{i=0}^{\infty} \lambda_i \int_T E\{\| A\Phi(t)e_i \|^2\} \, dt < \infty$.[4]

In view of Lemma 2.22, we have

$$A \sum_{i=0}^{n} \int_T \Phi(t)e_i \, d\beta_i = \sum_{i=0}^{n} \int_T A\Phi(t)e_i \, d\beta_i \,. \tag{2.24}$$

Now, it follows from Proposition 2.13 that $\sum_{i=0}^{\infty} \int_T \Phi(t)e_i \, d\beta_i$ tends to $\int_T \Phi \, dw$ wp1 as n approaches infinity. Since A is a closed linear operator, $\int_T \Phi \, dw \in \mathscr{D}(A)$ wp1. Moreover, $\sum_{i=0}^{\infty} \int_T A\Phi(t)e_i \, d\beta_i$ is well-defined by virtue of Proposition 2.14 and so, $A \int_T \Phi \, dw = \sum_{i=0}^{\infty} \int_T A\Phi e_i \, d\beta_i$.

As for continuity, we have

$$E\left\{\left\| A \int_{T_1}^{t} \Phi \, dw - \sum_{i=0}^{n} \int_{T_1}^{t} A\Phi(s)e_i \, d\beta_i \right\|^2\right\} \leqslant \sum_{n+1}^{\infty} \lambda_i E\left\{\int_T \| A\Phi(s)e_i \|^2 \, ds\right\} \tag{2.25}$$

by Proposition 2.14. It follows that

$$\mu\left\{\omega : \sup_t \left\| A \int_{T_1}^{t} \Phi \, dw - \sum_{i=0}^{n} \int_{T_1}^{t} A\Phi(s)e_i \, d\beta_i \right\| \neq 0\right\} \to 0 \qquad \text{as} \quad n \to \infty.$$

(Note that the indicated probability exists since a separable version of the norm expression can be used.) The result then is an immediate consequence of Proposition 2.12.

[4] The general case is readily deduced from the special case by use of the function $\chi^N(\omega)$ given by

$$\chi^N(\omega) = \begin{cases} 1 \text{ if } \int_T \| \Phi \|^2 \, dt \leqslant N & \text{and} \quad \sum_{i=0}^{\infty} \lambda_i \int_T \| A\Phi(t)e_i \|^2 \, dt \leqslant N \\ 0 \text{ otherwise} \end{cases}$$

3. Existence and Uniqueness Theorems

Let H be a separable Hilbert space and let K be a Hilbert space. We now turn our attention to the stochastic evolution equation

$$u(t) - u_0 + \int_{T_1}^{t} A(s)u(s)\,ds = \int_{T_1}^{t} \Phi(s)\,dw, \tag{3.1}$$

where $A(s)$ is a closed linear operator on K, $\Phi(\ )$ is an element of $\mathscr{M}_2(H, K)$, $w(\cdot)$ is an H-valued Wiener process, and u_0 is a K-valued random variable. We may write (3.1) in differential form as $du + A(t)\,u(t)\,dt = \Phi\,dw$, $u(T_1) = u_0$.

Definition 3.2. A K-valued random process $u(t)$ is a solution of (3.1) if (i) $u(t)$ satisfies (3.1) wp1 on T, (ii) $u(t)$ is measurable relative to $\mathscr{F}_t$ for all t, and (iii) $u(t)$ is continuous wp1 in t. Two solutions $u(t)$ and $\tilde{u}(t)$ of (3.1) are the "same" if $u(t)$ and $\tilde{u}(t)$ are uniformly stochastically equivalent (i.e., $\mu\{\omega : \sup_t \| u(t) - \tilde{u}(t)\| = 0\} = 1$).

Our existence theorems for (3.1) are all based on existence theorems for the corresponding abstract evolution equation

$$\dot{v} + A(t)v = 0, \qquad v(T_1) = v_0. \tag{3.3}$$

Existence and uniqueness theorems for (3.3) are given in [8–10, 12, and 13]. The basic idea of these theorems is to impose sufficiently strong conditions on $A(t)$ in order to insure the existence of an evolution operator $U(t, s)$. The solution of (3.3) is then obtained by a "variation of constants" formula. We essentially use the same idea here.

If the abstract evolution Eq. (3.3) has a *unique* solution, then it is easy to prove using the linearity of (3.1) that solutions of (3.1) are unique to within a uniform stochastic equivalence (i.e., are the "same"). Thus, we concentrate on establishing the existence of solutions of (3.1).

Theorem 3.4. *Suppose that* (i) $A(t) \equiv A$ *where* $-A$ *is the infinitesimal generator of a strongly continuous semigroup* $U(t)$; (ii) $w(t)$ *is an* H*-valued Wiener process with* (2.2) *as representation and* $\{\lambda_i\}$ *as associated eigenvalues*; (iii) $\Phi(\cdot)$ *is an element of* $\mathscr{M}_2(H, K)$ *with* $\Phi(t)e_i \in \mathscr{D}(A)$ wp1 *for all t and i and with* $A\Phi(\cdot)e_i \in \mathscr{M}_2(H, K)$ *for all i*; (iv) $\sup_{i,t} \| A\Phi(t)e_i \| \leqslant C$ wp1 *and* (v) $\Phi(\cdot)$ *is uniformly bounded on* T wp1. *Then* (3.1) *has the solution* $u(t) = U(t - T_1)u_0 + \int_{T_1}^{t} U(t - s)\,\Phi(s)\,dw$ *for all* u_0 *such that* $u_0 \in \mathscr{D}(A)$ wp1. *Moreover, the solution is unique to within a uniform stochastic equivalence.*

Theorem 3.4 is based on the standard semigroup result for (3.3) [8]. The following theorem is based on a perturbation result [13].

THEOREM 3.5. *Suppose that* (i) $A(t) = A_0 + B(t)$ *where* A_0 *is the infinitesimal generator of a strongly continuous semigroup,* $B(t) \in \mathscr{L}(K, K)$ *for all* t, *and* $B(\cdot)$ *is strongly continuously differentiable on* T; (ii) $w(t)$ *is an* H*-valued Wiener process with* (2.2) *as representation and* $\{\lambda_i\}$ *as associated eigenvalues;* (iii) $\Phi(\cdot)$ *is an element of* $\mathscr{M}_2(H, K)$ *with* $\Phi(t)e_i \in \mathscr{D}(A_0)$ wp1 *for all* t *and* i *and* $A_0\Phi(\cdot)e_i \in \mathscr{M}_2(H, K)$ *for all* i; (iv) $\sup_{i,t} \| A_0\Phi(t)e_i \| \leqslant C$ wp1 *and* (v) $\Phi(\cdot)$ *is uniformly bounded on* T wp1. *Then* (3.1) *has the solution* $u(t) = U(t, T_1)u_0 + \int_{T_1}^t U(t, s)\, \Phi(s)\, dw$ *for all* u_0 *with* $u_0 \in \mathscr{D}(A_0)$ wp1, *where* $U(t, s)$ *is the evolution operator generated by* $-A(t)$. *Moreover, the solution is unique to within a uniform stochastic equivalence.*

The following general theorem which we shall prove[5] is based on the general existence theorem for (3.3) [9, 10].

THEOREM 3.6. *Suppose that* (i) $-A(t)$ *generates an evolution operator* $U(t, s)$ *with the following properties* (a) $U(t, s)$ *is linear, bounded and strongly continuous in* s *and* t *for* $T_1 \leqslant s \leqslant t \leqslant T_2$, (b) $U(t, t) = I$ *and* $U(t, s) = U(t, r)\, U(r, s)$ *for* $T_1 \leqslant s \leqslant r \leqslant t \leqslant T_2$, (c) $U(t, s)$ *maps* $\mathscr{D}(A(s))$ *into* $\mathscr{D}(A(t))$ *for* $s \leqslant t$, (d) $U(t, s)$ *is strongly continuously differentiable in* t *for* $t > s$ *and* $\partial U/\partial t\, (t, s) + A(t)\, U(t, s) = 0$ *and*

$$\| A(t)U(t, s)\| \leqslant \frac{\eta_1}{| t - s |} \tag{3.7}$$

for $T_1 \leqslant s < t \leqslant T_2$ *where* η_1 *is a constant independent of* s *and* t, *and* (e) $U(t, s) = \exp(-(t - s)\, A(t)) + W(t, s)$ *for* $T_1 \leqslant s < t \leqslant T_2$ *where* $W(t, s) \in \mathscr{L}(K, K)$ *is strongly continuous in* t *and* $\exp(-\tau A(t))$ *is the analytic semigroup generated by* $-A(t)$ *and*

$$\| \exp(-(t - s)\, A(t))\| \leqslant \eta_2 , \tag{3.8}$$

$$\| A(t)\, W(t, s)\| \leqslant \eta_3/| t - s |^{\theta}, \tag{3.9}$$

where η_2, η_3 *and* θ *are constants* (*independent of* s *and* t) *with* $0 \leqslant \theta < 1/2$;[6] (ii) $w(t)$ *is an* H*-valued Wiener process with* (2.2) *as associated representation and* $\{\lambda_i\}$ *as associated eigenvalues*; (iii) $\Phi(\cdot)$ *is an element of* $\mathscr{M}_2(H, K)$ *with* $\Phi(s)e_i \in \mathscr{D}(A(t))$ wp1 *for all* $s \leqslant t$ *and all* i *and* $A(t)\, \Phi(\cdot)e_i \in \mathscr{M}_2(C, K)$ *for all* t *and* i; (iv) $\sup_{i, s \leqslant t < T}\{\| A(t)\, \Phi(s)e_i \|\} \leqslant C$ wp1; *and* (v) $\Phi(t)$ *is uniformly bounded in norm in* t wp1. *Then* (3.1) *has the solution*

$$u(t) = U(t, T_1)u_0 + \int_{T_1}^{t} U(t, s)\Phi(s)\, dw \qquad \textit{for all} \quad u_0 .$$

Moreover, the solution is unique to within a uniform stochastic equivalence.

[5] The proofs of Theorems 3.4 and 3.5 are entirely analogous and are therefore omitted.

[6] See [9 or 10] for conditions on $-A(t)$ which insure the existence of $U(t, s)$.

Before proving this theorem, we note that if we assume only that $\Phi(\cdot) \in \mathscr{M}_2(H, K)$ and $-A(t)$ generates an evolution operator, then we can prove an existence theorem for "weak" solutions of (3.1) based on the notion of "weak" solutions of (3.3) introduced in [10]. We now have

Proof of Theorem 3.6. Since (3.3) has a unique solution under the assumption (i), we may assume without loss of generality that $u_0 \equiv 0$ wp1.

We remark that assumptions (iii) and (iv) imply that $A(t)\, \Phi(\cdot)e_i \in \mathscr{M}_1(C, K)$ for all t and all i.

We begin by establishing the following:

$$U_0(t, \cdot)\, \Phi(\cdot) \in \mathscr{M}_1(H, K), \tag{3.10}$$

where

$$U_0(t, s) = \begin{cases} U(t, s) & s \leqslant t \\ 0 & s > t, \end{cases}$$

$$\sup_i \left\{ \int_{T_1}^{t} \| A(t)U(t, s)\Phi(s)e_i \|^2 \, ds \right\} < \infty \qquad \text{wp1} \tag{3.11}$$

uniformly in t, and

$$A(t)\, U_0(t, \cdot)\, \Phi(\cdot)e_i \in \mathscr{M}_1(C, K) \qquad \text{for all } t \text{ and } i. \tag{3.12}$$

Since $U(t, s)$ is strongly continuous in s for $s < t$, $\| U(t, \cdot)\|$ is measurable and bounded on $[T_1, t)$. Since $\Phi(\cdot) \in \mathscr{M}_1(H, K)$,

$$\int_T \| U_0(t, s)\Phi(s)\|^2 \, ds \leqslant \int_{T_1}^{t} \| U(t, s)\|^2 \, \| \Phi(s)\|^2 \, ds < \infty \qquad \text{wp1}.$$

In view of (i)(e), the properties of analytic semigroups and (iii), we have $A(t)\, U(t, s)\, \Phi(s)e_i = \exp(-(t - s)\, A(t))\, A(t)\, \Phi(s)e_i + A(t)\, W(t, s)\, \Phi(s)e_i$ wp1. It follows that

$$\| A(t)U(t, s)\Phi(s)e_i \|^2 \leqslant 2\eta_2{}^2 \| A(t)\Phi(s)e_i \|^2 + \frac{2\eta_3{}^2\alpha^2}{| t - s |^{2\theta}} \qquad \text{wp1}$$

(where $\| \Phi(s)\| \leqslant \alpha$ wp1 by (v)). Integrating this inequality and using the assumption (iv), we deduce that (3.11) holds.

Since $A(t)\, U_0(t, \cdot)$ is nonrandom, $A(t)\, U_0(t, s)\, \Phi(s)e_i$ is measurable relative to $\mathscr{F}_s$ and so, (3.12) follows from (3.11).

Let $\tau > 0$ be fixed. Since $U_0(t, \cdot)\, \Phi(\cdot) \in \mathscr{M}_1(H, K)$ by (3.10), we may define $u_\tau(t)$ by setting

$$u_\tau(t) = \begin{cases} \displaystyle\int_{T_1}^{t-\tau} U(t, s)\Phi(s)\, dw & t \geqslant T_1 + \tau \\ 0 & t < T_1 + \tau. \end{cases} \tag{3.13}$$

We claim that

$$A(t)u_\tau(t) = \begin{cases} \int_{T_1}^{t-\tau} A(t)U(t, s)\Phi(s)\, dw & t \geqslant T_1 + \tau \\ 0 & t < T_1 + \tau. \end{cases} \tag{3.14}$$

The claim is trivial if $t < T_1 + \tau$. So let us suppose that $t \geqslant T_1 + \tau$ and let us set $T^* = [T_1, t - \tau]$, $\Phi^*(\cdot) = U_0(t, \cdot)\,\Phi(\cdot)$ and $A^* = A(t)$. In view of (3.10)–(3.12) and the assumptions (i)(c), (iii) and (iv), we see that the conditions of Lemma 2.23 are satisfied by A^*, $\Phi^*(\cdot)$ and T^*. It follows that $\int_{T^*} \Phi^*(s)\, dw \in \mathscr{D}(A^*)$ wp1 and that

$$A^* \int_{T^*} A^*(s)\, dw = \sum_{i=0}^{\infty} \int_{T_1}^{t-\tau} A(t)U(t, s)\Phi(s)e_i\, d\beta_i\,.$$

In other words, $u_\tau(t) \in \mathscr{D}(A(t))$ wp1 and

$$A(t)u_\tau(t) = \sum_{i=0}^{\infty} \int_{T_1}^{t-\tau} A(t)U(t, s)\Phi(s)e_i\, d\beta_i \qquad \text{wp1}. \tag{3.15}$$

Since $\tau > 0$, we have

$$\int_{T_1}^{t-\tau} \| A(t)U(t, s)\Phi(s)\|^2\, ds \leqslant \frac{\eta_1{}^2\alpha^2}{\tau^2}(T_2 - T_1) \qquad \text{wp1} \tag{3.16}$$

by virtue of (3.7). Proposition 2.13 then allows us to conclude that (3.14) holds.

We shall now show that $A(s)\,u_\tau(s)$ is integrable on T for fixed τ. Let t be an element of $[T_1 + \tau, T_2)$ and let $\Psi_t(s, r)$ be given by

$$\Psi_t(s, r) = \begin{cases} A(s)U(s, r)\Phi(r) & \text{for } \begin{matrix} T_1 + \tau \leqslant s \leqslant t \\ T_1 \leqslant r \leqslant s - \tau \end{matrix} \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

Then $\Psi_t(s, r)$ is measurable in (s, r) and is measurable relative to $\mathscr{F}_s$ and $\mathscr{F}_r$ since $A(s)\,U(s, r)$ is nonrandom, and $\Phi(\cdot)$ is in $\mathscr{M}_2(H, K)$. But

$$\begin{aligned} \int_T \int_T \| \Psi_t(s, r)\|^2\, ds\, dr &= \int_{T_1+\tau}^{t} \left(\int_{T_1}^{s-\tau} \| A(s)U(s, r)\Phi(r)\|^2\, dr \right) ds \\ &\leqslant \eta_1{}^2\alpha^2(T_2 - T_1)^2/\tau^2 \qquad \text{wp1} \end{aligned} \tag{3.18}$$

by virtue of (3.7) and assumption (v). In other words, $\Psi_t(\cdot, \cdot) \in \mathscr{N}_2(H, K)$ and so,

$$\int_T \left(\int_T \Psi_t(s, r)\, ds \right) dw = \int_T \left(\int_T \Psi_t(s, r)\, dw \right) ds \qquad \text{wp1}. \tag{3.19}$$

It follows from (3.14) that

$$\int_T \left(\int_T \Psi_t(s, r)\, dw\right) ds = \int_{T_1}^{t} A(s) u_\tau(s)\, ds \qquad \text{wp1} \tag{3.20}$$

and hence, that $A(s)\, u_\tau(s)$ is integrable on $[T_1, t]$ for any t and fixed $\tau > 0$.

We now assert that $\int_{T_1}^t A(s)\, u_\tau(s)\, ds \to u(t) + \int_{T_1}^t \Phi(s)\, dw$ in probability as $\tau \to 0$. Using (3.19) and (3.20), we deduce that $\int_{T_1}^t A(s)\, u_\tau(s)\, ds = -\int_{t-\tau}^t U(r + \tau, r)\, \Phi(r)\, dw + \int_{T_1}^t U(r + \tau, r)\, \Phi(r)\, dw - \int_{T_1}^{t-\tau} U(t, r)\, \Phi(r)\, dw$. Now $\int_{T_1}^{t-\tau} U(t, r)\, \Phi(r)\, dw \to \int_{T_1}^t U(t, r)\, \Phi(r)\, dw$ wp1 as $\tau \to 0$ by Proposition 2.12. Since

$$\begin{aligned} E\left\{\left\| \int_{t-\tau}^{t} U(r+\tau, r)\Phi(r)\, dw \right\|^2\right\} &\leqslant \operatorname{Tr}(W) \int_{t-\tau}^{t} E\{\| U(r+\tau, r)\|^2 \| \Phi(r)\|^2\}\, dr \\ &\leqslant \operatorname{Tr}(W) c\alpha^2\tau \qquad \text{for some} \quad c > 0 \end{aligned}$$

(by (v)), $\int_{t-\tau}^t U(r + \tau, r)\, \Phi(r)\, dw \to 0$ in probability as $\tau \to 0$. Thus, to complete verification of the assertion, we show that

$$\int_{T_1}^{t} U(r+\tau, r)\Phi(r)\, dw \to \int_{T_1}^{t} \Phi(r)\, dw$$

in probability as $\tau \to 0$. Let $\Phi_n(r)$ be given by

$$\Phi_n(r) = \begin{cases} U(r + 1/n, r)\Phi(r) & T_1 \leqslant r \leqslant t - 1/n \\ 0 & r > t - 1/n. \end{cases} \tag{3.21}$$

Then $\Phi_n(r)$ converges to $\Phi(r)$ strongly and $\| \Phi_n(r)\| \leqslant c_1 \| \Phi(r)\|$ for some $c_1 > 0$ (by (i)(a)). Thus, Proposition 2.15 applies and we have shown that

$$\lim_{t \to 0} \int_{T_1}^{t} A(s) u_\tau(s)\, ds = -u(t) + \int_{T_1}^{t} \Phi(s)\, dw \tag{3.22}$$

in probability.

We show that $A(t)\, u(t)$ is continuous in expectation on T. Let $v_\tau(t) = A(t)\, u_\tau(t)$ with fixed $\tau > 0$. From 3.14, we deduce that

$$\begin{aligned} v_\tau(t+h) - v_\tau(t) &= \int_{t-\tau}^{t+h-\tau} A(t+h) U(t+h, s)\Phi(s)\, dw \\ &\quad + \int_{T_1}^{t-\tau} \{A(t+h)U(t+h, s) - A(t)U(t, s)\}\Phi(s)\, dw \end{aligned} \tag{3.25}$$

for $t \geqslant T_1 + \tau$. But

$$E\left\{\left\| \int_{t-\tau}^{t+h-\tau} A(t+h)U(t+h, s)\Phi(s)\, dw \right\|^2\right\} \leqslant \operatorname{Tr}(W) \frac{\eta_1{}^2\alpha^2 h}{\tau^2},$$

so that

$$\lim_{h\to 0} E\left\{\left\| \int_{t-\tau}^{t+h-\tau} A(t+h)U(t+h, s)\Phi(s)\, dw \right\|^2\right\} = 0. \tag{3.26}$$

Since $U(t, s)$ is an evolution operator,

$$\int_{T_1}^{t-\tau} \{A(t+h)U(t+h, s) - A(t)U(t, s)\}\Phi(s)\, dw$$
$$= \{A(t+h)U(t+h, t) - A(t)\}u_\tau(t) \qquad \text{wp1}.$$

In view of the strong continuity of $A(t)\, U(t, s)$ in t for $t > s$, we see that

$$\lim_{h\to 0} E\left\{\left\| \int_{T_1}^{t-\tau} \{A(t+h)U(t+h, s) - A\ t)U(t, s)\}\Phi(s)\, dw \right\|^2\right\} = 0. \tag{3.27}$$

As $v_\tau(t) = 0$ for $t < T_1 + \tau$, we conclude that

$$\lim_{h\to 0} E\{\| v_\tau(t+h) - v_\tau(t)\|^2\} = 0 \qquad \text{for all} \quad t \text{ in } T.$$

In other words, $v_\tau(t)$ is continuous in expectation. Thus, to show that $A(t)\, u(t)$ is continuous in expectation on T it will be sufficient to show that

$$\lim_{\tau\to 0} \sup_{t\in T} E\{\| A(t)u(t) - v \quad)\|^2\} = 0. \tag{3.28}$$

Suppose first that $t \in [T_1 + \tau, T_2]$. Now

$$E\{\| A(t)u(t) - A(t)u_\tau(t)\|^2\}$$
$$= E\left\{\left\| \sum_{i=0}^{\infty} \int_{t-\tau}^{t} A(t)U(t, s)\Phi(s)e_i\, d\beta_i(s) \right\|^2\right\} \qquad \text{by (2.13)}$$
$$\leqslant \sum_{i=0}^{\infty} \lambda_i E\left\{\int_{t-\tau}^{t} \| A(t)U(t, s)\Phi(s)e_i\|^2\, ds\right\} \qquad \text{by (2.14)}.$$

Since $A(t)U_0(t, \cdot)\Phi(\cdot) \in \mathscr{M}_1(C, K)$, the right hand side is

$$\leqslant \sum_{i=0}^{\infty} \lambda_i E\left(\int_{t-\tau}^{t} (2\eta_2{}^2\| A(t)\Phi(s)e_i\|^2 + \frac{2\eta_3{}^2\alpha^2}{| t-s |^\theta}\right) ds$$

by assumption (i)(e), the properties of analytic semigroups and assumption (iii);

$$\leqslant \operatorname{Tr} W \left(2\eta_2{}^2 C^2 \tau + \frac{2\eta_3{}^2\alpha^2}{(1-\theta)} \tau^{(1-\theta)}\right) \text{ using assumption (iv)}$$

$$\to 0 \text{ as } \tau \to 0 \text{ uniformly in } i \text{ and in } t \text{ on } T.$$

If now $t \in [T_1, T_1 + \tau]$, then $u_\tau(t) \equiv 0$ and so

$$E\{\| A(t)u(t) - A(t)u_\tau(t)\|^2\}$$

$$\leqslant \sum_{i=0}^{\infty} \lambda_i E \left\{\int_{T_1}^{T_1+\tau} \| A(\tau)U(t, s)\Phi(s)e_i \|^2 \, ds\right\}$$

$$\to 0 \text{ as } \tau \to 0 \text{ uniformly in } i \text{ and in } t \text{ on } T$$

as before, thus establishing (3.28). So

$$\int_{T_1}^{t} A(s)u_\tau(s)\, ds \to \int_{T_1}^{t} A(s)u(s)\, ds \text{ in expectation as } \tau \to 0 \text{ and}$$

$$\int_{T_1}^{t} A(s)u(s)\, ds \text{ is continuous in } t \text{ on } T \qquad \text{wp1}. \tag{3.29}$$

Equation (3.29) combined with (3.22) shows that $u(t)$ satisfies the equation wp1 and the continuity of $u(t)$ follows from the continuity of

$$\int_{T_1}^{t} \Phi(s)\, dw - \int_{T_1}^{t} A(s)u(s)\, ds$$

(from 3.29 and 2.12).

Hence $u(t)$ is a solution of (3.1) with $u_0 = 0$ wp1.

4. Stochastic Parabolic Partial Differential Equations

As an application of the results of Section 3, we consider the class of stochastic parabolic partial differential equations

$$du + A(t)\, u(t)\, dt = \Phi(t)\, dw, \qquad u(T_1) = u_0 \tag{4.1}$$

where $u_0 \in H = L_2(G)$, G a domain with smooth boundary, $w(t)$ is an H-valued Wiener process, $\Phi(\cdot) \in \mathscr{M}_2(L_2(G), L_2(G))$, $-A(t)$ is an elliptic operator on $L_2(G)$, and the conditions of Theorem 3.6 are satisfied.

In order to give specific examples of (4.1), we introduce the necessary machinery from the theory of parabolic partial differential equations. We let G be a bounded domain in E_n whose boundary, ∂G, is an $n-1$-dimensional variety which is locally C^∞. If $k = (k_1, ..., k_n)$ is a multiindex, then $|k| = k_1 + \cdots + k_n$ and $D^k = \partial^{|k|}/(\partial x_1^{k_1} \cdots \partial x_n^{k_n})$ (in the generalized sense). Letting $H = L_2(G)$, we define the spaces

$$H^m(G) = \{v \in H : D^k v \in H \text{ for } |k| \leqslant m\}; \tag{4.2}$$

$$H_0^m(G) = \overline{C_0^\infty(G)}\ ^7$$

$\Big($in the $H^m(G)$ topology which is generated by the inner product

$$\langle u, v\rangle_m = \sum_{|k|,|l| \leqslant m} \int_G D^k u\, \overline{D^l v}\, dx\Big); \tag{4.3}$$

$$H^0(\partial G) = L_2(\partial G); \tag{4.4}$$

and,

$$H^m(\partial G) = \{v \in L_2(\partial G) : D^k v \in L_2(\partial G), |k| \leqslant m\}. \tag{4.5}$$

We note that the dual spaces $H^\rho(\partial G)$ and $H^{-\rho}(\partial G)$ may be defined for $0 < \rho < m$. We also have

DEFINITION 4.6. Suppose that V and H are Hilbert spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_H$, respectively, and that $V \subset H$ both algebraically and topologically. Then a bilinear form $a(\cdot, \cdot)$ on $V \times V$ is coercive over a subspace V_0 of V if there are constants, $\eta_0 \geqslant 0$ and $\eta_1 > 0$ such that

$$\mathrm{Re}\{a(v, v)\} \geqslant \eta_1 \|v\|_V^2 - \eta_0 \|v\|_H^2 \tag{4.7}$$

for all v in V_0.

We now introduce the bilinear forms $a(t; u, v)$ and $b(t; u, v)$ on $H^m(G)$ given by

$$a(t; u, v) = \sum_{|k|,|l| \leqslant m} \int_G a_{kl}(t, x)\, D^k u\, \overline{D^l v}\, dx, \tag{4.8}$$

$$b(t; u, v) = a(t; u, v) + \sum_{j=0}^{n-1} \left(N_j(t)u, \overline{\frac{d^j v}{dv^j}}\right), \tag{4.9}$$

where $Da_{kl}(t, x) \in L_\infty(T \times G)$ and are Holder continuous in t uniformly

[7] $C_0^\infty(G)$ is the space of infinitely differentiable functions with compact support on G.

on G with exponent $>2/3$, $N_j(t)$ is a bounded linear map of $H^m(G)$ into $H^{-(m-j-1/2)}(\partial G)$, $(\cdot, \cdot)$ denotes the inner product in $H^{-(m-j-1/2)}(\partial G)$, $d^j/d\nu^j$ is a normal derivative (in a generalized sense), and the functions $t \to (N_j(t)u, v)$ are measurable with $|(N_j(t)u, v) - (N_j(s)u, v)| \leq c_j \, | t - s |^\alpha \| u \|_{H^m} \| v \|_{H^{m-j-1/2}}$ for $u \in H^m(G)$, $v \in H^{m-j-1/2}(\partial G)$, where c_j, $\alpha > 0$ are constants independent of t and s. These bilinear forms generate the differential operators $A_i(t)$, $i = 1, 2, 3$, given by

$$A_i(t)u = \sum_{|k|,|l| \leq m} (-1)^{|k|} D^k(a_{k_l}(t, x) \, D^l u), \; u \in \mathscr{D}(A_i(t)), \tag{4.10}$$

where

$$\mathscr{D}(A_1(t)) = \{u \in H_0{}^m(G) : A_1(t)u \in H, \; D^k u = 0 \text{ on } T \times \partial G \quad \text{for } | k | \leq m - 1\},$$

$$\mathscr{D}(A_2(t)) = \{u \in H^m(G) : A_2(t)u \in H, \; B_j(t)u = 0 \text{ on } T \times \partial G \quad \text{for } j = 0,..., m - 1\},$$

$$\mathscr{D}(A_3(t)) = \{u \in H^m(G) : A_3(t)u \in H, \; B_j(t)u = N_j(t)u \text{ on } T \times \partial G \quad \text{for } j = 0,..., m - 1\},$$

and $B_j(t)$ are the given differential operators. If we assume that $a(t; \cdot, \cdot)$ is coercive over $H_0{}^m(G)$ for all t, then $-A_1(t)$ generates an evolution operator having properties (a)–(c) of Theorem 3.6. Similarly, if we assume that $a(t; \cdot, \cdot)$ is coercive over $H^m(G)$ or that $b(t; \cdot, \cdot)$ is coercive over $H^m(G)$, then $-A_2(t)$ or $-A_3(t)$ generates an evolution operator having properties (a)–(c) of Theorem 3.6. (See [2] for details.) We note that under these conditions the partial differential equations $\dot{u} + A_i(t)u = f(t)$, $u(T_1) = u_0$ have unique solutions for f in $L_2(T; H)$.

Our general example now has the form: $du + A(t) \, u(t)dt = \Phi(t) \, dw$, $u(T_1) = u_0 \in H$ wp1 where $A(t) = A_i(t)$, $i = 1$, 2 or 3, $w(t) = \sum_{i=0}^\infty \beta_i(t)e_i$ with $e_i(x)$ an orthonormal basis of $H = L_2(G)$, and $\Phi(t)$ satisfies the conditions (i) $\Phi(t)$ is measurable in (t, ω) and is measurable relative to $\mathscr{F}_t$ for all t, (ii) $\sup_t \| \Phi(t) \| \leq \alpha$ wp1 for some $\alpha > 0$, (iii) $\Phi(s)e_i \in \mathscr{D}(A(t))$ wp1 for $s \leq t$, and (iv) $\sup_{i, s \leq t} \| A(t) \, \Phi(s)e_i \| \leq C$ wp1 for some $C > 0$. This general example satisfies the conditions of Theorem 3.6 and so has the "unique" solution

$$u(t) = U(t, T_1)u_0 + \int_{T_1}^t U(t, s)\Phi(s) \, dw$$

$$= U(t, T_1)u_0 + \sum_{k=0}^\infty \int_T^t U(t, s)\Phi(s)e_k \, d\beta_k ,$$

where $U(t, s)$ is the evolution operator generated by $-A(t)$. Various specific cases are given in [2].

APPENDIX: Infinite Dimensional Random Variables

We collect some of the standard definitions and results of the theory of Banach space-valued random variables in this appendix as a convenience for the reader. The treatment is along the lines of that given by Scalora [6].

Let $(\Omega, \mathscr{P}, \mu)$ be a probability space with $\mathscr{P}$ as Borel field and μ as measure. We assume that μ is complete. Also let X be a Banach space. We then have

Definition A.1. A strongly measurable mapping $x(\cdot)$ of Ω into X is called a random variable.

A random variable $x(\cdot)$ is *integrable* on Ω if and only if there is a sequence $\{x_n(\cdot)\}$ of finitely valued random variables such that (i) $x_n(\cdot)$ converges to $x(\cdot)$ almost everywhere, and (ii) $\lim_{m,n\to\infty} \int_\Omega \| x_n(\omega) - x_m(\omega)\| \, d\mu = 0$.

Definition A.2. If $x(\cdot)$ is integrable on Ω, then the expectation of x, $E\{x\}$, is the element of X given by

$$E\{x\} = \int_\Omega x(\omega) \, d\mu = \lim_{n\to\infty} \int_\Omega x_n(\omega) \, d\mu. \tag{A.3}$$

Definition A.4. Let $\mathscr{F}$ be a Borel field with $\mathscr{F} \subset \mathscr{P}$ and let $x(\cdot)$ be integrable on Ω. The conditional expectation of x relative to $\mathscr{F}$, $E\{x \mid \mathscr{F}\}$, is a random variable such that

$$\int_F x(\omega) \, d\mu = \int_F E\{x \mid \mathscr{F}\}(\omega) \, d\mu \tag{A.5}$$

for all F in $\mathscr{F}$.

We note that $E\{x \mid \mathscr{F}\}$ is unique wp1, is integrable on Ω, and is measurable relative to $\mathscr{F}$.

Definition A.6. Let $T = [T_1, T_2]$ be a finite interval. A mapping $x(t, \omega)$ of $T \times \Omega$ into X is called a stochastic process on T if $x(\cdot, \cdot)$ is measurable in the pair (t, ω) (using Lebesgue measure on T).

Definition A.6 is more restrictive than the usual one (cf., Doob [3]) but is adequate for our purposes. Also, we usually write $x(t)$ in place of $x(t, \omega)$ when discussing stochastic processes.

Definition A.7. Two measurable sets F_1 and F_2 in P are independent if $\mu(F_1 \cap F_2) = \mu(F_1)\, \mu(F_2)$. If $x(\cdot)$ is a random variable mapping Ω into X

and $y(\cdot)$ is a random variable mapping Ω into Y, then $x(\cdot)$ and $y(\cdot)$ are independent if the sets $\{\omega : x(\omega) \in A\}$ and $\{\omega : y(\omega) \in B\}$ are independent for all Borel sets A of X and all Borel sets B of Y. Finally, a random variable $x(\cdot)$ is independent of the Borel field $\mathscr{F} \subset \mathscr{P}$, if the sets F and $\{\omega : x(\omega) \in A\}$ are independent for all F in $\mathscr{F}$ and all Borel sets A of X.

The following propositions contain various results needed in the paper. These propositions are easy extensions of similar results for the ordinary case and are proven in detail in [2].

PROPOSITION A.8. *If $x(\cdot)$ and $y(\cdot)$ are independent X and Y valued random variables, respectively, and if f and g are nonrandom Baire functions mapping X and Y, respectively, into the complex numbers C, then $f(x(\cdot))$ and $g(y(\cdot))$ are independent random variables.*

PROPOSITION A.9. *Let $\mathscr{F}$ be a Borel field with $\mathscr{F} \subset \mathscr{P}$. Let f, x and Φ be random variables on Ω to C, X and $\mathscr{L}(X, Y)$, respectively. Then*

(i) *if $E\{\| x \|\} < \infty$, then $E\{E\{x \mid \mathscr{F}\}\} = E\{x\}$;*

(ii) *if $E\{\| x \|\} < \infty$ and x is measurable relative to $\mathscr{F}$, then $E\{x \mid \mathscr{F}\} = x$* wp1.

(iii) *if $E\{\| x \|\} < \infty$, $E\{| f | \| x \|\} < \infty$, and x is measurable relative to $\mathscr{F}$, then $E\{fx \mid \mathscr{F}\} = E\{f \mid \mathscr{F}\}x$* wp1;

(iv) *if $E\{\| x \|^2\} < \infty$, $E\{\| \Phi \|^2\} < \infty$ and Φ is measurable relative to $\mathscr{F}$, then $E\{\Phi x \mid \mathscr{F}\} = \Phi E\{x \mid \mathscr{F}\}$* wp1; *and*

(v) *if $E\{\| x \|\} < \infty$ and x is independent of $\mathscr{F}$, then $E\{x \mid \mathscr{F}\} = E\{x\}$* wp1.

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