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Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds ☆

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Abstract

We obtain formulas for the first and second cohomology groups of a general current Lie algebra with coefficients in the "current" module, and apply them to compute structure functions for manifolds of loops with values in compact Hermitian symmetric spaces.

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Introduction

We deal with the low-dimensional cohomology of current Lie algebras with coefficients in the "current module". Namely, let L be a Lie algebra, M an L-module, A an associative commutative algebra with unit, V a symmetric unital A-module. Then

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the Lie algebra structure on $L \otimes A$ and the $L \otimes A$-module structure on $M \otimes V$ are defined via obvious formulas:

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab,$$
$$(x \otimes a) \bullet (m \otimes v) = (x \bullet m) \otimes (a \bullet v)$$

for any $x, y \in L$, $m \in M$, $a, b \in A$, $v \in V$, where $\bullet$ denotes, by abuse of notation, a respective module action.

The aim of this paper is twofold. First, we want to demonstrate that the problem of description of such cohomology in terms of the tensor factors L and A probably does not have an adequate general solution, as even a partial answer for the two-dimensional cohomology seems to be overwhelmingly complex. Second, we want to demonstrate, nevertheless, computability of this cohomology in some cases and its application to some differential geometric questions.

In Section 1 we establish an elementary result from linear algebra which will be useful in the course of subsequent algebraic manipulations. In Section 2 we get a formula for the first cohomology group. In Section 3 we compute the second cohomology group in two cases—where L is abelian and where L acts trivially on the whole cohomology group $H^2(L \otimes A, M \otimes V)$. At the end of this section, we present a list of 13 types of 2-cocycles (so-called cocycles of rank 1, generated by decomposable elements in the tensor product) in the general case. However, this list is a priori not complete. In Section 4 a certain spectral sequence is sketched, which may provide a more conceptual framework for computations in preceding sections. However, we do not go into details and other sections are not dependent on that one. The last Section 5 is devoted to an application. We show how to derive from previous computations obstructions to integrability (structure functions) of certain canonical connections on the manifolds of loops with values in compact Hermitian symmetric spaces.

One should note that the result about the first cohomology group (in particular, about derivations of the current Lie algebra) can be found in different forms in the literature and is a sort of folklore, and partial results on the second cohomology were obtained by Cathelineau [3], Haddi [11], Lecomte and Roger [15] and the author [18]. However, all these results do not provide the whole generality we need, as various restrictions, notably the zero characteristic of the ground field and perfectness of the Lie algebra L were imposed. Moreover, as we see in Section 5, the case in a sense opposite to the case of perfect L, namely, the case of abelian L, does lead to some interesting application (first considered by Poletaeva).

The technique used is highly computational and linear-algebraic in nature and based on applying various symmetrization operators to the cocycle equation.

Notations

The ground field K is assumed to be arbitrary field of characteristic $\neq 2, 3$ in Sections 1–4, and $\mathbb{C}$ in Section 5.

$H^n(L, M)$, $C^n(L, M)$, $Z^n(L, M)$, $B^n(L, M)$ stand, respectively, for the spaces of cohomology, cochains, cocycles and coboundaries of a Lie algebra L with coefficients in a module M.

$M^L = \{m \in M \mid x \bullet m = 0$ or any $x \in L\}$ is a submodule of L-invariants.

If M, N are two L-modules, $\mathrm{Hom}(M, N)$ bears a standard L-module structure via $(x \bullet \varphi)(m) = \varphi(x \bullet m) - x \bullet \varphi(m)$ for $x \in L$, $m \in M$, and $Hom_L(M, N)$ is another notation for $\mathrm{Hom}(M, N)^L$.

$S^n(A, V)$ stands for the space of n-linear maps $A \times \cdots \times A \to V$, symmetric in all arguments.

$\wedge^n(V)$ and $T^n(V)$ stand, respectively, for the spaces of n-fold skew and tensor products of a module V.

$\mathrm{Har}^n(A, V)$ and $\mathscr{Z}^n(A, V)$ stands, respectively, for the spaces of Harrison cohomology and Harrison cocycles of an associative commutative algebra A with coefficients in a module V (for $n = 2$, these are just symmetric Hochschild cocycles; see [12], where this cohomology was introduced, and [6] for a more modern treatment).

$\mathrm{Der}(A)$ denotes the derivation algebra of an algebra A. More generally, $\mathrm{Der}(A, V)$ denotes the space of derivations of A with values in a A-module V.

All other (nonstandard and unavoidably numerous) notations for different spaces of multilinear mappings and modules are defined as they introduced in the text.

The symbol $\curvearrowright$ after an expression refers to the sum of all cyclic permutations (under $S(3)$) of letters and indices occurring in that expression.

1. A lemma from linear algebra

If either both L and M or both A and V are finite-dimensional, then each cocycle Φ in $Z^n(L \otimes A, M \otimes V)$ can be represented as an element of $\mathrm{Hom}(L^{\otimes n}, M) \otimes \mathrm{Hom}(A^{\otimes n}, V)$:

$$\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i, \tag{1.1}$$

where φ_i, α_i are n-linear mappings $L \times \cdots \times L \to M$ and $A \times \cdots \times A \to V$ respectively. We restrict our considerations to this case. The minimal possible number $|I|$ such that the cocycle Φ can be written in the form (1.1) will be called the *rank of cocycle*.

Representing $H^n(L \otimes A, M \otimes V)$ in terms of pairs (L, M) and (A, V), we encounter conditions such as

$$\sum_{i \in I} S\varphi_i \otimes T\alpha_i = 0, \tag{1.2}$$

where S and T are some linear operators defined on the spaces of n-linear mappings $L \times \cdots \times L \to M$ and $A \times \cdots \times A \to V$, respectively.

For example, the substitution $a_1 = \cdots = a_{n+1} = 1$ in the cocycle equation $d\Phi(x_1 \otimes a_1, \ldots, x_{n+1} \otimes a_{n+1}) = 0$, where Φ is as in (1.1), yields

$$\sum_{i \in I} d\varphi_i(x_1, \ldots, x_{n+1}) \otimes a_i(1, \ldots, 1) = 0.$$

Another example: applying the symmetrization operator Y with respect to the letters $x_1, \dots, x_{n+1}$, to the cocycle equation, we get:

$$\sum_{i\in I}\left(Y(x_1 \bullet \varphi_i(x_2,\dots,x_{n+1})) \otimes \sum_{j=1}^{n+1}(-1)^j a_j \bullet \alpha_i(a_1,\dots,\widehat{a_j},\dots,a_{n+1})\right) = 0.$$

So, suppose that a condition of type (1.2) holds. Since

$$\operatorname{Ker}(S \otimes T) = \operatorname{Hom}(A^{\otimes n}, V) \otimes \operatorname{Ker} T + \operatorname{Ker} S \otimes \operatorname{Hom}(L^{\otimes n}, M),$$

it follows that replacing α_i's and φ_i's by appropriate linear combinations, one can find a decomposition of the set of indices $I = I_1 \cup I_2$ such that

$$S\varphi_i = 0,\ \ i \in I_1 \quad \text{and} \quad T\alpha_i = 0,\ \ i \in I_2. \tag{1.3}$$

Suppose that another equality of type (1.2) holds:

$$\sum_{i\in I} S'\varphi_i \otimes T'\alpha_i = 0. \tag{1.2$'$}$$

Then it determines a new decomposition $I = I_1' \cup I_2'$ such that $S'\varphi_i = 0$ if $i \in I_1'$ and $T'\alpha_i = 0$ if $i \in I_2'$. It turns out that it is possible to replace φ_i's and α_i's by their linear combinations so that both decompositions will hold simultaneously.

Lemma 1.1. *Let U, W be two vector spaces, $S, S' \in \operatorname{Hom}(U, \cdot)$, $T, T' \in \operatorname{Hom}(W, \cdot)$. Then*

$$\operatorname{Ker}(S \otimes T) \cap \operatorname{Ker}(S' \otimes T') \simeq (\operatorname{Ker} S \cap \operatorname{Ker} S') \otimes W + \operatorname{Ker} S \otimes \operatorname{Ker} T' + \operatorname{Ker} S' \otimes \operatorname{Ker} T + U \otimes (\operatorname{Ker} T \cap \operatorname{Ker} T').$$

Proof. Since $\operatorname{Ker}(S \otimes T) = \operatorname{Ker} S \otimes W + U \otimes \operatorname{Ker} T$ and analogously for $\operatorname{Ker}(S' \otimes T')$, the equality to prove is a particular case of

$$(U_1 \otimes W + U \otimes W_1) \cap (U_2 \otimes W + U \otimes W_2) = (U_1 \cap U_2) \otimes W + U_1 \otimes W_2 + U_2 \otimes W_1 + U \otimes (W_1 \cap W_2) \tag{1.4}$$

provided U_1, U_2 and W_1, W_2 are subspaces of U and W respectively.

Assume for the moment that $U_1 \cap U_2 = W_1 \cap W_2 = 0$. Then expressing $U = U_1 \oplus U_2 \oplus U'$ and $W = W_1 \oplus W_2 \oplus W'$ for some subspaces U', W' and substituting this in the left side of (1.4), we get

$$(U_1 \otimes W \oplus U_2 \otimes W_1 \oplus U' \otimes W_1) \cap (U_1 \otimes W_2 \oplus U_2 \otimes W \oplus U' \otimes W_2) = U_1 \otimes W_2 \oplus U_2 \otimes W_1.$$

To prove (1.4) in the general case, pass to quotient modulo $(U_1 \cap U_2) \otimes W + U \otimes (W_1 \cap W_2)$ and obtain by the just proved $U_1 \otimes W_2 + U_2 \otimes W_1$. □

Below, in numerous applications of Lemma 1.1, we will, by abuse of language, say "by (1.2) and (1.2)$'$, one gets a decomposition $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that $S\varphi_i = S'\varphi_i = 0$ for $i \in I_1$, $S\varphi_i = T'\alpha_i = 0$ for $i \in I_2$, $S'\varphi_i = T\alpha_i = 0$ for $i \in I_3$ and $T\alpha = T'\alpha_i = 0$ for $i \in I_4$". This means that one can find a new expression $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$ with indicated properties (where the new φ_i's and α_i's are linear combinations of the old ones).

Unfortunately, for the "triple intersection" $\mathrm{Ker}(S \otimes T) \cap \mathrm{Ker}(S' \otimes T') \cap \mathrm{Ker}(S'' \otimes T'')$ the analogous decomposition is no longer true. That is why dealing with the second cohomology group in Section 3, we are unable to obtain a general result and restrict our considerations with cocycles of rank 1 or with some special cases. For the first cohomology group, however, Lemma 1.1 suffices to consider the general case, but at the end of the proof it turns out that it is possible to choose a basis consisting of cocycles of rank 1.

2. The first cohomology group

From now on (in this and subsequent sections), either both L and M or both A and V are finite-dimensional.

Theorem 2.1

$$H^1(L \otimes A, M \otimes V) \simeq H^1(L, M) \otimes V \oplus \mathrm{Hom}_L(L, M) \otimes \mathrm{Der}(A, V) \oplus \mathrm{Hom}(L/[L, L], M^L) \otimes \frac{\mathrm{Hom}(A, V)}{V + \mathrm{Der}(A, V)}. \quad (2.1)$$

Each cocycle in $Z^1(L \otimes A, M \otimes V)$ *is a linear combination of cocycles of the three following types* (*which correspond to the summands in* (2.1)):

(i) $x \otimes a \mapsto \varphi(x) \otimes (a \bullet v)$ *for some* $\varphi \in Z^1(L, M)$, $v \in V$;
(ii) $x \otimes a \mapsto \varphi(x) \otimes \alpha(a)$ *for some* $\varphi \in \mathrm{Hom}_L(L, M)$, $a \in \mathrm{Der}(A, V)$;
(iii) *as in* (ii) *with* $\varphi(L) \subseteq M^L$, $\varphi([L, L]) = 0, \alpha \in \mathrm{Hom}(A, V)$.

Remark. Theorem 2.1 was obtained earlier by Santharoubane [16] in the particular case where $M = L^*$, $V = A^*$ and L is 1-generated as $U(L)^+$-module, and by Haddi [11] (in homological form) in the case of characteristic zero and L perfect.

Proof. Let $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$ be a cocycle of

$$Z^1(L \otimes A, M \otimes V) \subset \mathrm{Hom}(L, M) \otimes \mathrm{Hom}(A, V).$$

The cocycle equation $d\Phi = 0$ reads

$$\sum_{i\in I}(x \bullet \varphi_i(y) \otimes a \bullet \alpha_i(b) - y \bullet \varphi_i(x) \otimes b \bullet \alpha_i(a) \\ -\varphi_i([x, y]) \otimes \alpha_i(ab)) = 0. \tag{2.2}$$

Symmetrizing this equation with respect to x, y, we get

$$\sum_{i\in I}(x \bullet \varphi_i(y) + y \bullet \varphi_i(x)) \otimes (a \bullet \alpha_i(b) - b \bullet \alpha_i(a)) = 0.$$

Substitute $a = b = 1$ in (2.2):

$$\sum_{i\in I} d\varphi_i(x, y) \otimes \alpha_i(1) = 0.$$

Applying Lemma 1.1 to the last two equations, we get a decomposition $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that

$$\begin{array}{lll} d\varphi_i = 0, & x \bullet \varphi_i(y) + y \bullet \varphi_i(x) = 0 & \text{for any } i \in I_1, \\ d\varphi_i = 0, & a \bullet \alpha_i(b) = b \bullet \alpha_i(a) & \text{for any } i \in I_2, \\ x \bullet \varphi_i(y) + y \bullet \varphi_i(x) = 0, & \alpha_i(1) = 0 & \text{for any } i \in I_3, \\ a \bullet \alpha_i(b) = b \bullet \alpha_i(a), & \alpha_i(1) = 0 & \text{for any } i \in I_4. \end{array}$$

It is easy to see that $\alpha_i(a) = a \bullet \alpha_i(1)$ for each $i \in I_2$, and the mappings $x \otimes a \mapsto \varphi_i(x) \otimes \alpha_i(a)$ are cocycles of type (i) from the statement of the Theorem 2.1, and that $\alpha_i = 0$ for each $i \in I_4$.

Substitute $b = 1$ in the cocycle equation (2.2):

$$\sum_{i\in I_1\cup I_3} (x \bullet \varphi_i(y) - \varphi_i([x, y])) \otimes (\alpha_i(a) - a \bullet \alpha_i(1)) = 0.$$

Now apply Lemma 1.1 again. For elements φ_i, where $i \in I_1$, the vanishing of $x \bullet \varphi_i(y) - \varphi_i([x, y])$ implies $\varphi_i = 0$, and the vanishing of $\alpha_i(a) - a \bullet \alpha_i(1)$ gives cocycles of type (i), an already considered case. We have $x \bullet \varphi_i(y) = \varphi_i([x, y])$ for all (remaining) $i \in I_3$.

Hence (2.2) can be rewritten as

$$\sum_{i\in I_3} \varphi_i([x, y]) \otimes (a \bullet \alpha_i(b) + b \bullet \alpha_i(a) - \alpha_i(ab)) = 0.$$

The vanishing of the first and second tensor factors gives rise to cocycles of type (iii) and (ii), respectively.

Hence we have

$$Z^1(L \otimes A, M \otimes V) = Z^1(L, M) \otimes V + \mathrm{Hom}_L(L, M) \otimes \mathrm{Der}(A, V) \\ + \mathrm{Hom}(L/[L, L], M^L) \otimes \mathrm{Hom}(A, V)$$

which can be rewritten as

$$Z^1(L \otimes A, M \otimes V) = Z^1(L, M) \otimes V \oplus \mathrm{Hom}_L(L, M) \otimes \mathrm{Der}(A, V) \\ \oplus \mathrm{Hom}(L/[L, L], M^L) \otimes \frac{\mathrm{Hom}(A, V)}{V + \mathrm{Der}(A, V)}.$$

From the considerations above we easily deduce:

$$B^1(L \otimes A, M \otimes V) = B^1(L, M) \otimes V$$

and (2.1) now follows. □

Corollary 2.2. *The derivation algebra of the current Lie algebra $L \otimes A$ is isomorphic to*

$$\mathrm{Der}(L) \otimes A \oplus \mathrm{Hom}_L(L, L) \otimes \mathrm{Der}(A)$$
$$\oplus \mathrm{Hom}(L/[L, L], Z(L)) \otimes \frac{\mathrm{End}(A)}{A + \mathrm{Der}(A)}.$$

This overlaps with [1, Theorem 7.1] and [2, Theorem 1.1].

Note that $\mathrm{Hom}_L(L, L)$ is nothing but a *centroid* of an algebra L (the set of all linear transformations in $\mathrm{End}(L)$ commuting with algebra multiplications).

Specializing to particular cases of L and A, we get on this way (largely known) results about derivations of some particular classes of Lie algebras. So, letting $L = \mathfrak{g}$, a classical Lie algebra over $\mathbb{C}$, and $A = \mathbb{C}[t, t^{-1}]$, the Laurent polynomial ring, we get a formula for derivation algebra of a loop algebra:

$$\mathrm{Der}(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \simeq \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus 1 \otimes W,$$

where $W = \mathrm{Der}(\mathbb{C}[t, t^{-1}])$ is the famous Witt algebra.

More generally, replacing the Laurent polynomial ring by an algebra of functions meromorphic on a compact Riemann surface and holomorphic outside the fixed finite set of punctures on the surface, we get a similar formula for derivation algebra of a Krichever–Novikov algebra of affine type, where Witt algebra is replaced by a Krichever–Novikov algebra of Witt type.

3. The second cohomology group

In this section we obtain some particular results on the second cohomology group $H^2(L \otimes A, M \otimes V)$. The computations go along the same scheme as for H^1 but are more complicated.

As we want to express H^2 in terms of the tensor products of modules depending on (L, M) and (A, V), it is natural to do so for underlying modules of the Chevalley–Eilenberg complex. We have (under the same finiteness assumptions as previous):

$$\begin{aligned} C^1(L \otimes A, M \otimes V) &\simeq C^1(L, M) \otimes C^1(A, V) \\ C^2(L \otimes A, M \otimes V) &\simeq C^2(L, M) \otimes S^2(A, V) \\ &\quad \oplus S^2(L, M) \otimes C^2(A, V). \end{aligned} \tag{3.1}$$

To obtain a similar decomposition in the third degree, let us denote (by abuse of language) the Young symmetrizer corresponding to tableau λ by the same symbol λ. We have decomposition of the unit element in the group algebra $K[S_3]$:

$$e = \frac{1}{6}\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} + \frac{1}{3}\left(\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}\right) + \frac{1}{6}\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\ \hline\end{array}.$$

Then, using the natural isomorphism $i : T^3(L \otimes A) \simeq T^3(L) \otimes T^3(A)$ and the projection $p : T^3(L \otimes A) \to \wedge^3(L \otimes A)$, one can decompose the third exterior power of the tensor product as follows:

$$\begin{aligned}\wedge^3(L \otimes A) = p \circ (e \times e) \circ i(T^3(L \otimes A)) &\simeq \wedge^3(L) \otimes S^3(A)\\ &\oplus \left(\begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}(L) \otimes \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array}(A)\right.\\ &+\left.\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array}(L) \otimes \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}(A)\right) \oplus S^3(L) \otimes \wedge^3(A)\end{aligned}$$

(all other components appearing in $T^3(L) \otimes T^3(A)$ vanish under the projection). One directly verifies that

$$\begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array}(u) = 0 \text{ if and only if } \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}(u) = 0$$

for each $u \in \wedge^3(L \otimes A)$.

Hence we get a (noncanonical) isomorphism:

$$\begin{aligned}\wedge^3(L \otimes A) &\simeq \wedge^3(L) \otimes S^3(A) \oplus \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array}(L) \otimes \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}(A)\\ &\oplus S^3(L) \otimes \wedge^3(A).\end{aligned}$$

Passing to $\mathrm{Hom}(\cdot, M \otimes V) \simeq \mathrm{Hom}(\cdot, M) \otimes \mathrm{Hom}(\cdot, V)$, one gets

$$\begin{aligned}C^3(L \otimes A, M \otimes V) &\simeq C^3(L, M) \otimes S^3(A, V) \oplus Y^3(L, M) \otimes \widetilde{Y}^3(A, V)\\ &\oplus S^3(L, M) \otimes C^3(A, V),\end{aligned} \tag{3.2}$$

where

$$Y^3(L, M) = \mathrm{Hom}\left(\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1}\end{array}(L), M\right),$$

$$\widetilde{Y}^3(A, V) = \mathrm{Hom}\left(\begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}(A), V\right).$$

According to (3.1)–(3.2) one can decompose H^2 as

$$H^2(L \otimes A, M \otimes V) = (H^2)' \oplus (H^2)'', \tag{3.3}$$

where $(H^2)'$ are the classes of cocycles lying in $C^2(L, M) \otimes S^2(A, V)$ and $(H^2)''$ are the classes of cocycles of the form $\Phi + \Psi$, where $\Phi \in S^2(L, M) \otimes C^2(A, V)$, $\Psi \in C^2(L, M) \otimes S^2(A, V)$, $\Phi \neq 0$. We will compute $(H^2)'$ and obtain some particular results on $(H^2)''$ (actually $(H^2)'$ and $(H^2)''$ are limit terms of a certain spectral sequence; see Section 4).

The differentials of the low degree in the piece

$$C^1(L \otimes A, M \otimes V) \overset{d^1}{\to} C^2(L \otimes A, M \otimes V) \overset{d^2}{\to} C^3(L \otimes A, M \otimes V)$$

of the standard Chevalley–Eilenberg complex can be decomposed as follows:

$$d^1 = d_1 + d_2,$$
$$d^2 = \sum_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 3}} d_{ij},$$

where

$$\begin{aligned}
d_1 &: C^1(L, M) \otimes C^1(A, V) \to C^2(L, M) \otimes S^2(A, V),\\
d_2 &: C^1(L, M) \otimes C^1(A, V) \to S^2(L, M) \otimes C^2(A, V),\\
d_{11} &: C^2(L, M) \otimes S^2(A, V) \to C^3(L, M) \otimes S^3(A, V),\\
d_{12} &: C^2(L, M) \otimes S^2(A, V) \to Y^3(L, M) \otimes \widetilde{Y}^3(A, V),\\
d_{13} &: C^2(L, M) \otimes S^2(A, V) \to S^3(L, M) \otimes C^3(A, V),\\
d_{21} &: S^2(L, M) \otimes C^2(A, V) \to C^3(L, M) \otimes S^3(A, V),\\
d_{22} &: S^2(L, M) \otimes C^2(A, V) \to Y^3(L, M) \otimes \widetilde{Y}^3(A, V),\\
d_{23} &: S^2(L, M) \otimes C^2(A, V) \to S^3(L, M) \otimes C^3(A, V).
\end{aligned}$$

Direct computations show:

$$\begin{aligned}
&d_1(\varphi \otimes \alpha)(x_1 \otimes a_1, x_2 \otimes a_2)\\
&\quad = \tfrac{1}{2}(x_2 \bullet \varphi(x_1) - x_1 \bullet \varphi(x_2)) \otimes (a_1 \bullet \alpha(a_2) + a_2 \bullet \alpha(a_1))\\
&\quad\quad - \varphi([x_1, x_2]) \otimes \alpha(a_1 a_2);
\end{aligned}$$

$$\begin{aligned}
&d_2(\varphi \otimes \alpha)(x_1 \otimes a_1, x_2 \otimes a_2)\\
&\quad = \tfrac{1}{2}(x_1 \bullet \varphi(x_2) + x_2 \bullet \varphi(x_1)) \otimes (a_2 \bullet \alpha(a_1) - a_1 \bullet \alpha(a_2));
\end{aligned}$$

$$\begin{aligned}
&d_{11}(\varphi \otimes \alpha)(x_1 \otimes a_1, x_2 \otimes a_2, x_3 \otimes a_3)\\
&\quad = \tfrac{1}{3}(\varphi([x_1, x_2], x_3) + \curvearrowright) \otimes (\alpha(a_1 a_2, a_3) + \curvearrowright)\\
&\quad\quad - \tfrac{1}{3}(x_1 \bullet \varphi(x_2, x_3) + \curvearrowright) \otimes (a_1 \bullet \alpha(a_2, a_3) + \curvearrowright);
\end{aligned}$$

$$\begin{aligned}
&d_{12}(\varphi \otimes \alpha)(x_1 \otimes a_1, x_2 \otimes a_2, x_3 \otimes a_3) \\
&\quad = (2\varphi([x_1, x_2], x_3) + \varphi([x_1, x_3], x_2) - \varphi([x_2, x_3], x_1)) \\
&\qquad \otimes (\alpha(a_1a_2, a_3) - \alpha(a_2a_3, a_1)) + (-x_1 \bullet \varphi(x_2, x_3) + x_2 \bullet \varphi(x_1, x_3) \\
&\qquad + 2x_3 \bullet \varphi(x_1, x_2)) \otimes (a_1 \bullet \alpha(a_2, a_3) - a_3 \bullet \alpha(a_1, a_2));
\end{aligned}$$

$$d_{13}(\varphi \otimes \alpha)(x_1 \otimes a_1, x_2 \otimes a_2, x_3 \otimes a_3) = 0;$$

$$\begin{aligned}
&d_{22}(\varphi \otimes \alpha)(x_1 \otimes a_1, x_2 \otimes a_2, x_3 \otimes a_3) \\
&\quad = (2\varphi([x_1, x_2], x_3) + \varphi([x_1, x_3], x_2) - \varphi([x_2, x_3], x_1)) \\
&\qquad \otimes (\alpha(a_1a_2, a_3) - \alpha(a_2a_3, a_1)) + (-x_1 \bullet \varphi(x_2, x_3) + x_2 \bullet \varphi(x_1, x_3)) \\
&\qquad \otimes (a_1 \bullet \alpha(a_2, a_3) + a_3 \bullet \alpha(a_1, a_2) + 2a_2 \bullet \alpha(a_1, a_3));
\end{aligned}$$

$$\begin{aligned}
&d_{23}(\varphi \otimes \alpha)(x_1 \otimes a_1, x_2 \otimes a_2, x_3 \otimes a_3) \\
&\quad = \tfrac{1}{3}(x_1 \bullet \varphi(x_2, x_3) + \curvearrowright) \otimes (a_1 \bullet \alpha(a_2, a_3) + \curvearrowright)
\end{aligned}$$

(the absence of d_{21} in this list is merely a technical matter: at a relevant stage of computations, it will be convenient to use the entire differential d rather than d_{21}).

Now the reader should be prepared for a bunch of tedious and cumbersome definitions. We apologize for this, but our excuse is that all this stuff provides building blocks for $H^2(L \otimes A, M \otimes V)$ and one can hardly imagine that it may be defined in a simpler way. Taking a glance at the expressions below, one can believe that the general formula for $H^n(L \otimes A, M \otimes V)$ hardly exists – if it does, one should give correct n-dimensional generalizations of definitions below (in a few cases this is evident – like Harrison or cyclic cohomology, but in most it is not).

Definitions

(i) Define $d^{[\,]}, d^{\bullet} : \mathrm{Hom}(L^{\otimes 2}, M) \to \mathrm{Hom}(L^{\otimes 3}, M)$ as follows:

$$d^{[\,]}\varphi(x, y, z) = \varphi([x, y], z) + \curvearrowright,$$
$$d^{\bullet}\varphi(x, y, z) = x \bullet \varphi(y, z) + \curvearrowright.$$

(ii) Define $\wp, D : \mathrm{Hom}(A^{\otimes 2}, V) \to \mathrm{Hom}(A^{\otimes 3}, V)$ as follows:

$$\wp\alpha(a, b, c) = \alpha(ab, c) + \curvearrowright,$$
$$D\alpha(a, b, c) = a \bullet \alpha(b, c) + \curvearrowright.$$

(iii) $\mathscr{B}(L, M) = \{\varphi \in C^2(L, M) \mid \varphi([x, y], z) + z \bullet \varphi(x, y) = 0;\ d^{[\,]}\varphi(x, y, z) = 0\}$.

(iv) $Q^2(L, M) = \{d\psi \mid \psi \in \mathrm{Hom}(L, M);\ x \bullet \psi(y) = y \bullet \psi(x)\}$;
$H^2_M(L) = (Z^2(L, M^L) + Q^2(L, M))/Q^2(L, M)$.

(v) $\mathscr{K}(L, M) = \{\varphi \in C^2(L, M) \mid d^{[\,]}\varphi(x, y, z) = 2x \bullet \varphi(y, z)\}$;
$\mathscr{J}(L, M) = \{\varphi \in C^2(L, M) \mid \varphi(x, y) = \psi([x, y]) - \frac{1}{2}x \bullet \psi(y) + \frac{1}{2}y \bullet \psi(x)$ for $\psi \in \mathrm{Hom}(L, M)\}$; $\mathscr{H}(L, M) = (\mathscr{K}(L, M) + \mathscr{J}(L, M))/\mathscr{J}(L, M)$.

(vi) $\mathscr{X}(L, M) = \{\varphi \in C^2(L, M) \mid 2\varphi([x, y], z) = z \bullet \varphi(x, y);$
$\varphi([x, y], z) = \varphi([z, x], y)\}$.
(vii) $\mathscr{T}(L, M) = \{\varphi \in C^2(L, M) \mid 3\varphi([x, y], z) = 2z \bullet \varphi(x, y);$
$\varphi([x, y], z) = \varphi([z, x], y)\}$.
(viii) $\mathrm{Poor}_-(L, M) = \{\varphi \in C^2(L, M^L) \mid \varphi([L, L], L) = 0\}$;
$\mathrm{Poor}_+(L, M) = \{\varphi \in S^2(L, M^L) \mid \varphi([L, L], L) = 0\}$.
(ix) $\mathrm{Sym}^2(L, M) = \{\varphi \in S^2(L, M) \mid x \bullet \varphi(y, z) = y \bullet \varphi(x, z)\}$;
$SB^2(L, M) = \{\varphi \in S^2(L, M) \mid \varphi(x, y) = x \bullet \psi(y) + y \bullet \psi(x)$ for $\psi \in \mathrm{Hom}(L, M)\}$; $SH^2(L, M) = (\mathrm{Sym}^2(L, M) + SB^2(L, M))/SB^2(L, M)$.
(x) Define an action of L on $\mathrm{Hom}(L^{\otimes 2}, M)$ via

$$z \circ \varphi(x, y) = z \bullet \varphi(x, y) + \varphi([x, z], y) + \varphi(x, [y, z]).$$

$$\mathscr{S}^2(L, M) = \{\varphi \in S^2(L, M)^L \mid \varphi([x, y], z) + \curvearrowright = 0\}.$$

(xi) $D(A, V) = \{\beta \in \mathrm{Hom}(A, V) \mid \beta(abc) = a \bullet \beta(bc) - bc \bullet \beta(a) + \curvearrowright\}$.
(xii) $HC^1(A, V) = \{\alpha \in C^2(A, V) \mid \wp\alpha = 0\}$.
(xiii) $\mathscr{C}^2(A, V) = \{\alpha \in C^2(A, V) \mid \alpha(ac, b) - \alpha(bc, a) + a \bullet \alpha(b, c) - b \bullet \alpha(a, c) + 2c \bullet \alpha(a, b) = 0\}$.
(xiv) $\mathscr{P}_-(A, V) = \{\alpha \in C^2(A, V) \mid \alpha(ab, c) = a \bullet \alpha(b, c) + b \bullet \alpha(a, c)\}$;
$\mathscr{P}_+(A, V) = \{\alpha \in S^2(A, V) \mid \alpha(ab, c) = a \bullet \alpha(b, c) + b \bullet \alpha(a, c)\}$.
(xv) $\mathscr{A}(A, V) = \{\alpha \in S^2(A, V) \mid 2D\alpha = \wp\alpha\}$.

The spaces defined in (xi), (xv) are relevant in computation of $Kerd_{11}$ (Lemma 3.2), the spaces defined in (iii)–(viii), (xiv) are relevant in computation of $\mathrm{Ker}\, d_{11} \cap \mathrm{Ker}\, d_{12}$ (see (3.6)), the spaces defined in (ix) are relevant in computation for the particular case where L is abelian (Proposition 3.5), and the spaces defined in (x), (xii)–(xiii) are relevant in computation of the relative cohomology group $H^2(L \otimes A; L, M \otimes V)$ (Proposition 3.8).

Remarks

(i) d (the Chevalley–Eilenberg differential) $= d^{[\,]} + d^{\bullet}$.
(ii) As $B^2(L, M^L) \subseteq Q^2(L, M)$, there is a surjection

$$H^2(L) \otimes M^L \to H^2_M(L).$$

(iii) If $V = K$, then $HC^1(A, V)$ is just the first-order cyclic cohomology $HC^1(A)$.
(iv) The following relations hold:

$$\mathrm{Poor}_-(L, M) \subseteq \mathscr{B}(L, M) \subseteq Z^2(L, M),$$
$$\mathscr{B}(L, M) \cap Z^2(L, M^L) = \mathrm{Poor}_-(L, M),$$
$$\mathscr{S}^2(L, M) \cap S^2(L, M^L)^L = \mathrm{Poor}_+(L, M),$$
$$\mathscr{C}^2(A, V) \cap HC^1(A, V) = \mathscr{P}_-(A, V),$$
$$\mathscr{L}^2(A, V) \cap \mathscr{A}(A, V) = \mathscr{P}_+(A, V),$$
$$\mathrm{Der}(A, V) \subseteq D(A, V).$$

Proposition 3.1

$$
\begin{aligned}
(H^2)' \simeq H^2(L, M) \otimes V \oplus H^2_M(L) \otimes \frac{\mathrm{Hom}(A, V)}{V \oplus \mathrm{Der}(A, V)} \\
\oplus \mathscr{H}(L, M) \otimes \mathrm{Der}(A, V) \oplus \mathscr{B}(L, M) \otimes \frac{\mathrm{Har}^2(A, V)}{\mathscr{P}_+(A, V)} \\
\oplus C^2(L, M)^L \otimes \mathscr{P}_+(A, V) \oplus \mathscr{X}(L, M) \otimes \frac{\mathscr{A}(A, V)}{\mathscr{P}_+(A, V)} \\
\oplus \mathscr{T}(L, M) \otimes \frac{D(A, V)}{\mathrm{Der}(A, V)} \\
\oplus \mathrm{Poor}_-(L, M) \otimes \frac{S^2(A, V)}{\mathrm{Hom}(A, V) + D(A, V) + \mathrm{Har}^2(A, V) + \mathscr{A}(A, V)}.
\end{aligned}
$$

Each cocycle which lies in $C^2(L, M) \otimes S^2(A, V)$ is a linear combination of cocycles of the eight following types (*which correspond to the respective direct summands in the isomorphism*):

(i) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes ab \bullet v$, *where* $\varphi \in Z^2(L, M)$ *and* $v \in V$;
(ii) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes \beta(ab)$, *where* $\varphi \in Z^2(L, M^L)$ *and* $\beta \in \mathrm{Hom}(A, V)$;
(iii) *as in* (ii) *with* $\varphi \in \mathscr{K}(L, M)$ *and* $\beta \in \mathrm{Der}(A, V)$;
(iv) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes \alpha(a, b)$, *where* $\varphi \in \mathscr{B}(L, M)$ *and* $\alpha \in \mathscr{Z}^2(A, V)$;
(v) *as in* (iv) *with* $\varphi \in C^2(L, M)^L$ *and* $\alpha \in \mathscr{P}_+(A, V)$;
(vi) *as in* (iv) *with* $\varphi \in \mathscr{X}(L, M)$ *and* $\alpha \in \mathscr{A}(A, V)$;
(vii) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes (3a \bullet \beta(b) + 3b \bullet \beta(a) - 2\beta(ab))$, *where* $\varphi \in \mathscr{H}(L, M)$ *and* $\beta \in D(A, V)$;
(viii) *as in* (iv) *with* $\varphi \in \mathrm{Poor}_-(L, M)$ *and* $\alpha \in S^2(A, V)$.

Proof. We have

$$
(H^2)' = \frac{\mathrm{Ker}\, d_{11} \cap \mathrm{Ker}\, d_{12}}{\mathrm{Im}\, d_1}. \tag{3.4}
$$

We compute the relevant spaces in the subsequent series of lemmas.

Lemma 3.2

$$
\begin{aligned}
\mathrm{Ker}\, d_{11} = Z^2(L, M) \otimes V \\
+ \{\varphi \in C^2(L, M) \mid 2d^{[\,]}\varphi + d^{\bullet}\varphi = 0\} \otimes \mathscr{A}(A, V)
\end{aligned}
$$

$$+\{\varphi \in C^2(L, M) \mid 3d^{[\,]}\varphi + 2d^{\bullet}\varphi = 0\} \otimes D(A, V)$$
$$+\{\varphi \in C^2(L, M) \mid d^{[\,]} = d^{\bullet}\varphi = 0\} \otimes S^2(A, V).$$

Proof. Substituting $a_1 = a_2 = a_3 = 1$ into the equation $d_{11}\Phi = 0$ (as usual, $\Phi = \sum_{i\in I} \varphi_i \otimes \alpha_i$), one derives the equality

$$\sum_{i\in I} d\varphi_i(x_1, x_2, x_3) \otimes \alpha_i(1, 1) = 0 \tag{3.5}$$

and a decomposition $I = I_1 \cup I_2$ with $d\varphi_i = 0$ for $i \in I_1$ and $\alpha_i(1, 1) = 0$ for $i \in I_2$.

Substituting then $a_2 = a_3 = 1$ into the same equation, one gets

$$\sum_{i\in I}(3d^{[\,]}\varphi_i + 2d^{\bullet}\varphi_i) \otimes (\alpha_i(1, a_1) - a_1 \bullet \alpha_i(1, 1)) = 0$$

and by Lemma 1.1 there is a decomposition $I = I_{11} \cup I_{12} \cup I_{21} \cup I_{22}$ with

$$\begin{array}{lll} d\varphi_i = 0, & 3d^{[\,]}\varphi_i + 2d^{\bullet}\varphi_i = 0 & \text{for any } i \in I_{11}, \\ d\varphi_i = 0, & \alpha_i(1, a) = \alpha_i(1, 1) & \text{for any } i \in I_{12}, \\ 3d^{[\,]}\varphi_i + 2d^{\bullet}\varphi_i = 0, & \alpha_i(1, 1) = 0 & \text{for any } i \in I_{21}, \\ \alpha_i(1, 1) = 0, & \alpha_i(1, a) = a \bullet \alpha_i(1, 1) & \text{for any } i \in I_{22}. \end{array}$$

Obviously $d\varphi_i = d^{\bullet}\varphi_i = 0$ for any $i \in I_{11}$, so components with $i \in I_{11}$ lie in $\operatorname{Ker} d_{11}$, and $\alpha_i(1, a) = 0$ for any $i \in I_{22}$.

Further, substituting $a_3 = 1$ in our equation, we get

$$\sum_{i\in I}(2d^{[\,]}\varphi_i + d^{\bullet}\varphi_i) \otimes (\alpha_i(a_1, a_2) - 3a_1 \bullet \alpha_i(1, a_2) - 3a_2 \bullet \alpha_i(1, a_1)$$
$$+ 2\alpha_i(1, a_1a_2) + 3a_1a_2 \bullet \alpha_i(1, 1)) = 0.$$

In order to apply Lemma 1.1 again, we join the sets I_{12} and I_{22} (with the common defining condition $\alpha_i(1, a) = a \bullet \alpha_i(1, 1)$) and obtain a decomposition $I = I'_1 \cup I'_2 \cup I'_3 \cup I'_4$ such that

$$\begin{array}{lll} 2d^{[\,]}\varphi_i + d^{\bullet}\varphi_i = 0, & \alpha_i(1, a) = a \bullet \alpha_i(1, 1) & \text{for any } i \in I'_1 \\ & \alpha_i(a, b) = ab \bullet \alpha_i(1, 1) & \text{for any } i \in I'_2 \\ d^{[\,]}\varphi_i = d^{\bullet}\varphi_i = 0, & \alpha_i(1, 1) = 0 & \text{for any } i \in I'_3 \\ 3d^{[\,]}\varphi_i + 2d^{\bullet}\varphi_i = 0, & \alpha_i(a, b) = 3a \bullet \alpha_i(1, b) & \\ & +3b \bullet \alpha_i(1, a) - 2\alpha_i(1, ab) & \text{for any } i \in I'_4. \end{array}$$

Note that components $\varphi_i \otimes \alpha_i$ with $i \in I'_3$ are among those with $i \in I_{11}$ (and lie in $\operatorname{Ker} d_{11}$).

Now, since the contribution of terms with $i \in I'_4$ to the left side of (3.5) vanishes, we may apply Lemma 1.1 again, and obtain a decomposition $I'_1 \cup I'_2 = I'_{11} \cup I'_{12} \cup I'_{21} \cup I'_{22}$ such that

$$\begin{array}{lll} 2d^{[\,]}\varphi_i + d^{\bullet}\varphi_i = 0, & \alpha_i(1, a) = 0 & \text{for any } i \in I'_{12} \\ d\varphi_i = 0, & \alpha_i(a, b) = ab \bullet \alpha_i(1, 1) & \text{for any } i \in I'_{21}, \end{array}$$

and the two remaining types of components do not contribute to the whole picture: those with indices from I'_{11} satisfy $d^{[\,]}\varphi_i = d^{\bullet}\varphi_i = 0$, the case covered by previous cases, and those with indices from I'_{22} vanish, as $\alpha_i(a,b) = ab \bullet \alpha_i(1,1) = 0$. Moreover, the components with indices from I'_{21} lie in $\operatorname{Ker} d_{11}$.

The remaining part of the equation $d_{11}\Phi = 0$ now reads

$$\sum_{i \in I'_{12} \cup I'_4} d^{[\,]}\varphi_i(x_1,x_2,x_3) \otimes (\wp\alpha(a_1,a_2,a_3) - 2D\alpha_i(a_2,a_3) \\ + 3a_1a_2 \bullet \alpha_i(1,a_3) - a_3 \bullet \alpha_i(1,a_1a_2) + \curvearrowright) = 0.$$

Applying Lemma 1.1 again, and noting that the vanishing of the first tensor factor in each summand above yields the already considered case $d^{[\,]}\varphi_i = d^{\bullet}\varphi_i = 0$, we obtain that the second tensor factor vanishes for all $i \in I'_{12} \cup I'_4$.

Consequently, we obtain two types of components $\varphi_i \otimes \alpha_i$ lying in $\operatorname{Ker} d_{11}$:

$$2d^{[\,]}\varphi_i + d^{\bullet}\varphi_i = 0; \quad \wp\alpha_i = 2D\alpha_i$$

and

$$3d^{[\,]}\varphi_i + 2d^{\bullet}\varphi_i = 0; \quad \wp\alpha_i = \frac{3}{2}D\alpha_i;$$

α_i satisfies the defining condition for$i \in I'_4$.

The last two conditions imposed on α_i imply $\alpha_i(1,\cdot) \in D(A,V)$.

Summarizing all this, we obtain the statement of the lemma. □

Lemma 3.3

$$\operatorname{Ker} d_{12} = C^2(L,M) \otimes V + \{\varphi \in C^2(L,M) \mid x \bullet \varphi(y,z) \\ = z \bullet \varphi(x,y)\} \otimes \operatorname{Hom}(A,V) + \{\varphi \in C^2(L,M) \mid \varphi([x,y],z) \\ - \varphi([y,z],x) - x \bullet \varphi(y,z) + z \bullet \varphi(x,y) = 0\} \otimes \mathscr{Z}^2(A,V) \\ + \{\varphi \in C^2(L,M) \mid x \bullet \varphi(y,z) = z \bullet \varphi(x,y); \\ \varphi([x,y],z) = \varphi([y,z],x)\} \otimes S^2(A,V).$$

Proof. Let $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i \in \operatorname{Ker} d_{12}$. Substituting $a_2 = 1$ in the equation $d_{12}\Phi = 0$, one gets

$$\sum_{i \in I} (-x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi(x_1,x_3) + 2x_3 \bullet \varphi_i(x_1,x_2)) \\ \otimes (a_1 \bullet \alpha_i(1,a_3) - a_3 \bullet \alpha_i(1,a_1)) = 0.$$

Hence we have a decomposition $I = I_1 \cup I_2$ such that, for $i \in I_1$, the first tensor factor in each summand above vanishes, and, for $i \in I_2$, the second one vanishes. Elementary transformations show that

$$x \bullet \varphi_i(y,z) = z \bullet \varphi_i(x,y) \quad \text{for any } i \in I_1,$$
$$\alpha_i(1,a) = a \bullet \alpha_i(1,1) \quad \text{for any } i \in I_2.$$

Then substituting $a_3 = 1$ into the same initial equation $d_{12}\Phi = 0$, one gets

$$\sum_{i\in I}(2\varphi_i([x_1,x_2],x_3) + \varphi_i([x_1,x_3],x_2) - \varphi_i([x_2,x_3],x_1) - x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi_i(x_1,x_3) + 2x_3 \bullet \varphi_i(x_1,x_2)) \otimes (\alpha_i(1,a_1a_2) - \alpha_i(a_1,a_2)) = 0.$$

Applying Lemma 1.1 and the fact that the vanishing of the first tensor factor here is equivalent to the condition $\varphi_i([x,y],z) - \varphi_i([y,z],x) - x \bullet \varphi_i(y,z) + z \bullet \varphi_i(x,y) = 0$, we get a decomposition $I = I_{11} \cup I_{12} \cup I_{21} \cup I_{22}$ such that

$$\begin{array}{lll} x \bullet \varphi_i(y,z) = z \bullet \varphi_i(x,y), & \varphi_i([x,y],z) = \varphi_i([y,z],x) & \text{for any } i \in I_{11}, \\ x \bullet \varphi_i(y,z) = z \bullet \varphi_i(x,y), & \alpha_i(a,b) = \alpha_i(1,ab) & \text{for any } i \in I_{12}, \\ \varphi_i([x,y],z) - \varphi_i([y,z],x) & & \\ \quad - x \bullet \varphi_i(y,z) & & \\ \quad + z \bullet \varphi_i(x,y) = 0, & \alpha_i(1,a) = a \bullet \alpha_i(1,1) & \text{for any } i \in I_{21}, \\ \alpha_i(a,b) = ab \bullet \alpha_i(1,1) & & \text{for any } i \in I_{22}. \end{array}$$

It is easy to see that the components $\varphi_i \otimes \alpha_i$ with indices belonging to I_{11}, I_{12} and I_{22}, already lie in $\operatorname{Ker} d_{12}$.

The remaining part of the equation $d_{12}\Phi = 0$ becomes

$$\sum_{i\in I_{21}}(2\varphi_i([x_1,x_2],x_3) + \varphi_i([x_1,x_3],x_2) - \varphi_i([x_2,x_3],x_1)) \otimes \delta\alpha_i(a_1,a_2,a_3) = 0,$$

where δ is Harrison(=Hochschild) differential. Thus there is a decomposition $I_{21} = I'_1 \cup I'_2$, where φ_i for $i \in I'_1$ satisfies the same relations as for $i \in I_{11}$, and $\alpha_i \in \mathscr{Z}^2(A,V)$ for any $i \in I'_2$.

Putting all these computations together yields the formula desired (the four summands there correspond to the defining conditions for I_{22}, I_{12}, I'_2 and I_{11}, respectively; the sum, in general, is not direct). □

Elementary but tedious transformations of expressions entering in defining conditions of summands of $\operatorname{Ker} d_{11}$ and $\operatorname{Ker} d_{12}$, allow us to write their intersection as the following direct sum:

$$\operatorname{Ker} d_{11} \cap \operatorname{Ker} d_{12} \simeq Z^2(L,M) \otimes V \oplus Z^2(L,M^L) \otimes \frac{\operatorname{Hom}(A,V)}{V \oplus \operatorname{Der}(A,V)} \oplus \mathscr{K}(L,M) \otimes \operatorname{Der}(A,V) \oplus \mathscr{B}(L,M) \otimes \frac{\mathscr{Z}^2(A,V)}{\mathscr{P}_+(A,V)}$$

$$\oplus C^2(L,M)^L \otimes \mathscr{P}_+(A,V) \oplus \mathscr{X}(L,M) \otimes \frac{\mathscr{A}(A,V)}{\mathscr{P}_+(A,V)}$$
$$\oplus \mathscr{T}(L, M \otimes \frac{D(A,V)}{\mathrm{Der}(A,V)}$$
$$\oplus \mathrm{Poor}_-(L,M) \otimes \frac{S^2(A,V)}{\mathrm{Hom}(A,V) + D(A,V) + \mathscr{Z}^2(A,V) + \mathscr{A}(A,V)}. \quad (3.6)$$

According to (3.4), to compute $(H^2)'$, we must consider the equation $\Phi = d_1\Psi$, where $\Phi \in \mathrm{Ker}\, d_{11} \cap \mathrm{Ker}\, d_{12}$ and $\Psi \in \mathrm{Hom}(L \otimes A, M \otimes V)$, which is equivalent to elucidation of all possible cohomological dependencies between the obtained classes of cocycles.

Lemma 3.4. *Let*:

$\{\varphi_i\}$ *be cohomologically independent cocycles in* $Z^2(L,M)$,
$\{\theta_i\}$ *be cocycles in* $Z^2(L,M^L)$ *independent modulo* $Q^2(L,M)$,
$\{\kappa_i\}$ *be elements of* $\mathscr{K}(L,M)$ *independent modulo* $\mathscr{T}(L,M)$,
$\{\varepsilon_i\}$ *be linearly independent cocycles in* $\mathscr{B}(L,M)$,
$\{\rho_i\}$ *be linearly independent elements in* $C^2(L,M)^L$,
$\{\chi_i\}$ *be linearly independent elements in* $\mathscr{X}(L,M)$,
$\{\tau_i\}$ *be linearly independent elements in* $\mathscr{T}(L,M)$,
$\{\xi_i\}$ *be linearly independent cocycles in* $\mathrm{Poor}_-(L,M)$,
$\{v_j\}$ *be linearly independent elements in* V,
$\{\delta_j\}$ *be linearly independent derivations in* $\mathrm{Der}(A,V)$,
$\{\beta_j\}$ *be mappings in* $D(A,V)$ *independent modulo* $\mathrm{Der}(A,V)$,
$\{\gamma_j\}$ *be mappings in* $\mathrm{Hom}(A,V)$ *independent both modulo* $\mathrm{Der}(A,V)$ *and modulo mappings* $a \mapsto a \bullet v$ *for all* $v \in V$,
$\{F_j\}$ *be cocycles in* $\mathscr{Z}^2(A,V)$ *independent both cohomologically and modulo* $\mathscr{P}_+(A,V)$,
$\{P_j\}$ *be linearly independent elements in* $\mathscr{P}_+(A,V)$,
$\{A_j\}$ *be elements in* $\mathscr{A}(A,V)$ *independent modulo* $\mathscr{P}_+(A,V)$,
$\{G_j\}$ *be mappings in* $S^2(A,V)$ *independent simultaneously modulo*:
mappings $a \wedge b \mapsto \gamma(ab)$ *for all* $\gamma \in \mathrm{Hom}(A,V)$,
mappings $a \wedge b \mapsto 3a \bullet \beta(b) + 3b \bullet \beta(a) - 2\beta(ab)$ *for all* $\beta \in D(A,V)$, *and* $\mathscr{Z}^2(A,V) + \mathscr{A}(A,V)$.

Then the elements of $\mathrm{Ker}\, d_{11} \cap \mathrm{Ker}\, d_{12}$:

$$\varphi_i \otimes (R_{v_j} \circ m), \quad \theta_i \otimes (\gamma_j \circ m), \quad \kappa_i \otimes (\delta_j \circ m), \quad \varepsilon_i \otimes F_j,$$
$$\rho_i \otimes P_j, \quad \chi_i \otimes A_j, \quad \tau_i \otimes (3\delta\beta_j - \beta_j \circ m), \quad \xi_i \otimes G_j$$

(*m stands for multiplication in A and R_v is an element in* $\mathrm{Hom}(A, V)$ *defined by* $a \mapsto a \bullet v$), *are independent modulo* $\mathrm{Im}\, d_1$.

Proof. We must prove that if

$$
\begin{aligned}
&\sum \varphi_i(x, y) \otimes ab \bullet v_j + \sum \theta_i(x, y) \otimes \gamma_j(ab) + \sum \kappa_i(x, y) \otimes \delta_j(ab) \\
&\quad + \sum \varepsilon_i(x, y) \otimes F_j(a, b) + \sum \rho_i(x, y) \otimes P_j(a, b) \\
&\quad + \sum \chi_i(x, y) \otimes A_j(a, b) \\
&\quad + \sum \tau_i(x, y) \otimes (3a \bullet \beta_j(b) + 3b \bullet \beta_j(a) - 2\beta_j(ab)) \\
&\quad + \sum \xi_i(x, y) \otimes G_j(a, b) \\
&\quad = \sum_{i \in I} \big(\psi_i([x, y]) \otimes \alpha_i(ab) + \tfrac{1}{2}(-x \bullet \psi_i(y) + y \bullet \psi_i(x)) \\
&\qquad \otimes (a \bullet \alpha_i(b) + b \bullet \alpha_i(a))\big)
\end{aligned}
\tag{3.7}
$$

for some $\sum_{i \in I} \psi_i \otimes \alpha_i \in \mathrm{Hom}(L, M) \otimes \mathrm{Hom}(A, V)$ (the right side here is the generic element in $\mathrm{Im}\, d_1$), then all terms in the left side vanish.

One has $\delta_j(1) = \beta_j(1) = P_j(1, a) = A_j(1, a) = 0$ and one may assume that $\gamma_j(1) = F_j(1, a) = G_j(1, a) = 0$. Substitute $a = b = 1$ in (3.7):

$$\sum \varphi_i(x, y) \otimes v_j = \sum_{i \in I} d\psi_i(x, y) \otimes \alpha_i(1).$$

As φ_i's are cohomologically independent and v_j's are linearly independent, the last equality implies that all summands $\varphi_i(x, y) \otimes v_j$ vanish and there is a decomposition $I = I_1 \cup I_2$ with $d\psi_i = 0$ for $i \in I_1$ and $\alpha_i(1) = 0$ for $i \in I_2$.

Now substitute $b = 1$ in (3.7):

$$
\begin{aligned}
&\sum \theta_i(x, y) \otimes \gamma_j(a) + \sum \kappa_i(x, y) \otimes \delta_j(a) + \sum \tau_i(x, y) \otimes \beta_j(a) \\
&\quad = \sum_{i \in I} (\psi_i([x, y]) + \tfrac{1}{2}(-x \bullet \psi_i(y) + y \bullet \psi_i(x))) \otimes (\alpha_i(a) - a \bullet \alpha_i(1)).
\end{aligned}
\tag{3.8}
$$

Substituting (3.8) in (3.7), one gets

$$
\begin{aligned}
&\sum \varepsilon_i(x, y) \otimes F_j(a, b) + \sum \rho_i(x, y) \otimes P_j(a, b) + \sum \chi_i(x, y) \otimes A_j(a, b) \\
&\quad + 3 \sum \tau_i(x, y) \otimes \delta\beta_j(a, b) + \sum \xi_i(x, y) \otimes G_j(a, b) \\
&\quad = \frac{1}{2} \sum (-x \bullet \psi_i(y) + y \bullet \psi_i(x)) \otimes (\delta\alpha_i(a, b) - ab \bullet \alpha_i(1)).
\end{aligned}
$$

The independence conditions of Lemma imply that all summands in the left side vanish and, due to Lemma 1.1, for $\sum_{i\in I}\psi_i\otimes\beta_i$, there exists a decomposition $I = I_{11}\cup I_{12}\cup I_{21}\cup I_{22}$ with

$$\begin{array}{lll} d\psi_i = 0, & x\bullet\psi_i(y) = y\bullet\psi_i(x) & \text{for any } i\in I_{11},\\ d\psi_i = 0, & \delta\alpha_i(a,b) = ab\bullet\alpha_i(1) & \text{for any } i\in I_{12},\\ x\bullet\psi_i(y) = y\bullet\psi_i(x), & \alpha_i(1) = 0 & \text{for any } i\in I_{21},\\ \alpha_i(1) = 0, & \delta\alpha_i(a,b) = ab\bullet\alpha_i(1) & \text{for any } i\in I_{22}. \end{array}$$

Denoting $\alpha'_i(a) = \alpha_i(a) - a\bullet\alpha_i(1)$ for $i\in I_{12}$, we get $\alpha'_i\in\mathrm{Der}(A,V)$.

Substituting all this information back into (3.8), one finally obtains

$$\begin{aligned}\sum\theta_i(x,y)\otimes\gamma_j(a) + \sum\kappa_i(x,y)\otimes\delta_j(a)\\ = \frac{1}{2}\sum_{i\in I_{12}}\psi_i([x,y])\otimes(\alpha_i(a) - a\bullet\alpha_i(1)) + \sum_{i\in I_{21}}\psi_i([x,y])\otimes\alpha_i(a)\\ + \sum_{i\in I_{22}}(\psi_i([x,y]) + \frac{1}{2}(-x\bullet\psi_i(y) + y\bullet\psi_i(x)))\otimes\alpha_i(a).\end{aligned}$$

The independence conditions of Lemma imply that all terms appearing in the last equality vanish, and the desired assertion follows. □

Conclusion of the proof of Proposition 3.1

Lemma 3.3 implies that

$$\begin{aligned}\mathrm{Im}\, d_1 \simeq B^2(L,M)\otimes V \oplus (Q^2(L,M)\cap Z^2(L,M^L))\otimes\frac{\mathrm{Hom}(A,V)}{V\oplus\mathrm{Der}(A,V)}\\ \oplus(\mathscr{T}(L,M)\cap\mathscr{K}(L,M))\otimes\mathrm{Der}(A,V)\oplus\mathscr{B}(L,M)\otimes\mathrm{Der}(A,V)\end{aligned}$$

which together with (3.6) entails the asserted isomorphism. □

Now we turn to computation of the second summand in (3.3), $(H^2)''$.

We are unable to compute it in general (and are in doubt about the existence of a closed general formula for $(H^2)''$) and confine ourselves to two particular cases (in both of them it turns out that $(H^2)''$ coincides with the classes of cocycles lying in $S^2(L,M)\otimes C^2(A,V)$).

Proposition 3.5. *Suppose L is abelian. Then*

$$\begin{aligned}(H^2)'' \simeq S^2(L,M^L)\otimes\frac{C^2(A,V)}{\{a\bullet\beta(b) - b\bullet\beta(a)\mid\beta\in\mathrm{Hom}(A,V)\}}\\ \oplus SH^2(L,M)\otimes\{a\bullet\beta(b) - b\bullet\beta(a)\mid\beta\in\mathrm{Hom}(A,V)\}.\end{aligned}$$

First we establish a lemma valid in the general situation (where L is not necessarily abelian).

Lemma 3.6

(i) $\operatorname{Ker} d_{23} = \operatorname{Ker} d^\bullet \otimes C^2(A, V) + S^2(L, M) \otimes \{\alpha \in C^2(A, V) \mid \alpha(a, b) = a \bullet \beta(b) - b \bullet \beta(a)\}$;
(ii) $\operatorname{Im} d_2 = \{\varphi \in S^2(L, M) \mid \varphi(x, y) = x \bullet \psi(y) + y \bullet \psi(x)\} \otimes \{\alpha \in C^2(A, V) \mid \alpha(a, b) = a \bullet \beta(b) - b \bullet \beta(a)\}$.

Proof. The only thing which perhaps needs a proof here is the equality

$$\operatorname{Ker} D = \left\{\alpha \in C^2(A, V) \mid \alpha(a, b) = a \bullet \beta(b) - b \bullet \beta(a)\right\}.$$

The validity of it is verified by appropriate substitution of 1's. □

Proof of Proposition 3.5. Let $\Phi = \sum_{i\in I} \varphi_i \otimes \alpha_i \in \operatorname{Ker} d_{23}$, with a decomposition on the set of indices $I = I_1 \cup I_2$ such that

$$\begin{aligned} &d^\bullet\varphi_i(x_1, x_2, x_3) = 0 && \text{for any } i \in I_1,\\ &\alpha_i(a, b) = -a \bullet \beta_i(b) + b \bullet \beta_i(a) && \text{for any } i \in I_2. \end{aligned}$$

By Lemma 3.6(i), we may also assume that elements α_i, where $i \in I_1$, are independent modulo $\{a \bullet \beta(b) - b \bullet \beta(a)\}$, and hence $\alpha_i(1, a) = 0$ for each $i \in I_1$.

Suppose there is

$$\Psi = \sum_{i\in I'} \varphi'_i \otimes \alpha'_i \in C^2(L, M) \otimes S^2(A, V) \tag{3.9}$$

such that the class of $\Phi - \Psi$ belongs to $(H^2)''$. This, in particular, means that $d_{22}\Phi = d_{12}\Psi$:

$$\begin{aligned} &\sum_{i\in I}(-x_1 \bullet \varphi_i(x_2, x_3) + x_2 \bullet \varphi_i(x_1, x_3))\\ &\quad\otimes (a_1 \bullet \alpha_i(a_2, a_3) + a_3 \bullet \alpha_i(a_1, a_2) + 2a_2 \bullet \alpha_i(a_1, a_3))\\ &\quad= \sum_{i\in I'}(-x_1 \bullet \varphi'_i(x_2, x_3) + x_2 \bullet \varphi'_i(x_1, x_3) + 2x_3 \bullet \varphi'_i(x_1, x_2))\\ &\quad\otimes (a_1 \bullet \alpha'_i(a_2, a_3) - a_3 \bullet \alpha'_i(a_1, a_2)). \end{aligned}$$

Substituting here $a_2 = 1$, one gets

$$\begin{aligned} &2\sum_{i\in I_1}(-x_1 \bullet \varphi_i(x_2, x_3) + x_2 \bullet \varphi_i(x_1, x_3)) \otimes \alpha_i(a_1, a_3)\\ &\quad+ 3\sum_{i\in I_1}(-x_1 \bullet \varphi_i(x_2, x_3) + x_2 \bullet \varphi_i(x_1, x_3)) \otimes \alpha_i(a_1, a_3)\\ &\quad= \sum_{i\in I'}(-x_1 \bullet \varphi'_i(x_2, x_3) + x_2 \bullet \varphi'_i(x_1, x_3) + 2x_3 \bullet \varphi'_i(x_1, x_2))\\ &\quad\otimes (a_1 \bullet \alpha'_i(1, a_3) - a_3 \bullet \alpha'_i(1, a_1)). \end{aligned}$$

Hence, due to the independence condition imposed on α_i for $i \in I_1$,

$$- x_1 \bullet \varphi_i(x_2, x_3) + x_2 \bullet \varphi_i(x_1, x_3) = 0, \quad i \in I_1. \tag{3.10}$$

This, together with condition $\varphi_i \in \operatorname{Ker} d^\bullet$, evidently implies $\varphi_i(L, L) \subseteq M^L$ for each $i \in I_1$. Note that the terms from $S^2(L, M^L) \otimes C^2(A, V)$ lie in $Z^2(L \otimes A, M \otimes V)$.

Now write the cocycle equation for elements from $S^2(L, M) \otimes \{a \bullet \beta(b) - b \bullet \beta(a)\}$:

$$\begin{aligned}\sum_{i \in I_2} &(x_1 \bullet \varphi_i(x_2, x_3) \otimes (a_1 a_2 \bullet \beta_i(a_3) - a_1 a_3 \bullet \beta_i(a_2)) \\ &+ x_2 \bullet \varphi_i(x_1, x_3) \otimes (-a_1 a_2 \bullet \beta_i(a_3) + a_2 a_3 \bullet \beta_i(a_1)) \\ &+ x_3 \bullet \varphi_i(x_1, x_2) \otimes (a_1 a_3 \bullet \beta_i(a_2) - a_2 a_3 \bullet \beta_i(a_1))) = 0.\end{aligned}$$

Substituting $a_2 = a_3 = 1$, we get

$$\sum_{i \in I_2} (x_2 \bullet \varphi_i(x_1, x_3) - x_3 \bullet \varphi_i(x_1, x_2)) \otimes (\beta_i(a_1) - a_1 \bullet \beta_i(1)) = 0.$$

As the vanishing of the second tensor factor here leads to the vanishing of the whole α_i, we see that the condition (3.10) holds also in this case, i.e., for all $i \in I_2$. Conversely, if (3.10) holds, then the cocycle equation is satisfied. Thus the space of cocycles in $Z^2(L \otimes A, M \otimes V)$ whose cohomology classes lie in $(H^2)''$, coincides with

$$\begin{aligned}&S^2(L, M^L) \otimes \frac{C^2(A, V)}{\{a \bullet \beta(b) - b \bullet \beta(a) \mid \beta \in \operatorname{Hom}(A, V)\}} \\ &\quad \oplus \frac{\operatorname{Sym}^2(L, M) + SB^2(L, M)}{SB^2(L, M)} \otimes \{a \bullet \beta(b) - b \bullet \beta(a) \mid \beta \in \operatorname{Hom}(A, V)\}\end{aligned}$$

(note that we can always take $\Psi = 0$ in (3.9)).

To conclude the proof, one can observe that all these cocycles are cohomologically independent. This is proved in a pretty standard way, as in Lemma 3.4. □

Summarizing Proposition 3.1 (for the case where L is abelian) and Proposition 3.5, we obtain

Theorem 3.7. *Let L be an abelian Lie algebra. Then*

$$\begin{aligned}&H^2(L \otimes A, M \otimes V) \\ &\quad \simeq H^2(L, M) \otimes V \oplus \mathscr{H}(L, M) \otimes \operatorname{Der}(A, V) \\ &\quad\quad \oplus C^2(L, M^L) \otimes \frac{S^2(A, V)}{V \oplus \operatorname{Der}(A, V)}\end{aligned}$$

$$\oplus\ S^2(L, M^L) \otimes \frac{C^2(A, V)}{\{a \bullet \beta(b) - b \bullet \beta(a) \mid \beta \in \mathrm{Hom}(A, V)\}}$$
$$\oplus\ SH^2(L, M) \otimes \{a \bullet \beta(b) - b \bullet \beta(a) \mid \beta \in \mathrm{Hom}(A, V)\}.$$

Each cocycle in $Z^2(L \otimes A, M \otimes V)$ is a linear combination of cocycles of the four following types (*which correspond respectively to the first*, *the sum of the second and the third*, *the fourth and the fifth summands in the isomorphism*):

(i) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes ab \bullet v$ *for some* $\varphi \in Z^2(L, M)$ *and* $v \in V$;
(ii) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes \alpha(a, b)$ *for some* $\varphi \in C^2(L, M^L)$ *and* $\alpha \in S^2(A, V)$;
(iii) *as in* (ii) *with* $\varphi \in S^2(L, M^L)$ *and* $\alpha \in C^2(A, V)$;
(iv) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes (a \bullet \beta(b) - b \bullet \beta(a))$ *for some* $\varphi \in \mathrm{Sym}^2(L, M)$ *and* $\beta \in \mathrm{Hom}(A, V)$.

Remark. It is easy to see that if L is abelian, then there is inclusion $\mathscr{H}(L, M) \subseteq H^2(L, M)$ ($\mathscr{H}(L, M)$ consists of classes of cocycles taking values in M^L). Hence, singling out appropriate terms from the first three direct summands in the isomorphism above, we obtain $\mathscr{H}(L, M) \otimes S^2(A, V)$ as a direct summand of $H^2(L \otimes A, M \otimes V)$.

Now we want to perform another particular computation of the second cohomology group, namely, to compute the relative cohomology $H^2(L \otimes A; L, M \otimes V)$.

We easily see that all constructions can be restricted to the relative complex $\mathrm{Hom}(\wedge^\star(L \otimes A/K1), M \otimes V)$ with a single (but greatly simplifying the matter) difference that all mappings from $C^3(A, V)$, $Y^3(A, V)$ and $S^3(A, V)$ vanish whenever one of their arguments is 1.

We write $(H^2_L)'$ and $(H^2_L)''$ to denote the corresponding components of $H^2(L \otimes A; L, M \otimes V)$.

Proposition 3.8

$$(H^2_L)'' \simeq S^2(L, M^L)^L \otimes HC^1(A, V) \oplus \mathscr{S}^2(L, M) \otimes \frac{\mathscr{C}^2(L, M)}{\mathscr{P}_-(A, V)}$$
$$\oplus\ \frac{S^2(L, M)^L}{S^2(L, M^L)^L} \otimes \mathscr{P}_-(A, V)$$
$$\oplus\ \mathrm{Poor}_+(L, M) \otimes \frac{C^2(A, V)}{HC^1(A, V) + \mathscr{C}^2(A, V)}.$$

Proof. The proof goes along the same scheme as of Proposition 3.1. By Lemma 3.6(i), $\mathrm{Ker}\, d_{23} = \mathrm{Ker}\, d^\bullet \otimes C^2(A, V)$ (as the second tensor factor in the second component there vanishes in this case).

The condition $d_{22}\Phi = d_{12}\Psi$ for $\Phi = \sum \varphi_i \otimes \alpha_i \in \operatorname{Ker} d_{23}$ and $\Psi = \sum \varphi_i' \otimes \alpha_i' \in C^2(L, M) \otimes S^2(L, M)$ reads

$$\begin{aligned}
&\sum_{i\in I}(2\varphi_i([x_1, x_2], x_3) + \varphi_i([x_1, x_3], x_2) - \varphi_i([x_2, x_3], x_1))\\
&\quad\otimes (\alpha_i(a_1a_2, a_3) - \alpha_i(a_2a_3, a_1)) + (-x_1 \bullet \varphi_i(x_2, x_3) + x_2 \bullet \varphi_i(x_1, x_3))\\
&\quad\otimes (a_1 \bullet \alpha_i(a_2, a_3) + a_3 \bullet \alpha_i(a_1, a_2) + 2a_2 \bullet \alpha_i(a_1, a_3))\\
&\quad= \sum_{i\in I'}(2\varphi_i'([x_1, x_2], x_3) + \varphi_i'([x_1, x_3], x_2) - \varphi_i'([x_2, x_3], x_1))\\
&\qquad\otimes (\alpha_i'(a_1a_2, a_3) - \alpha_i'(a_2a_3, a_1))\\
&\qquad+ (-x_1 \bullet \varphi_i'(x_2, x_3) + x_2 \bullet \varphi_i'(x_1, x_3) + 2x_3 \bullet \varphi_i'(x_1, x_2))\\
&\qquad\otimes (a_1 \bullet \alpha_i'(a_2, a_3) - a_2 \bullet \alpha_i'(a_1, a_2)).
\end{aligned}$$

Substituting here $a_2 = 1$, we obtain (remember about vanishing of all α's if one of arguments is 1):

$$\begin{aligned}
&\sum_{i\in I}(2\varphi_i([x_1, x_2], x_3) + \varphi_i([x_1, x_3], x_2) - \varphi_i([x_2, x_3], x_1)\\
&\quad-x_1 \bullet \varphi_i(x_2, x_3) + x_2 \bullet \varphi_i(x_1, x_3)) \otimes \alpha_i(a_1, a_3) = 0.
\end{aligned}$$

This implies

$$\begin{aligned}
&2\varphi_i([x_1, x_2], x_3) + \varphi_i([x_1, x_3], x_2) - \varphi_i([x_2, x_3], x_1)\\
&\quad-x_1 \bullet \varphi(x_2, x_3) + x_2 \bullet \varphi_i(x_1, x_3) = 0, \quad i \in I.
\end{aligned} \tag{3.11}$$

Since $\varphi \in \operatorname{Ker} d^\bullet$,

$$x_1 \bullet \varphi_i(x_2, x_3) + x_3 \bullet \varphi_i(x_1, x_3) + x_3 \bullet \varphi_i(x_1, x_2) = 0, \quad i \in I. \tag{3.12}$$

With the help of elementary transformations, (3.11) and (3.12) yield

$$\varphi_i([x_1, x_3], x_2) + \varphi_i([x_2, x_3], x_1) + x_3 \bullet \varphi_i(x_1, x_2) = 0$$

or, in other words, $\varphi_i \in S^2(L, M)^L$ for each $i \in I$.

Now, writing the cocycle equation for $\sum_{i\in I} \varphi_i \otimes \alpha_i \in S^2(L, M)^L \otimes C^2(A, V)$, one gets

$$\begin{aligned}
&\sum_{i\in I} \varphi_i([x_1, x_2], x_3) \otimes (\alpha_i(a_1a_2, a_3) - a_1 \bullet \alpha_i(a_2, a_3) - a_2 \bullet \alpha_i(a_1, a_3))\\
&\quad+ \varphi_i([x_1, x_3], x_2) \otimes (-\alpha_i(a_1a_3, a_2) - a_1 \bullet \alpha_i(a_2, a_3) + a_3 \bullet \alpha_i(a_1, a_2))\\
&\quad+ \varphi_i([x_2, x_3], x_1) \otimes (\alpha_i(a_2a_3, a_1) + a_2 \bullet \alpha_i(a_1, a_3) + a_3 \bullet \alpha_i(a_1, a_2)) = 0.
\end{aligned}$$

Antisymmetrize this expression with respect to a_1, a_2:

$$\begin{aligned}
&\sum_{i\in I}(\varphi_i([x_1, x_3], x_2) + \varphi_i([x_2, x_3], x_1)) \otimes (-\alpha_i(a_1a_3, a_2) + \alpha_i(a_2a_3, a_1)\\
&\quad-a_1 \bullet \alpha_i(a_2, a_3) + a_2 \bullet \alpha_i(a_1, a_3) + 2a_3 \bullet \alpha_i(a_1, a_2)) = 0.
\end{aligned}$$

Consequently, we have a decomposition $I = I_1 \cup I_2$ with

$$\varphi_i([x_1, x_3], x_2) + \varphi_i([x_2, x_3], x_1) = 0, \quad i \in I_1 \tag{3.13}$$

$$\alpha_i \in \mathscr{C}^2(A, V), \quad i \in I_2. \tag{3.14}$$

Note that (3.13) together with condition $\varphi_i \in S^2(L, M)^L$ implies $\varphi_i(L, L) \subseteq M^L$ for any $i \in I_1$. Applying to the condition (3.14) the symmetrizer $e - (13) + (123)$, we get

$$\begin{aligned} &a_1 \bullet \alpha_i(a_2, a_3) + a_2 \bullet \alpha_i(a_1, a_3) \\ &\quad = \tfrac{1}{3}(2\alpha_i(a_1a_2, a_3) - \alpha_i(a_2a_3, a_1) - \alpha_i(a_1a_3, a_2)), \quad i \in I_2. \end{aligned} \tag{3.15}$$

Taking into account (3.13)–(3.15), the cocycle equation can be rewritten as

$$\begin{aligned} &\sum_{i \in I}(\varphi_i([x_1, x_2], x_3) - \varphi_i([x_1, x_3], x_2) + \varphi_i([x_2, x_3], x_1)) \\ &\quad \otimes (\alpha_i(a_1a_2, a_3) + \alpha_i(a_1a_3, a_2) + \alpha_i(a_2a_3, a_1)) = 0. \end{aligned}$$

By Lemma 1.1, there is a decomposition $I = I_{11} \cup I_{12} \cup I_{21} \cup I_{22}$ such that

$$\begin{array}{lll} \varphi_i([x, y], z) = \varphi_i(x, [y, z]), & \varphi_i([x, y], z) + \curvearrowright = 0 & \text{for any } i \in I_{11}, \\ \varphi_i([x, y], z) = \varphi_i(x, [y, z]), & \alpha_i \in HC^1(A, V) & \text{for any } i \in I_{12}, \\ \varphi_i([x, y], z) + \curvearrowright = 0, & \alpha_i \in \mathscr{C}^2(A, V) & \text{for any } i \in I_{21}, \\ \alpha_i \in \mathscr{C}^2(A, V) \cap HC^1(A, V) & & \text{for any } i \in I_{22}. \end{array}$$

Evidently, $\varphi_i([L, L], L) = 0$ for any $i \in I_{11}$ and $\alpha_i \in \mathscr{P}_-(A, V)$ for any $i \in I_{22}$. All these four types of components are cocycles in $Z^2(L \otimes A, M \otimes V)$.

Therefore, the space of cocycles whose cohomology classes lie in $(H_L^2)''$ is as follows:

$$\begin{aligned} {}^L Z^{02} \simeq\ & S^2(L, M^L)^L \otimes HC^1(A, V) + \mathscr{S}^2(L, M) \otimes \mathscr{C}^2(A, V) \\ & + S^2(L, M)^L \otimes \mathscr{P}_-(A, V) + \mathrm{Poor}_+(L, M) \otimes C^2(A, V). \end{aligned}$$

(the four summands here correspond to the components indexed by I_{12}, I_{21}, I_{22} and I_{11} respectively; note that, in this case, we may let $\Psi = 0$ again).

Rewriting this as a direct sum, we get

$$\begin{aligned} & S^2(L, M^L)^L \otimes HC^1(A, V) \oplus \mathscr{S}^2(L, M) \otimes \frac{\mathscr{C}^2(A, V)}{\mathscr{P}_-(A, V)} \\ & \oplus \frac{S^2(L, M)^L}{S^2(L, M^L)^L} \otimes \mathscr{P}_-(A, V) \oplus \mathrm{Poor}_+(L, M) \otimes \frac{C^2(A, V)}{HC^1(A, V) + \mathscr{C}^2(A, V)}. \end{aligned}$$

And finally, one may show in the same fashion as previously, that all these cocycles are cohomologically independent, and the assertion of the Proposition follows. □

Summarizing Propositions 3.1 and 3.8, we obtain

$$H^2(L \otimes A; L, M \otimes V) \simeq (H_L^2)' \oplus (H_L^2)'',$$

where

$$\begin{aligned}(H_L^2)' \simeq \mathscr{B}(L, M) \otimes \frac{\mathrm{Har}^2(A, V)}{\mathscr{P}_+(A, V)} \\ \oplus C^2(L, M)^L \otimes \mathscr{P}_+(A, V) \oplus \mathscr{X}(L, M) \otimes \frac{\mathscr{A}(A, V)}{\mathscr{P}_+(A, V)} \\ \oplus \mathrm{Poor}_-(L, M) \\ \otimes \frac{S^2(A, V)}{\mathrm{Hom}(A, V) + D(A, V) + \mathrm{Har}^2(A, V) + \mathscr{A}(A, V)}\end{aligned}$$

and $(H_L^2)''$ is described by Proposition 3.8.

We conclude this section with enumeration (for the case of generic L) of all possible cocycles of rank 1, i.e. those which can be written in a form $\varphi \otimes \alpha \in \mathrm{Hom}(L^{\otimes 2}, M) \otimes \mathrm{Hom}(A^{\otimes 2}, V)$.

In view of (3.3), Propositions 3.1 and 3.8, it suffices to consider cocycles of rank 1 whose cohomology classes lie in $(H^2)''$ and which are independent modulo $(H_L^2)''$. Let us denote this space of cocycles by Z''.

Proposition 3.9. *Each element of Z'' is cohomologic to a sum of cocycles of the following two types*:

(i) $x \otimes a \wedge y \otimes b \mapsto \varphi(x, y) \otimes (a \bullet \beta(b) - b \bullet \beta(a))$, *where* $\varphi \in \mathrm{Sym}^2(L, M)$ *is such that* $\varphi([L, L], L) = 0$, *and* $\beta \in \mathrm{Hom}(A, V)$;

(ii) *as in* (i) *with* $\varphi \in S^2(L, M)$, *where* $2\varphi([x, y], z) = x \bullet \varphi(y, z) - y \bullet \varphi(x, z)$, *and* $\beta \in \mathrm{Der}(A, V)$.

Proof. Mainly repetition of previous arguments. □

Therefore, there are, in general, 13 types of cohomologically independent cocycles of rank 1 (7 coming from Proposition 3.1 + 4 coming from Proposition 3.8 + 2 coming from Proposition 3.9). Of course, in particular cases some of these types of cocycles may vanish.

We see that, for $H^2(L \otimes A; L, M \otimes V)$ and for $H^2(L \otimes A, M \otimes V)$, L abelian, it is possible (in both cases) to choose a basis consisting of rank 1 cocycles. In general this is, however, not true. The case of $H^2(W_1(n) \otimes A, W_1(n) \otimes A)$, where $W_1(n)$ is the Zassenhaus algebra of positive characteristic, treated in [18], shows that there are cocycles of rank 2 not cohomologic to (any sum of) cocycles of rank 1.

4. A sketch of a spectral sequence

The computations performed in preceding sections can be described (and generalized) in terms of a certain spectral sequence. Let us indicate briefly the main idea (hopefully, the full treatment with further applications will appear elsewhere).

One has a Cauchy formula

$$\wedge^n(L \otimes A) \simeq \bigoplus_{\lambda \vdash n} Y_\lambda(L) \otimes Y_{\lambda^\sim}(A),$$

where Y_λ is the Schur functor associated with the Young diagram λ, and $\lambda^\sim$ is the Young diagram obtained from λ by interchanging its rows and columns (see, e.g., [5, p. 121]).

Applying the functor $\mathrm{Hom}(\cdot, M \otimes V) \simeq \mathrm{Hom}(\cdot, M) \otimes \mathrm{Hom}(\cdot, V)$ to both sides of this isomorphism one gets a decomposition of the underlying modules in the Chevalley–Eilenberg complex:

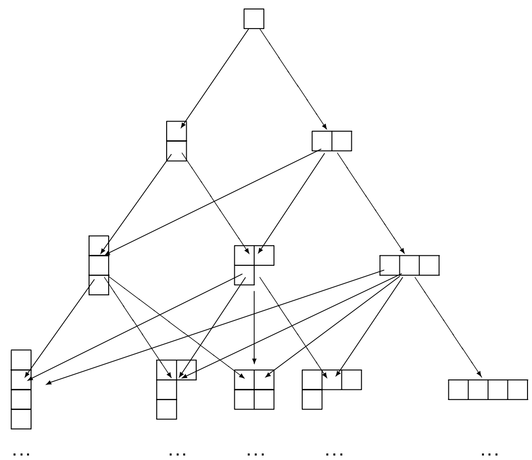
$$C^n(L \otimes A, M \otimes V) \simeq \sum_{\lambda \vdash n} C_\lambda(L, M) \otimes C_{\lambda^\sim}(A, V), \tag{4.1}$$

where $C_\lambda(U, W) = \mathrm{Hom}(Y_\lambda(U), W)$. The two extreme terms here are $C^n(L, M) \otimes S^n(A, V)$ and $S^n(L, M) \otimes C^n(A, V)$.

So each differential $d : C^n(L \otimes A, M \otimes V) \to C^{n+1}(L \otimes A, M \otimes V)$ in the Chevalley–Eilenberg complex decomposes according to (4.1) into components

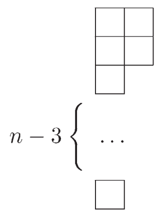
$$d_\lambda^{\lambda'} : C_{\lambda'}(L, M) \otimes C_{\lambda'^\sim}(A, V) \to C_\lambda(L, M) \otimes C_{\lambda^\sim}(A, V)$$

for each pair $\lambda' \vdash n$ and $\lambda \vdash (n + 1)$. Therefore the following graph of all Young diagrams



may be interpreted in the following way: each Young diagram λ of size n designates a module $C_\lambda(L, M) \otimes C_{\lambda^\sim}(A, V)$ and an arrow from λ' to λ represents $d_\lambda^{\lambda'}$.

One can prove that nonzero arrows $d_{\lambda}^{\lambda'}$ are exactly the following: all arrows going "from right to left" and those going "from left to right" for which either λ' is a column of height n and λ is a diagram of size $n+1$ and of the following shape:



or λ' is included in λ.

Using this, we can define a decreasing nonnegative filtration $F^k C^\star$ on the complex $(C^\star(L \otimes A, M \otimes V), d)$ as the sum of all terms $C_\lambda(L, M) \otimes C_{\lambda^\sim}(A, V)$ with λ belonging to a "closure" under nonzero arrows of a single column of height $k+1$.

Now we may consider a (first quadrant) spectral sequence $\{E_r^{\star\star}, d_r\}$ associated with this filtration. Since the filtration is finite in each degree, the spectral sequence converges to the desired cohomology group $H^\star(L \otimes A, M \otimes V)$.

Then $E_\infty^{20} = 0$ and $(H^2)'$ and $(H^2)''$ from Section 3 are nothing but E_∞^{11} and E_∞^{02}, respectively.

5. Structure functions

In this section we show how the result from Section 3 may be applied to the geometric problem of calculation of structure functions on manifolds of loops with values in compact Hermitian symmetric spaces.

Recall that the base field in this section is $\mathbb{C}$, what is stipulated by a geometric nature of the question considered. However, all algebraic considerations remain true over any field of characteristic 0.

Let us briefly recall the necessary notions and results. Let M be a complex manifold endowed with a G-structure (so G is a complex Lie group). *Structure functions* are sections of certain vector bundles over M. Their importance stems from the fact that they constitute the complete set of obstructions to integrability (=possibility of local flattening) of a given G-structure. In the case $G = \mathrm{O}(n)$ structure functions are known under the, perhaps, more common name *Riemann tensors* (and constitute one of the main objects of study in the Riemannian geometry).

A remarkable fact is that structure functions admit a purely algebraic description. Starting with $\mathfrak{g}_{-1} = T_m(M)$, the tangent space at a point $m \in M$, and $\mathfrak{g}_0 = \mathrm{Lie}(G)$, one may construct, via apparatus of Cartan prolongations, a graded Lie algebra $\mathfrak{g} = \bigoplus_{i \geqslant -1} \mathfrak{g}_i$. Namely, for $i > 0$, we have

$$\mathfrak{g}_i = \left\{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) \mid [X(v), w] = [X(w), v] \text{ for all } v, w \in \mathfrak{g}_{-1}\right\}. \tag{5.1}$$

For any such graded Lie algebra, one may define the *Spencer cohomology groups* $H^{pq}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$. Then the space of structure functions of order k, i.e. obstructions to identification of the kth infinitesimal neighborhood of a point $m \in M$ with that of a point of the manifold with a flat G-structure, is isomorphic to the group $H^{k2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$. Note that since $H^2(\mathfrak{g}_{-1}, \mathfrak{g}) = \bigoplus_{k \geqslant 1} H^{k2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, to compute structure functions for a given G-structure on a manifold, one merely needs to evaluate the usual Chevalley–Eilenberg cohomology group $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ of an abelian Lie algebra $\mathfrak{g}_{-1}$ with coefficients in the whole $\mathfrak{g}$ and to identify structure functions of order k with the graded component $\{\overline{\varphi} \in H^2(\mathfrak{g}_{-1}, \mathfrak{g}) \mid \operatorname{Im} \varphi \subseteq \mathfrak{g}_{k-2}\}$, $k \geqslant 1$. We refer for the classical text [17, Chapter VII] for details.

One of the nice examples of manifolds endowed with a G-structure are (irreducible) compact Hermitian symmetric spaces (CHSS). There are two naturally distinguishable cases: rank $M = 1$ and rank $M > 1$.

If rank $M = 1$, then $X = \mathbb{C}P^n$, a complex projective space. In this case $\mathfrak{g}$ turns out to be a general (infinite-dimensional) Lie algebra of Cartan type $W(n)$ with a standard grading of depth 1 (recall that $W(n)$ may be defined as a Lie algebra of derivations of the polynomial ring in n indeterminates, and consists of differential operators of the form $\sum f_i(x_1, \dots, x_n)\partial/\partial x_i$, $f_i(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$). The result of Serre about cohomology of involutive Lie algebras of vector fields (see [10] for the original Serre's letter and [13], Theorem 1 or [14, p. 9] for a more explicit formulation) implies that structure functions in this case vanish. We will refer for this case as a *rank one case*.

If rank $M > 1$, $\mathfrak{g}$ turns out to be a classical simple Lie algebra with a grading of depth 1 and length 1: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In particular, Cartan prolongations of order > 1 vanish, so we might only have structure functions of orders 1, 2 and 3 only (see [7], Proposition 4 or [8], Proposition 4.2). Corresponding structure functions were determined by Goncharov ([7, Theorem 1] or [8, Theorem 4.5]). We will refer for this case as a *general case*.

Remark. In the sequel we will need the following well-known fact: for any rank,

$$\{x \in \mathfrak{g}_i \mid [x, \mathfrak{g}_{-1}] = 0\} = 0, \quad i = 0, 1 \tag{5.2}$$

(this condition sometimes is referred as *transitivity* of the corresponding graded Lie algebra; see, e.g., [4] and references therein). In particular, $\mathfrak{g}_{-1}$ is a faithful $\mathfrak{g}_0$-module.

During the last decade, there was a big amount of activity by Grozman, Leites, Poletaeva, Serganova and Shchepochkina in determining structure functions of various classes of (super)manifolds and G-structures on them (see, e.g., [9,13,14] with a transitive closure of references therein).

Here we describe structure functions of manifolds M^{S^1} of loops with values in a (finite-dimensional) CHSS M. The group G here is formally no longer a Lie group, but its infinite-dimensional analogue, the group of loops, and the corresponding Lie algebra is a loop Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with a grading inherited from $\mathfrak{g}$:

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{i \geqslant -1} \mathfrak{g}_i \otimes \mathbb{C}[t, t^{-1}].$$

The last statement follows from the next simple but handy observation:

Proposition 5.1. *Let* $\bigoplus_{i \geqslant -1} \mathfrak{g}_i$ *be the Cartan prolongation of a pair* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$, *where* $\mathfrak{g}_{-1}, \mathfrak{g}_0$ *are finite-dimensional. Then* $\bigoplus_{i \geqslant -1}(\mathfrak{g}_i \otimes A)$ *is the Cartan prolongation of a pair* $(\mathfrak{g}_{-1} \otimes A, \mathfrak{g}_0 \otimes A)$.

Proof. Induction on i. As all $\mathfrak{g}_i$ are finite-dimensional, an element $X \in \mathrm{Hom}(\mathfrak{g}_{-1} \otimes A, \mathfrak{g}_{i-1} \otimes A)$ in inductive definition (5.1) of Cartan prolongation may be expressed in the form $\sum_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1})$, $\alpha_i \in \mathrm{End}(A)$. The rest goes as in the proof of Theorem 2.1. □

Thus, we shall obtain, so to speak, a "loopization" of Serre's and Goncharov's results.

In November 1993, Dmitry Leites showed to author a handwritten note by Elena Poletaeva containing computations of structure functions of manifolds of loops corresponding to the following two cases: the (rank one) case $\mathfrak{g} = W(1)$ and the (general) case $\mathfrak{g} = sl(4)$ with graded components $\mathfrak{g}_{-1} = V \otimes V^\star$, $\mathfrak{g}_0 = sl(2) \oplus gl(2)$, $\mathfrak{g}_1 = V^\star \otimes V$, where V is a standard two-dimensional $gl(2)$-module. Unfortunately, this note has never been published and seems to be lost, and more than 10 years later nobody from the involved parties cannot recollect the details. Though formally the main results of this section are generalizations of those Poletaeva's forgotten results, it should be noted that Poletaeva considered already the typical representatives in both—rank one and general—cases and observed all the main components and phenomena occurring in cohomology under consideration.

Definitions

(i) Structure functions (identified with elements of the second cohomology group) generated by cocycles of the form

$$(x \otimes a) \wedge (y \otimes b) \mapsto \varphi(x, y) \otimes abu, \quad x, y \in \mathfrak{g}_{-1}, a, b \in \mathbb{C}[t, t^{-1}],$$

where φ is a structure function of CHSS and $u \in \mathbb{C}[t, t^{-1}]$, will be called *induced*.

(ii) Structure functions generated by cocycles of the form

$$(x \otimes a) \wedge (y \otimes b) \mapsto \varphi(x, y) \otimes \alpha(a, b), \quad x, y \in \mathfrak{g}_{-1}, a, b \in \mathbb{C}[t, t^{-1}],$$

where $\varphi \in C^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ and $\alpha \in S^2(\mathbb{C}[t, t^{-1}], \mathbb{C}[t, t^{-1}])$, will be called *almost induced*.

(iii) Define a symmetric analogue of $H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, denoted as $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, to be a quotient space

$$\frac{S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})}{\{\varphi \in S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) | \varphi(x, y) = [x, \psi(y)] + [y, \psi(x)] \text{ for some } \psi \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0)\}}.$$

Clearly, induced and almost induced structure functions arise respectively from the direct summands $H^2(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes \mathbb{C}[t, t^{-1}]$ and $\mathscr{H}(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes S^2(\mathbb{C}[t, t^{-1}], \mathbb{C}[t, t^{-1}])$ of the cohomology group $H^2(\mathfrak{g}_{-1} \otimes \mathbb{C}[t, t^{-1}], \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])$ (see Remark after Theorem 3.7 and compare with a paragraph after the proof of Proposition 2.2 in [18]).

Theorem 5.2. *For the manifold M^{S^1} of loops with values in a CHSS M, the following hold*:

(i) *Structure functions can be only of order* 1, 2 *or* 3.
(ii) *The space of structure functions of order* 1 *modulo almost induced structure functions is isomorphic to*

$$B^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \frac{S^2(\mathbb{C}[t, t^{-1}], \mathbb{C}[t, t^{-1}])}{(\mathbb{C}1 \oplus \mathbb{C}\frac{d}{dt}) \otimes \mathbb{C}[t, t^{-1}]}$$
$$\oplus\, S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \otimes \frac{C^2(\mathbb{C}[t, t^{-1}], \mathbb{C}[t, t^{-1}])}{\mathrm{End}(\mathbb{C}[t, t^{-1}])}$$
$$\oplus\, SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \frac{\mathrm{End}(\mathbb{C}[t, t^{-1}])}{\mathbb{C}[t, t^{-1}]}.$$

(iii) *If* rank $M = 1$, *the third direct summand in the last expression vanish*.
(iv) *If* rank $M = 1$, *almost induced structure functions of order* 1 *and all structure functions of order* 2 *and* 3 *vanish*.
(v) *If* rank $M > 1$, *all structure functions of order* 2 *and* 3 *are induced*.

Remarks

(i) $B^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ is the space of corresponding Spencer coboundaries, i.e., the space of mappings $\varphi \in C^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ of the form $\varphi(x, y) = [x, \psi(y)] - [y, \psi(x)]$ for some $\psi \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.
(ii) Theorem 3.7 suggests the way in which denominator is embedded into numerator in the three quotient spaces involving $\mathbb{C}[t, t^{-1}]$ in (ii). In the first quotient space, the element $(\lambda 1 + \mu\frac{d}{dt})t^n \in (\mathbb{C}1 \oplus \mathbb{C}\frac{d}{dt}) \otimes \mathbb{C}[t, t^{-1}]$ corresponds to the mapping $\alpha \in S^2(\mathbb{C}[t, t^{-1}], \mathbb{C}[t, t^{-1}])$ defined by

$$\alpha(t^i, t^j) = \lambda t^{i+j+n} + \mu(i + j)t^{i+j+n-1}.$$

In the second one, the mapping $\beta(t^i) = \sum_n \lambda_{in} t^n \in \mathrm{End}(\mathbb{C}[t, t^{-1}])$ corresponds to the mapping $\alpha \in C^2(\mathbb{C}[t, t^{-1}], \mathbb{C}[t, t^{-1}])$ defined by

$$\alpha(t^i, t^j) = \sum_n (\lambda_{j,n-i} - \lambda_{i,n-j}) t^n.$$

In the third one, the element $t^n \in \mathbb{C}[t, t^{-1}]$ corresponds to the mapping $\beta \in \operatorname{End}(C[t, t^{-1}])$ which is multiplication by t^n:

$$\beta(t^i) = t^{i+n}.$$

Proof. Our task is to compute $H^2(\mathfrak{g}_{-1} \otimes \mathbb{C}[t, t^{-1}], \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])$ for an appropriate $\mathfrak{g}$. It turns out that the concrete structure of the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ is not important in our approach, and for notational convenience we replace it by an arbitrary (associative commutative unital) algebra A.

Substitute our specific data into the equation of Theorem 3.7:

$$\begin{aligned} H^2(\mathfrak{g}_{-1} \otimes A, \mathfrak{g} \otimes A) \simeq {} & H^2(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes A \oplus \mathscr{H}(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes \operatorname{Der}(A) \\ & \oplus C^2(\mathfrak{g}_{-1}, \mathfrak{g}^{\mathfrak{g}_{-1}}) \otimes \frac{S^2(A, A)}{A \oplus \operatorname{Der}(A)} \\ & \oplus S^2(\mathfrak{g}_{-1}, \mathfrak{g}^{\mathfrak{g}_{-1}}) \otimes \frac{C^2(A, A)}{\{a\beta(b) - b\beta(a) \mid \beta \in \operatorname{End}(A)\}} \\ & \oplus SH^2(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes \{a\beta(b) - b\beta(a) \mid \beta \in \operatorname{End}(A)\}. \end{aligned} \tag{5.3}$$

The next technical lemma is devoted to determination of components appearing in this isomorphism.

Lemma 5.3

(i) $\mathfrak{g}^{\mathfrak{g}_{-1}} = \mathfrak{g}_{-1}$,
(ii) $\mathscr{H}(\mathfrak{g}_{-1}, \mathfrak{g}) = H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$,
(iii) $SH^2(\mathfrak{g}_{-1}, \mathfrak{g}) = SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$.

Proof

(i) Evident in view of (5.2).
(ii) Follows from definitions of appropriate spaces, (5.2) and part (i).
(iii) The grading of $\mathfrak{g}$ induces grading of $SH^2(\mathfrak{g}_{-1}, \mathfrak{g})$:

$$SH^2(\mathfrak{g}_{-1}, \mathfrak{g}) = \bigoplus_{i \geqslant -1} SH^2_i(\mathfrak{g}_{-1}, \mathfrak{g}),$$

where

$$SH^2_i(\mathfrak{g}_{-1}, \mathfrak{g}) = (\operatorname{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) + SB^2(\mathfrak{g}_{-1}, \mathfrak{g}_i))/SB^2(\mathfrak{g}_{-1}, \mathfrak{g}_i),$$

$$\operatorname{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) = \big\{\varphi \in S^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \mid [x, \varphi(y, z)] = [y, \varphi(x, z)] \text{ for all } x, y, z \in \mathfrak{g}_{-1}\big\},$$

$$SB^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) = \big\{\varphi \in S^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \mid \varphi(x, y) = [x, \psi(y)] + [y, \psi(x)] \text{ for some } \psi \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i+1})\big\}.$$

We immediately see that $\mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) = S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$, and $\varphi(\cdot, y)$ belongs to the $(i + 1)$st Cartan prolongation of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ for each $\varphi \in \mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i)$, where $i \geqslant 0$, and $y \in \mathfrak{g}_{-1}$. Hence each $\varphi \in \mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i)$ can be written in the form

$$\varphi(x, y) = [x, F(\varphi, y)] \quad \text{for all } x, y \in \mathfrak{g}_{-1}, \tag{5.4}$$

for a certain bilinear map $F : \mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \times \mathfrak{g}_{-1} \to \mathfrak{g}_{i+1}$.

But the symmetry of φ implies that $F(\varphi, \cdot) \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i+1})$ belongs to the $(i + 2)$nd Cartan prolongation of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. Hence

$$F(\varphi, y) = [y, G(F, \varphi)] \quad \text{for all } \varphi \in \mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i),\ y \in \mathfrak{g}_{-1}, \tag{5.5}$$

for a certain bilinear map $G : \mathrm{Hom}(\mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \times \mathfrak{g}_{-1}, \mathfrak{g}_{i+1}) \times \mathfrak{g}_{-1} \to \mathfrak{g}_{i+2}$.

Combining (5.4) and (5.5) together, one gets $\varphi(x, y) = [x, [y, H(\varphi, y)]]$ for a certain bilinear map $H : \mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \times \mathfrak{g}_{-1} \to \mathfrak{g}_{i+2}$. Applying again symmetry of φ, we see that H is constant in a second argument, and hence each element $\varphi \in \mathrm{Sym}^2(\mathfrak{g}_{-1}, \mathfrak{g}_i)$ can be written in the form $\varphi(x, y) = [x, [y, h]]$ for an appropriate $h = H(\varphi, \cdot) \in \mathfrak{g}_{i+2}$.

But then $\varphi(x, y) = [x, \psi(y)] + [y, \psi(x)]$ for $\psi = -\frac{ad(h)}{2}$, and $SH^2_i(\mathfrak{g}_{-1}, \mathfrak{g}) = 0$ for $i \geqslant 0$.

Therefore, $SH^2(\mathfrak{g}_{-1}, \mathfrak{g})$ does not vanish only in the (-1)st graded component, and the desired equality follows. □

Continuation of the proof of Theorem 5.2. Substituting the results of Lemma 5.3 into (5.3), decomposing the Chevalley–Eilenberg cohomology $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ into the direct sum of corresponding Spencer cohomologies, and rearranging the summands as indicated in Remark after Theorem 3.7, we obtain

$$\begin{aligned} H^2(\mathfrak{g}_{-1} \otimes A, \mathfrak{g} \otimes A) \simeq\ & H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes S^2(A, A) \\ & \oplus \left(\bigoplus_{k>1} H^{k2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \right) \otimes A \\ & \oplus B^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \frac{S^2(A, A)}{A \oplus \mathrm{Der}(A)} \\ & \oplus S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \otimes \frac{C^2(A, A)}{\{a\beta(b) - b\beta(a) \mid \beta \in \mathrm{End}(A)\}} \\ & \oplus SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \{a\beta(b) - b\beta(a) \mid \beta \in \mathrm{End}(A)\}. \end{aligned}$$

The first tensor product here consists of almost induced structure functions of order 1 and the second one consists of induced structure functions of order > 1, what implies (ii).

As was noted earlier, in the rank one case the first and second tensor product vanishes (what follows from the Serre's theorem), what implies (iv). In the general case, the second tensor product reduces to structure functions of order 2 and 3 – that is, to $(H^{2,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \oplus H^{3,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})) \otimes A$. This proves (i) and (v) (well, after the final substitution $A = \mathbb{C}[t, t^{-1}]$).

Part (iii) follows from

Lemma 5.4 *For* $\mathfrak{g} = W(n)$ *with the standard grading*, $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) = 0$.

Proof. Denoting $\mathfrak{g}_{-1}$ as V, we have $\mathfrak{g}_0 = gl(V)$, and the statement reduces to the following: for any $\varphi \in S^2(V, V)$, there is a $\psi \in \mathrm{Hom}(V, gl(V))$ such that $\varphi(x, y) = \psi(x)(y) + \psi(y)(x)$. But this is obvious: take $\psi(x)(y) = \frac{1}{2}\varphi(x, y)$. □

Remark. In fact, this trivial reasoning shows that *any* linear mapping $V \times V \to V$, not necessarily symmetric one, may be represented in the form $\varphi(x, y) = \psi(x)(y) + \psi(y)(x)$ for certain $\psi \in \mathrm{Hom}(V, gl(V))$. In particular, it shows that the second Spencer cohomology $H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ vanishes for $\mathfrak{g} = W(n)$, which is a particular case of the Serre's theorem.

This completes the proof of the Theorem 5.2. □

Theorem 5.2 tells how to describe structure functions of manifolds of loops with values in CHSS in terms of structure functions of underlying CHSS (Spencer cohomology groups), and the space $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, which is a sort of a symmetric analogue of the Spencer cohomology group.

The thorough treatment of the latter symmetric analogue, including its calculation for various $\mathfrak{g}$'s, as well as related construction of a symmetric analogue of Cartan prolongation and some questions pertained to Jordan algebras and Leibniz cohomology, will, hopefully, appear elsewhere. Here we only briefly outline how $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ can be determined in the general case (i.e., for classical simple Lie algebras $\mathfrak{g}$) in terms of the corresponding root system.

All gradings of length 1 and depth 1 of classical simple Lie algebras may be obtained in the following way (see, e.g., [4]). Let R be a root system of $\mathfrak{g}$ corresponding to a Cartan subalgebra $\mathfrak{h}$, B a basis of R, $\{h_\beta, e_\alpha \mid \beta \in B, \alpha \in R\}$ a Chevalley basis of $\mathfrak{g}$. Let $N_{\alpha,\alpha'}$ be structure constants in this basis: $[e_\alpha, e_{\alpha'}] = N_{\alpha,\alpha'} e_{\alpha+\alpha'}$, $\alpha + \alpha' \in R$. Fix a root $\beta \in B$ such that β enters in decomposition of each root only with coefficients $-1, 0, 1$ (the existence of such root implies that R is not of type G_2, F_4 or E_8). Denote by R_i, $i = -1, 0, 1$, the set of roots in which β enters with coefficient i. Then

$$\mathfrak{g}_{-1} = \bigoplus_{\alpha \in R_{-1}} \mathbb{C}e_\alpha, \quad \mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_0} \mathbb{C}e_\alpha, \quad \mathfrak{g}_1 = \bigoplus_{\alpha \in R_1} \mathbb{C}e_\alpha.$$

Now, consider the mapping

$$T : \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0) \to S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$$
$$\psi(x) \mapsto (T\psi)(x, y) = [x, \psi(y)] + [y, \psi(x)].$$

The question of determining $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ evidently reduces to evaluation of $\operatorname{Ker} T$.

Writing

$$\psi(e_r) = \sum_{\alpha \in B} \lambda^r_\alpha h_\alpha + \sum_{\alpha \in R_0} \mu^r_\alpha e_\alpha$$

for $r \in R_{-1}$ and parameters $\lambda^r_\alpha, \mu^r_\alpha \in \mathbb{C}$, we see that the equation $[x, \psi(y)] + [y, \psi(x)] = 0$ is equivalent to the following three conditions:

$$\sum_{\alpha \in B} \lambda^s_\alpha r(h_\alpha) = \mu^r_{r-s} N_{s,r-s} \quad \text{for all } r, s \in R_{-1} \text{ such that } r - s \in R_0;$$
$$\sum_{\alpha \in B} \lambda^s_\alpha r(h_\alpha) = 0 \quad \text{for all } r, s \in R_{-1} \text{ such that } r - s \notin R;$$
$$\mu^r_\alpha N_{s,\alpha} = 0 \quad \text{for all } r, s \in R_{-1}, \alpha \in R_0 \text{ such that } r - s \neq \alpha$$

which serve as (linear) defining relations for the space $\operatorname{Ker} T$ and may be computed in each particular case.

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