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ON STRONG COMPACTNESS AND SUPERCOMPACTNESS *

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Introduction

The purpose of this paper is to study concepts relevant to a refutation of Solovay's conjecture that every strongly compact cardinal is supercompact. We obtain a counterexample to the conjecture by proving that every measurable cardinal that is a limit of strongly compact cardinals is itself strongly compact [Theorem 2.21], and that the least cardinal with this property (if it exists) is not supercompact. We in fact prove the stronger statement that the least cardinal with this property is not the limit of a stationary subset of measurable cardinals [Theorem 2.22]. [By a theorem of Solovay [15], if κ is 2^κ-supercompact, then the set $\{\alpha < \kappa: \alpha \text{ is a measurable cardinal}\}$ is a stationary subset of κ.] We then develop a general method for preserving certain strongly compact cardinals in suitable Cohen extensions, and present an outline of a proof that it is consistent relative to certain large cardinals that the first strongly compact is not supercompact. Jacques Stern [21] has used this method to obtain consistency results concerning the second strongly compact cardinal.

Shortly after the distribution of a preliminary version of our results, Magidor obtained by entirely different methods the much stronger result that it is consistent, relative to the consistency of the existence

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of a strongly compact cardinal, that the first strongly compact is in fact the first measurable [11]. This settles a problem of Tarski.

Our counterexample to Solovay's conjecture does not require a lengthy proof. However, many of the temporally preceding concepts are of interest in their own right, and we present them in Part 2 of this paper. Part 1 is a brief study of Jech's notion of a closed unbounded subset of $p_\kappa\lambda$. We present therein a characterization of the stationary subsets of $p_\kappa\lambda$.

Notation

We generally use standard set-theoretical notation throughout. Exceptions and expressions that do not enjoy uniform approval are clearly stated.

If A is a set,

$$\bigcap A = \{x : (\forall y \in A)(x \in y)\},$$

$$\bigcup A = \{x : (\exists y \in A)(x \in y)\}.$$

If A and B are sets, ${}^AB$ is the set of all functions with domain A and range a subset of B. For $f \in {}^AB$ and $W \subseteq A$,

$$f[W] = \{x : (\exists y \in W)(f(y) = x)\}.$$

$p(A)$ is the set of all subsets of A unless otherwise stated.

Small Greek letters almost always denote ordinals. Exceptions are clearly stated. Cardinals are initial ordinals and are usually denoted by the letters "κ", "ν", and "λ". If A is a set, $|A|$ is the cardinality of A. If κ and λ are cardinals, $p_\kappa\lambda$ is the set of all subsets of λ of cardinality less than κ, $\kappa^\lambda = |{}^\lambda\kappa|$, and $\lambda^{<\kappa} = |\bigcup\{{}^\alpha\lambda : \alpha < \kappa\}|$. We reserve the term "inaccessible" for strongly inaccessible cardinals. A cardinal κ is a strong limit if it is a limit cardinal and if in addition $2^\nu < \kappa$ for all $\nu < \kappa$. Henceforth κ, ν, and λ will be cardinals with κ regular and $\lambda \geqslant \kappa > \omega$.

We shall say that μ is a two-valued measure on a non-empty set X if $\mu : p(X) \to 2$ is a measure in the usual sense of the word. μ is non-principle if there is no $x \in X$ so that for every $A \subseteq X$, $\mu(A) = 1$ iff $x \in A$. μ is κ-additive if for every $\alpha < \kappa$ and $f : \alpha \to p(X)$ so that $\mu(f(\beta)) = 1$ for every $\beta < \alpha$, $\mu(\bigcap_{\beta<\alpha} f(\beta)) = 1$.

Let $\mathfrak{M}$ be a model of ZFC and $\tau(v_0, ..., v_n)$ a term of ZF. Define $\phi(v_0, ..., v_n, v_{n+1})$ to be the formula "$\tau(v_0, ..., v_n) = v_{n+1}$" of ZF. Then for $x_0, ..., x_n, x_{n+1}$ in $\mathfrak{M}$, we say that "in $\mathfrak{M}$, $\tau(x_0, ..., x_n) = x_{n+1}$" or "$\mathfrak{M} \models \tau(x_0, ..., x_n) = x_{n+1}$ to mean that $\mathfrak{M} \models \phi(x_0, ..., x_n, x_{n+1})$. Sometimes, when we are not working in $\mathfrak{M}$, we write "$x_{n+1} = \tau^{\mathfrak{M}}(x_0, ..., x_n)$" for "$\mathfrak{M} \models \tau(x_0, ..., x_n) = x_{n+1}$".

PART 1. THE CLOSED UNBOUNDED FILTER ON $p_\kappa\lambda$

In a paper on combinatorial properties of sets [3], Jech extends the concept of a closed unbounded subset of a regular cardinal κ to the broader notion of a closed unbounded subset of $p_\kappa\lambda$ for any cardinal $\lambda \geq \kappa$, and shows that many of the standard arguments concerning the former concept extend to the latter concept as well. In the same paper, Jech translates Jensen's combinatorial property $\diamond$ for a cardinal κ to the context of $p_\kappa\lambda$.

We briefly consider in Section 1 of this part a more stringent notion of closure and use it to obtain a characterization of the stationary subsets of $p_\kappa\lambda$. It is a well-known fact that if S is a subset of a regular cardinal κ, then S is not stationary iff there is a function $f: S \to \kappa$ so that $f(\alpha) < \alpha$ for all α in S and for every unbounded subset T of S, f is not constant on T. We prove that for every unbounded subset S of $p_\kappa\lambda$, S is not stationary iff there is a function $f: S \to \lambda \times \lambda$ so that $f(y) \in y \times y$ for all y in S and for every unbounded subset T of S, f is not constant on T. We also prove that the condition on f cannot be weakened to allow f to be a function from S into λ.

In Section 2 we use the characterization of the stationary subsets of $p_\kappa\lambda$ and the technique which Kunen developed [4] to show that every subtle cardinal has the diamond property, to prove the somewhat surprising result that if κ is subtle and λ is any cardinal $\geq \kappa$, then $p_\kappa\lambda$ has the generalized diamond property.

§1. A characterization of the stationary subsets of $p_\kappa\lambda$

1.1. Definition (Jech). Let U be a subset of $p_\kappa\lambda$. U is *unbounded* if for all z in $p_\kappa\lambda$, there is an x in U so that $z \subseteq x$; U is *directed* if for all x and y in U, there is a z in U so that $x \cup y \subseteq z$; U is *closed* if for all $A \subseteq U$ such that A is directed and $|A| < \kappa$, $\bigcup A$ is in U; and U is *stationary* if for all closed, unbounded subsets A of $p_\kappa\lambda$, $U \cap A \neq \emptyset$. If $\mathcal{W}$ is a family of subsets of $p_\kappa\lambda$, $\mathcal{W}$ is *closed under less than κ intersections* if for all $\nu<\kappa$ and $f:\nu\to\mathcal{W}$, $\bigcap_{\alpha<\nu} f(\alpha)\in\mathcal{W}$. $\mathcal{W}$ is *closed under diagonal intersections* if for all sequences $\langle A_\alpha;\alpha<\lambda\rangle$ such that $A_\alpha\in\mathcal{W}$ for all $\alpha<\lambda$, the set $\{x\in p_\kappa\lambda\colon (\forall\alpha\in x)(x\in A_\alpha)\}$ is in $\mathcal{W}$.

1.2. Theorem (Jech [3]). (i) *The family of closed unbounded subsets of $p_\kappa\lambda$ is closed under less than κ intersections and diagonal intersections.*

(ii) *If S is a stationary subset of $p_\kappa\lambda$ and $f:S\to\lambda$ is such that $f(x)\in x$ for all x in S, there is a stationary subset T of S so that f is constant on T.*

1.3. Definition. A subset U of $p_\kappa\lambda$ is *strongly closed* if it is closed and if in addition, for all non-empty $A\subseteq U$, $\bigcap A$ is in U. If $\mathcal{W}\subseteq\bigcup_{n<\omega}{}^{({}^n\lambda)}p_\kappa\lambda$ with $|\mathcal{W}|<\kappa$, $\mathcal{C}(\mathcal{W})=\{x\in p_\kappa\lambda\colon$ for all υ in $\mathcal{W}$ if υ is n-ary and σ is an n-tuple of x, then $\upsilon(\sigma)\subseteq x\}$.

Note that the set $\{x\in p_\kappa\lambda\colon x\neq\emptyset\}$ is a closed unbounded subset of $p_\kappa\lambda$ which is not strongly closed. It is easy to check that the family of strongly closed unbounded subsets of $p_\kappa\lambda$ is closed under less than κ intersections and diagonal intersections, and that if $\mathcal{W}\subseteq\bigcup_{n<\omega}{}^{({}^n\lambda)}p_\kappa\lambda$ is such that $|\mathcal{W}|<\kappa$, then $\mathcal{C}(\mathcal{W})$ is strongly closed and unbounded.

1.4. Proposition. *Let U be a strongly closed unbounded subset of $p_\kappa\lambda$. Then there is a $\mathcal{W}\subseteq{}^{({}^n\lambda)}p_\kappa\lambda$ so that $|\mathcal{W}|\leqslant\omega$ and $\mathcal{C}(\mathcal{W})=U$.*

Proof. For every $n<\omega$, define $\upsilon_n:{}^n\lambda\to p_\kappa\lambda$ so that for $\alpha_0,\ldots,\alpha_{n-1}$ in λ,

$$\upsilon_n(\alpha_0,\ldots,\alpha_{n-1})=\bigcap\{x\in U\colon\{\alpha_0,\ldots,\alpha_{n-1}\}\subseteq x\}\ .$$

Let $\mathcal{W}=\{\upsilon_n\colon n<\omega\}$. That $U\subseteq\mathcal{C}(\mathcal{W})$ is clear. If $x\in\mathcal{C}(\mathcal{W})$, then

$$A=\{y\in p_\kappa\lambda\colon(\exists n<\omega)(\exists\alpha_0,\ldots,\exists\alpha_{n-1}\in x)(\upsilon_n(\alpha_0,\ldots,\alpha_{n-1})=y\}$$

is a directed subset of U and $x=\bigcup A$. By closure of U, x is in U.

Before proceeding with the next theorem, we note that if $\upsilon:{}^n\lambda\to p_\kappa\lambda$ and $n>2$, there is a $\upsilon^*:\lambda\times\lambda\to p_\kappa\lambda$ so that $\mathcal{C}(\{\upsilon^*\})\subseteq\mathcal{C}(\{\upsilon\})$.

1.5. Theorem. *Let U be a closed unbounded subset of $p_\kappa\lambda$. There is a $w : \lambda \times \lambda \to p_\kappa\lambda$ such that $\mathcal{C}(\{w\}) \subseteq U$.*

Proof. For every $n < \omega$, define $v_n : {}^{n+1}\lambda \to p_\kappa\lambda$ by induction on n. For $n = 0$ and $\alpha < \lambda$, let $v_0(\alpha)$ be an element of U such that $\alpha \in v_0(\alpha)$. If v_k has been defined for all $k < n+1$ and $\alpha_0, ..., \alpha_n$ are in λ, let $v_{n+1}(\alpha_0, ..., \alpha_n)$ be an element of U such that

$$\bigcup\{x \in p_\kappa\lambda : (\exists k < n+1)(\exists \beta_0, ..., \beta_{k-1} \in \{\alpha_0, ..., \alpha_n\})(x = v_k(\beta_0, ..., \beta_{\kappa-1}))\}$$
$$\subseteq v_{n+1}(\alpha_0, ..., \alpha_n) .$$

Set $\mathcal{W} = \{v_n : n < \omega\}$.

If $x \in \mathcal{C}(\mathcal{W})$, then $A = \{y \in p_\kappa\lambda$: there is a $k < \omega$ and $\beta_0, ..., \beta_{k-1}$ in x so that $y = v_k(\beta_0, ..., \beta_{k-1})\}$ is a directed subset of U and $x = \bigcup A$. Hence, x is in U.

For $n < \omega$, let $v_n^* : \lambda \times \lambda \to p_\kappa\lambda$ be such that $\mathcal{C}(\{v_n^*\}) \subseteq \mathcal{C}(\{v_n\})$, and define $w : \lambda \times \lambda \to p_\kappa\lambda$ so that for α and β in λ, $w(\alpha, \beta) = \bigcup_{n<\omega} v_n^*(\alpha, \beta)$. Then $\mathcal{C}(\{w\}) \subseteq U$.

1.6. Corollary. *An unbounded subset S of $p_\kappa\lambda$ is not stationary iff there is a function $f : S \to \lambda \times \lambda$ so that $f(y) \in y \times y$ for all y in S, and for all unbounded $T \subseteq S$, f is not constant on T.*

Proof. The implication from right to left follows by applying Theorem 1.2(ii) successively to the two components of f.

For the reverse implication, let S be an unbounded non-stationary subset of $p_\kappa\lambda$. By the theorem, there is a $w : \lambda \times \lambda \to p_\kappa\lambda$ such that $\mathcal{C}(\{w\}) \subseteq p_\kappa\lambda - S$. Define $f : S \to \lambda \times \lambda$ so that for y in S, $f(y) = (\alpha_y, \beta_y)$ is a tuple in $y \times y$ such that $w(\alpha_y, \beta_y) \nsubseteq y$. Suppose there is an unbounded subset T of S such that f is constant on T, i.e., $f[T] = \{(\alpha, \beta)\}$, for some α and β in λ. Let z be any element of T such that $w(\alpha, \beta) \subseteq z$. Since z is in T, $f(z) = (\alpha, \beta)$ and by the definition of f, $w(\alpha, \beta) \nsubseteq z$, which is a contradiction.

Remark. If U is a closed unbounded subset of $p_\kappa\lambda$ so that for all x and y in U, $x \cup y$ is in U, then there is a $w : \lambda \to p_\kappa\lambda$ such that $\mathcal{C}(\{w\}) \subseteq U$. [For α in λ, let $w(\alpha)$ be an element of U such that $\alpha \in w(\alpha)$.]

The following proposition shows that the function f in Corollary 1.6 can not be assumed to have range in λ.

1.7. Proposition. *If $\lambda > \kappa$, there is a non-stationary unbounded subset S of $p_\kappa\lambda$ such that for all $f : S \to \lambda$ with $f(x) \in x$ for all x in S, there is an unbounded subset T of S with f constant on T.*

Proof. Let $l : \lambda \times \lambda \to \lambda$ be the standard Gödel enumeration of all ordered pairs of λ with the property that for any cardinal ν less than λ, $l[\nu \times \nu] = \nu$. We show that $S = p_\kappa\lambda - \mathcal{C}(\{l\})$ has the desired properties.

We first prove that there is no function $v : \lambda \to p_\kappa\lambda$ such that $\mathcal{C}(\{v\}) \subseteq \mathcal{C}(\{l\})$. Note that it suffices to prove this for $\lambda = \kappa^+$. (For if $v : \lambda \to p_\kappa\lambda$ is such that $\mathcal{C}(\{v\}) \subseteq \mathcal{C}(\{l\})$, then $v^* : \kappa^+ \to p_\kappa\kappa^+$ defined for $\alpha < \kappa^+$ by

$$v^*(\alpha) = \bigcap\{x \in \mathcal{C}(\{v\}) : \alpha \in x\} \cap \kappa^+$$

has the property that $\mathcal{C}(\{v^*\}) \subseteq \mathcal{C}(\{l \mid \kappa^+ \times \kappa^+\})$.) So assume that there is such a v and let

$$g(\beta) = (\text{least } \alpha < \kappa)(l(\beta, \alpha) \notin \bigcap\{x \in \mathcal{C}(\{v\}) : \beta \in x\})$$

for $\beta < \kappa^+$. There is a stationary subset A of κ^+ and an $\alpha < \kappa^+$, such that $g[A] = \{\alpha\}$. But then for β in A, $l(\beta, \alpha)$ is in $\bigcap\{x \in \mathcal{C}(\{v\}) : \alpha \in x\}$, which is absurd since $|A| = \kappa^+$.

Now suppose that $f : S \to \lambda$ is such that $f(x) \in x$ for all x in S and that for all $\alpha < \lambda$, $A_\alpha = \{x \in S : f(x) = \alpha\}$ is not unbounded. Define $v : \lambda \to p_\kappa\lambda$ so that for all $\alpha < \lambda$ there is no z in A_α with $v(\alpha) \subseteq z$. By the above observation, $\mathcal{C}(\{v\}) \nsubseteq \mathcal{C}(\{l\})$. If x is in $\mathcal{C}(\{v\}) - \mathcal{C}(\{l\})$, then x is in S and $v(f(x)) \subseteq x$, which contradicts the fact that x is in $A_{f(x)}$.

By Theorem 1.5, for every closed unbounded subset U of $p_\kappa\lambda$ there is a closed unbounded subset U^* of U, such that for all x and y in U^*, $x \cap y$ is in U^*. In contrast to this, Proposition 1.7 and the remark preceding it show the existence (for $\lambda > \kappa$) of a closed unbounded subset U of $p_\kappa\lambda$ such that for every closed unbounded subset U^* of U, there are x and y in U^* with $x \cup y$ not in U^*.

Let κ, ν and λ be cardinals such that $\kappa \leqslant \nu < \lambda$. If U is a closed unbounded subset of $p_\kappa\nu$, the set $\{y \in p_\kappa\lambda : y \cap \nu \in U\}$ is a closed unbounded subset of $p_\kappa\lambda$. However it is not difficult to construct a closed unbounded subset U of $p_\kappa\lambda$ so that $U_\nu = \{x \in p_\kappa\nu : (\exists y \in U)(y \cap \nu = x)\}$ is not closed. The following proposition shows that U_ν is closed if U is strongly closed.

1.8. Proposition. *If U is a strongly closed unbounded subset of $p_\kappa\lambda$, the set $U_\nu = \{x \in p_\kappa\nu : (\exists y \in U)(y \cap \nu = x)\}$ is a strongly closed unbounded subset of $p_\kappa\nu$.*

Proof. By a lemma of Solovay [18], it suffices to show that U_ν is closed under increasing sequences of length less than κ.

Let $\langle x_\alpha ; \alpha < \delta\rangle$ be an increasing sequence of elements of U_ν, where $\delta < \kappa$ (for $\alpha < \beta < \delta$, $x_\alpha \subseteq x_\beta$). For every $\alpha < \delta$ select an element y_α of U such that $y_\alpha \cap \nu = x_\alpha$ and define $z_\alpha = \bigcap_{\beta > \alpha} y_\beta$. By the strong closure of U, z_α is in U for all $\alpha < \delta$, and for $\alpha < \beta < \delta$, $z_\alpha \subseteq z_\beta$. Then $z = \bigcup_{\alpha<\delta} z_\alpha$ is in U and $z \cap \nu = \bigcup_{\alpha<\delta} x_\alpha$ is in U_ν.

1.9. Corollary. *If U is a stationary subset of $p_\kappa\nu$, $\{y \in p_\kappa\lambda : y \cap \nu \in U\}$ is a stationary subset of $p_\kappa\lambda$.*

1.10. We remark without proof that if $2^{<\kappa} < \lambda^{<\kappa}$, there are exactly $2^{\lambda^{<\kappa}}$ closed unbounded and 2^λ strongly closed unbounded subsets of $p_\kappa\lambda$. Theorem 1.5 shows that the two families generate the same filter.

§ 2. The generalized diamond property

In a set of notes on the constructible universe and on combinatorial properties of cardinals, jointly authored by Jensen and Kunen, Kunen introduces the concept of subtlety and proves that every subtle cardinal has the diamond property. A cardinal κ is *subtle* if for every function $s : \kappa \to p(\kappa)$ such that $s(\alpha) \subseteq \alpha$ for all α in κ, and for every closed unbounded subset U of κ, there are $\alpha < \beta$ in U such that $s(\alpha) = s(\beta) \cap \alpha$. The diamond property is due to Jensen, who first used it to construct a Suslin tree in the constructible universe. A cardinal κ *has the diamond property*, $\diamond(\kappa)$, if there is a function $s : \kappa \to p(\kappa)$ such that for all α in κ, $s(\alpha) \subseteq \alpha$, and for all subsets A of κ, $\{\alpha \in \kappa : A \cap \alpha = s(\alpha)\}$ is a stationary subset of κ. Jech translates this to the context of $p_\kappa\lambda$ as follows: $p_\kappa\lambda$

has the diamond property, $\diamond(\kappa, \lambda)$, if there is a $t : p_\kappa\lambda \to p_\kappa\lambda$ such that $t(x) \subseteq x$ for all x in $p_\kappa\lambda$, and $\{x \in p_\kappa\lambda : t(x) = x \cap A\}$ is a stationary subset of $p_\kappa\lambda$ for all subsets A of λ. We will show in Theorem 1.14 that if κ is subtle and $\lambda \geqslant \kappa$, $p_\kappa\lambda$ has the diamond property.

1.11. Definition. κ is λ-*subtle* if for all functions $s : p_\kappa\lambda \to p_\kappa\lambda$ such that $s(x) \subseteq x$ for every x in $p_\kappa\lambda$, and for every closed unbounded subset U of $p_\kappa\lambda$, there are x and y in U such that $x \not\subseteq y$ and $s(x) = s(y) \cap x$.

1.12. Lemma. *If κ is subtle, then κ is λ-subtle for all cardinals $\lambda > \kappa$.*

Proof. Let U be a closed unbounded subset of $p_\kappa\lambda$ and $s : p_\kappa\lambda \to p_\kappa\lambda$ so that $s(x) \subseteq x$ for all x in $p_\kappa\lambda$. Select a sequence $\langle x_\alpha ; \alpha < \kappa\rangle$ of elements of U so that for all $\alpha < \beta < \kappa$, $x_\alpha \not\subseteq x_\beta$, and for all limit ordinals $\alpha < \kappa$, $x_\alpha = \bigcup_{\beta<\alpha} x_\beta$. Set $x = \bigcup_{\alpha<\kappa} x_\alpha$. Define a bijection $k : x \to \kappa$ and a function $h : \kappa \to \kappa$ so that for $\alpha < \kappa$, $k[x_\alpha]$ is an ordinal and $h(\alpha) = k[x_\alpha]$.

Since h is a monotone increasing continuous function, there is a closed unbounded subset C of κ, such that if α is in C, then α is a limit ordinal and $h(\alpha) = \alpha$. For α in C, define $t(\alpha) = k[s(x_\alpha)]$. By the subtlety of κ there are $\alpha < \beta$ in C such that $t(\alpha) = t(\beta) \cap \alpha$. Translating, we have $s(x_\alpha) = s(x_\beta) \cap x_\alpha$ and $x_\alpha \not\subseteq x_\beta$.

1.13. Lemma. *Let U be a closed unbounded subset of $p_\kappa\lambda$ and $l : \lambda \times \lambda \to \lambda$ a bijection. There are functions $h : \lambda \to \kappa$ and $g : \lambda \to \lambda$ such that for all x in $p_\kappa\lambda$, if $x \cap \kappa$ is an ordinal and x is in $\mathcal{C}(\{h, l, g\})$, then x is in U.*

Proof. By Theorem 1.5, there is a $w : \lambda \times \lambda \to p_\kappa\lambda$ such that $\mathcal{C}(\{w\}) \subseteq U$.

Define functions $\pi_0, \pi_1 : \lambda \times \lambda \to \lambda$ and $h : \lambda \to \kappa$ such that for all α and β in λ, $\pi_0(l(\alpha, \beta)) = \alpha$, $\pi_1(l(\alpha, \beta)) = \beta$, and $h(l(\alpha, \beta)) = |w(\alpha, \beta)|$. For α and β in λ, let $k_{l(\alpha,\beta)} : h(l(\alpha, \beta)) \to w(\alpha, \beta)$ be a bijection. For all α and β in λ, let $g(l(\alpha, \beta)) = k_\alpha(\beta)$ if defined, otherwise $g(l(\alpha, \beta)) = 0$.

The proof of the following theorem is a generalization of Kunen's proof that every subtle cardinal has the diamond property.

1.14. Theorem. *If κ is subtle and $\lambda > \kappa$, $p_\kappa\lambda$ has the diamond property.*

Proof. Let $L = \{x \in p_\kappa\lambda : x \in \mathcal{C}(\{l\})$ and $x \cap \kappa$ is an ordinal$\}$, where $l : \lambda \times \lambda \to \lambda$ is a bijection. L is a closed unbounded subset of $p_\kappa\lambda$. For x and y in L, define $x < y$ if $x \subseteq y$ and $x \cap \kappa < y \cap \kappa$.

We will define for all x in L, $h_x : x \to x \cap \kappa$, $g_x : x \to x$, and $t(x) \subseteq x$ by induction on $\kappa \cap x$. Suppose y is in L and h_x, g_x and $t(x)$ have been defined for all x in L with $x < y$.

Case I. There are $h : y \to y \cap \kappa$, $g : y \to y$, and $a \subseteq y$ such that for all x in $L \cap \mathcal{C}(\{h, g\})$, if $x < y$, then $t(x) \neq a \cap x$. Set $h_y = h$, $g_y = g$, and $t(y) = a$.

Case II. Suppose Case I does not hold. Then let $h_y : y \to \{0\}$, $g_y : y \to \{0\}$, and $t(y) = \emptyset$.

We claim that for all subsets A of λ, the set $S_A = \{x \in p_\kappa\lambda : x \cap A = t(x)\}$ is stationary. Suppose not. Then by Lemma 1.13, there is a subset A of λ and functions $h' : \lambda \to \kappa$ and $g' : \lambda \to \lambda$ such that

$$E = L \cap \mathcal{C}(\{h', g'\}) \subseteq p_\kappa\lambda - S_A .$$

For all x in L define

$$s(x) = \{l(\alpha, \gamma) : (\exists \beta \in x)\, ((\alpha = l(0, \beta) < h_x(\beta) = \gamma)$$
$$\vee\, (\alpha = l(1, \beta) < g_x(\beta) = \gamma) \vee (\alpha = l(2, \beta) \wedge \gamma \in t(x)))\}.$$

By Lemma 1.12, there are x and y in E such that $x < y$ and $s(x) = s(y) \cap x$. [In the proof of Lemma 1.12 we could take the sequence $\langle x_\alpha : \alpha < \kappa\rangle$ so that for all $\alpha < \beta < \kappa$, $x_\alpha \cap \kappa < x_\beta \cap \kappa$.] Then $h_y \restriction x = h_x$, $g_y \restriction y = g_x$, and $t(x) = t(y) \cap x$.

Let $h'_y = h' \restriction y$, $g'_y = g' \restriction y$, and $a = A \cap y$. By our assumptions on h', g', and A, for all z in $L \cap \mathcal{C}(\{h'_y, g'_y\})$ if $z < y$, then $t(z) \neq a \cap z$. It follows that in the definition of h_y, g_y, and $t(y)$, case I holds. This contradicts the fact that $t(x) = t(y) \cap x$.

Theorem 1.14 is somewhat surprising because it shows that the diamond property of $p_\kappa\lambda$ can not in general be destroyed by forcing conditions that are κ^+-closed.

1.15. Corollary. *If κ is subtle and λ is any cardinal $\geqslant \kappa$, there are 2^λ almost disjoint stationary subsets of $p_\kappa\lambda$.*

[Two subsets A and B of $p_\kappa\lambda$ are *almost disjoint* if $A \cap B$ is not stationary.]

Proof. Jech [6] shows that if $p_\kappa\lambda$ has the diamond property, there are 2^λ almost disjoint stationary subsets of $p_\kappa\lambda$.

1.16. Corollary. *If κ is subtle and λ is any cardinal $\geqslant \kappa$, $p_\kappa\lambda$ is the disjoint union of $\lambda^{<\kappa}$ stationary subsets.*

Proof. For every x in $p_\kappa\lambda$, let $A_x = \{y \supseteq x\colon t_y = x\}$. By the theorem, the A_x's are stationary. Also if x and y are in $p_\kappa\lambda$ and $x \neq y$, then $A_x \cap A_y = \emptyset$.

If λ is regular, $p_\kappa\lambda$ can always be split into λ disjoint stationary subsets. Also Jech has shown that if κ is a successor cardinal and λ is regular, then any stationary subset of $p_\kappa\lambda$ is the disjoint union of λ stationary subsets. A theorem of Solovay shows that if λ is regular, any stationary subset of λ is the disjoint union of λ stationary subsets. We conjecture that if κ is regular and λ is any cardinal greater than κ, any stationary subset of $p_\kappa\lambda$ can be split into $\lambda^{<\kappa}$ disjoint stationary subsets.

Part 2. FINE MEASURES ON $p_\kappa\lambda$

In [6], Keisler and Tarski introduce the concept of strong compactness as an extension of the ω-compactness property of first order logic and study several equivalent notions. Working in a different vein, Solovay introduces in [15] the concept of supercompactness as an extension of a property of normal measures on a measurable cardinal and conjectures that the two concepts are essentially the same. We prove in this part that Solovay's conjecture is false and study fine, minimal and normal measures on $p_\kappa\lambda$.

Preliminaries are to be found in Section1. In Section 2 we consider certain related conjectures and produce counterexamples. We also prove that the existence of an extendible cardinal implies the existence of a great number of strongly compact cardinals that are not supercompact. In Section 3 we show how to preserve strong compactness in certain Cohen extensions and use this technique to outline the construction of a countable standard model of ZFC in which there is exactly one strongly compact cardinal and no supercompacts.

§ 1. Preliminaries

2.1. Definition. A two-valued measure μ on $p_\kappa\lambda$ is *fine* if it is κ-additive and if for all $\alpha < \lambda$, $\mu(\{x \in p_\kappa\lambda \colon \alpha \in x\}) = 1$. κ is λ-*strongly compact* if there is a fine measure on $p_\kappa\lambda$ and *strongly compact* if there is a fine measure on $p_\kappa\lambda$ for all $\lambda \geqslant \kappa$.

There are several other equivalent formulations of strong compactness, the most notable among these being the κ-compactness of the infinitary language $L_{\kappa,\kappa}$. We refer the reader to Keisler–Tarski [6] and to Jech [3].

2.2. Definition. A fine measure μ on $p_\kappa\lambda$ is *normal* if for every function f from $p_\kappa\lambda$ into λ, if $\mu(\{x \in p_\kappa\lambda \colon f(x) \in x\}) = 1$, then for some $\alpha < \lambda$,

$\mu(\{x \in p_\kappa\lambda : f(x) = \alpha\}) = 1$. κ is *λ-supercompact* if there is a normal measure on $p_\kappa\lambda$ and *supercompact* if there is a normal measure on $p_\kappa\lambda$ for all $\lambda \geqslant \kappa$.

2.3. Definition. A set M is closed under *λ-sequences* if every function from λ into M is in M. If μ is an $\aleph_1$-additive measure on a set X, the *closure* of μ (abbreviated to "clos (μ)") is the least cardinal λ such that the transitive collapse of the ultrapower of the universe with respect to μ is closed under ν-sequences for every $\nu < \lambda$. [We assume that the reader is familiar with the elementary ultrapower techniques. If μ is an $\aleph_1$-additive measure on a set X, "$j : V \to M \approx V^X/\mu$" will always mean that j is the canonical elementary embedding of the universe into the transitive collapse M of the ultrapower of the universe with respect to μ. All elementary embeddings will be assumed to be with respect to the relation $\in$.]

2.4. Theorem (Reinhardt–Solovay [15]). *κ is supercompact iff for all $\lambda \geqslant \kappa$, there is an elementary embedding $j : \langle V;\in\rangle \to \langle M;\in\rangle$ so that M is closed under λ-sequences, $j(\kappa) > \lambda$, and κ is the least ordinal so that $j(\kappa) > \kappa$.*

2.5. Let μ be a fine measure on $p_\kappa\lambda$ and $j : V \to M \simeq V^{p_\kappa\lambda}/\mu$. If f is a function with domain $p_\kappa\lambda$, $\ulcorner f \urcorner^\mu$ will always be the element of M that corresponds to the equivalence class of f with respect to μ. We omit the "μ" when no confusion results. If c is in M, $\langle c_x ; x \in p_\kappa\lambda\rangle$ will be some function such that $\ulcorner\langle c_x ; x \in p_\kappa\lambda\rangle\urcorner = c$.

Let ν be a cardinal so that $\kappa \leqslant \nu \leqslant \lambda$, and $q : p_\kappa\lambda \to p_\kappa\nu$ so that for all α in ν, $\mu(\{x \in p_\kappa\lambda : \alpha \in q(x)\}) = 1$. Then $q_*(\mu)$ is a fine measure on $p_\kappa\nu$ ($q_*(\mu)$ is defined so that for any subset A of $p_\kappa\nu$, $q_*(\mu)(A) = 1$ iff $\mu(\{x \in p_\kappa\lambda : q(x) \in A\}) = 1$). Let $j_0 : V \to M_0 \approx V^{p_\kappa\lambda}/q_*(\mu)$, and define $k : M_0 \to M$ so that for any function f with domain $p_\kappa\nu$, $k(\ulcorner f \urcorner^{q_*(\mu)}) = \ulcorner f \circ g \urcorner^\mu$. k is an elementary embedding of M_0 into M, and $k \circ j_0 = j$.

Now suppose that μ is normal. A theorem of Solovay [15] shows that $\mathrm{clos}(\mu) = (\lambda^{<\kappa})^+$. A simple induction shows that for all $\alpha \leqslant \lambda$, $\ulcorner\langle \circ(x \cap \alpha); x \in p_\kappa\lambda\rangle\urcorner = \alpha$ (where for y a set of ordinals, $\circ(y)$ is the order type of y with respect to the well ordering induced by $\in$). We define $(2^{\lambda^{<\kappa}})^M$ and $(2^{\lambda^{<\kappa}})^{+M}$ to be the ordinals γ and δ, respectively, such that $M \models$ (γ is the cardinality of the power set of $\lambda^{<\kappa}$, and δ is the least cardinal greater than γ). Since M is closed under $\lambda^{<\kappa}$-sequences and $j(\kappa)$ is inaccessible in M, $2^{\lambda^{<\kappa}} \leqslant (2^{\lambda^{<\kappa}})^M$ and $(2^{\lambda^{<\kappa}})^{+M} < j(\kappa)$. By the usual cardinality arguments, $j(\kappa) < (2^{\lambda^{<\kappa}})^+$. It follows that $(2^{\lambda^{<\kappa}})^{+M} < (2^{\lambda^{<\kappa}})^+$.

To continue the discussion, suppose that $\nu < \lambda$ and that $q(y) = y \cap \nu$ for all y in $p_\kappa\lambda$. Then $q_*(\mu)$ is a normal measure on $p_\kappa \nu$.

2.6. Proposition. *If* $\lambda \geqslant 2^{\nu^{<\kappa}}$, *the least ordinal moved by* k *is* $(2^{\nu^{<\kappa}})^{+M_0}$.

Proof. First note that for all $\alpha \leqslant \nu$,

$$k(\alpha) = k(\ulcorner\langle \mathrm{o}(x \cap \alpha); x \in p_\kappa \nu\rangle\urcorner^{q_*(\mu)})$$

$$= \ulcorner\langle \mathrm{o}((y \cap \nu) \cap \alpha); y \in p_\kappa \lambda\rangle\urcorner^{\mu}$$

$$= \ulcorner\langle \mathrm{o}(y \cap \alpha); y \in p_\kappa \lambda\rangle\urcorner^{\mu} = \alpha.$$

It then follows that for all x in $p_\kappa\nu$, $k(x) = x$, and for all subsets A of $p_\kappa\nu$, $k(A) = A$. Also, if W is a subset of $p(p_\kappa\nu)$ and W is in M_0, $k(W) = W$.

Now suppose that there is an ordinal less than $(2^{\nu^{<\kappa}})^{+M_0}$ that is moved by k and that α is the least ordinal with this property. There is a subset W of $p(p_\kappa\nu)$ in M_0 and a bijection $f: W \to \alpha$ also in M_0. $M \models$ ($k(f)$ is a bijection from $k(W)$ into $k(\alpha)$. Since $k(W) = W$, there is an $x \in W$ so that $k(f)(x) = \alpha$. Also there is a $\beta < \alpha$ so that $f(x) = \beta$. Since $k(x) = x$, $k(f)(x) = k(\beta)$ and $k(\beta) = \alpha$. Then β is moved by k, which is a contradiction.

Since $\lambda \geqslant 2^{\nu^{<\kappa}}$ and M is λ-closed we have $(2^{\nu^{<\kappa}})^{+M} = (2^{\nu^{<\kappa}})^+$. Hence $(2^{\nu^{<\kappa}})^{+M} = (2^{\nu^{<\kappa}})^+ > (2^{\nu^{<\kappa}})^{+M_0}$ (by the remarks at the end of 2.5); thus $k((2^{\nu^{<\kappa}})^{+M_0}) > (2^{\nu^{<\kappa}})^{+M_0}$.

A theorem of Solovay states that every measurable cardinal κ has a normal measure μ so that $\mu(\{\alpha < \kappa: \alpha \text{ is not measurable}\}) = 1$. Another way of expressing this is that κ is not measurable in the ultrapower of the universe with respect to μ. The following proposition is an analogous result for the case when κ is λ-supercompact.

2.7. Proposition. *Suppose κ is λ-supercompact. There is a normal measure μ on $p_\kappa\lambda$ such that κ is not λ-supercompact in the transitive collapse of the ultrapower of the universe with respect to μ.*

Proof. We get the desired measure μ by taking μ normal on $p_\kappa\lambda$ such that $j(\kappa)$ is minimal. (Here $j : V \to M \simeq V^{p_\kappa\lambda}/\mu$.)

Suppose that $M \models$ (μ^* is a normal measure on $p_\kappa\lambda$ and for $j^* : V \to M^* \simeq V^{p_\kappa\lambda}/\mu^*$, $j^*(\kappa) = \gamma$). Since M is closed under $\lambda^{<\kappa}$-sequences, μ^* is a normal measure on $p_\kappa\lambda$ and $j'(\kappa) = \gamma$ for the canonical $j' : V \to M' \simeq V^{p_\kappa\lambda}/\mu^*$. But $\gamma < (2^{\lambda^{<\kappa}})^{+M} < j(\kappa)$, which is a contradiction.

§2. The existence of a strongly compact cardinal that is not supercompact

Let κ be a measurable cardinal and μ a fine measure on $p_\kappa\kappa$. Then two important properties that hold for any such κ and μ are that there is a function $q : p_\kappa\kappa \to p_\kappa\kappa$ such that $q_*(\mu)$ is a normal measure on $p_\kappa\kappa$ and that $\operatorname{clos}(\mu) = \kappa^+$. These had been two of the more cogent arguments by analogy for the conjecture that every strongly compact cardinal is supercompact.

We generalize these properties to the context of $p_\kappa\lambda$ as follows:

2.8. Definition. Suppose that κ is λ-strongly compact. We say that $A(\kappa, \lambda)$ holds if for every fine measure μ on $p_\kappa\lambda$, there is a $q : p_\kappa\lambda \to p_\kappa\lambda$ such that $q_*(\mu)$ is a normal measure on $p_\kappa\lambda$; and that $B(\kappa, \lambda)$ holds if for every fine measure μ on $p_\kappa\lambda$, $\operatorname{clos}(\mu) > \lambda$.

Let us first note that $B(\kappa, \lambda)$ implies $A(\kappa, \lambda)$. Suppose that μ is a fine measure on $p_\kappa\lambda$ with $\operatorname{clos}(\mu) > \lambda$. Let $j : V \to M \simeq V^{p_\kappa\lambda}/\mu$ be the canonical embedding and let q be a function with domain $p_\kappa\lambda$ such that $\ulcorner q \urcorner = j[\lambda]$. An easy check establishes that $q_*(\mu)$ is a normal measure on $p_\kappa\lambda$.

In subsection 2.2, we prove that $B(\kappa, \lambda)$ fails whenever κ is λ-strongly compact and $\lambda > \kappa$. We do not know if $A(\kappa, \lambda)$ ever holds for $\lambda > \kappa$. But we prove in subsection 2.3 that it fails whenever κ is strongly compact and κ is a limit of strongly compact cardinals. In subsection 2.1 we study the concept of a minimal fine measure on $p_\kappa\lambda$, which is related to the question of the relative strength of $A(\kappa, \lambda)$ and $B(\kappa, \lambda)$.

§2.1. *Minimal fine measures*

2.9. Definition. A fine measure μ on $p_\kappa\lambda$ is minimal if for all functions q from $p_\kappa\lambda$ into $p_\kappa\lambda$ such that $q_*(\mu)$ is a fine measure on $p_\kappa\lambda$, q is injective on a set of measure one (i.e., there is a subset A of $p_\kappa\lambda$ such that $\mu(A) = 1$ and for all x and y in A if $x \neq y$, $q(x) \neq q(y)$).

The relation of minimality to the conjectures $A(\kappa, \lambda)$ and $B(\kappa, \lambda)$ turns on the fact that for every minimal measure μ on $p_\kappa\lambda$ the existence of a function q from $p_\kappa\lambda$ into $p_\kappa\lambda$ such that $q_*(\mu)$ is a normal measure on $p_\kappa\lambda$, implies that $\text{clos}(\mu) > \lambda$. For the elementary embedding $k : M_0 \to M$ discussed in 2.5 (where $M \simeq V^{p_\kappa\lambda}/\mu$ and $M_0 \simeq V^{p_\kappa\lambda}/q_*(\mu)$) is in fact an isomorphism. Hence $\text{clos}(\mu) = \text{clos}(q_*(\mu)) = (\lambda^{<\kappa})^+$.

A recent result of Solovay states that the class of $\aleph_1$-additive measures is well-founded with respect to the Keisler ordering on measures (this ordering is discussed in [7]). It follows that if κ is λ-strongly compact, there is a minimal fine measure on $p_\kappa\lambda$.

For regular cardinals λ we will show how to obtain "canonical" minimal measures from fine measures on $p_\kappa\lambda$. We will also prove that if λ is regular or the cofinality of λ is less than κ, then every normal measure on $p_\kappa\lambda$ is minimal.

Remark. Let κ be a regular cardinal and $\lambda > \kappa$ so that $2^{<\kappa} < \lambda$. If A is any unbounded subset of $p_\kappa\lambda$, then $p_\kappa\lambda = \bigcup_{x\in A} p(x)$ and $|p(x)| < \lambda$ for every $x \in A$. It follows that $|A| = \lambda^{<\kappa}$. This observation, due to Solovay [19], will be tacitly used throughout the chapter.

The next proposition establishes a criterion for minimality.

2.10. Proposition. *Let λ be a regular cardinal, μ a fine measure on $p_\kappa\lambda$, $j : V \to M \simeq V^{p_\kappa\lambda}/\mu$, and $S : p_\kappa\lambda \to \lambda$ so that $\ulcorner S \urcorner = \sup(f[\lambda])$. If S is injective on a set of measure one, μ is minimal.*

Proof. Let A be a subset of $p_\kappa\lambda$ of measure one so that S is injective on

A, and let q be a function from $p_\kappa\lambda$ into $p_\kappa\lambda$ so that $q_*(\mu)$ is a fine measure on $p_\kappa\lambda$. Define a function $g : p_\kappa\lambda \to \lambda$ by

$$g(x) = \begin{cases} S(y) & \text{if there is a } y \text{ in } A \text{ such that} \\ & S(y) < S(x) \text{ and } q(y) = q(x), \\ 0 & \text{otherwise} \end{cases}$$

for all x in $p_\kappa\lambda$.

Suppose $g(x) > 0$ a.e. Since $g(x) < S(x)$ a.e., there is an $\alpha < \lambda$ so that for $E = \{x \in A: 0 < g(x) < \alpha\}$, $\mu(E) = 1$. The cardinality of $q[E]$ is $\lambda^{<\kappa}$, because it is an unbounded subset of $p_\kappa\lambda$. But $q[E] \subseteq q[\{y \in A: s(y) < \alpha\}]$, which has cardinality $\leq \alpha$.

We conclude that $g(x) = 0$ a.e. The set $\{x \in A: g(x) = 0\}$ has measure one with respect to μ and q is injective on it.

By a theorem of Solovay [18], every stationary subset of a regular cardinal ν is the disjoint union of ν stationary subsets. Henceforth we will assume that for every regular cardinal λ a function A_λ from λ into the family of all stationary subsets of λ has been chosen so that $\bigcup_{\alpha<\lambda} A_\lambda(\alpha) = \{\alpha < \lambda: \mathrm{cf}(\alpha) = \omega\}$ and for all α and β in λ, if $\alpha \neq \beta$, $A(\alpha) \cap A(\beta) = \emptyset$. We omit the subscript "λ" when no confusion results.

2.10. Definition. Let λ be a regular cardinal, μ a fine measure on $p_\kappa\lambda$, $j : V \to M \simeq V^{p_\kappa\lambda}/\mu$, and s a function from $p_\kappa\lambda$ into λ so that $\ulcorner s \urcorner = \sup(j[\lambda])$. The *minimal cover for* μ is the function q from $p_\kappa\lambda$ into $p_\kappa\lambda$ defined by $q(x) = \{\alpha < s(x): A_\lambda(\alpha) \cap s(x)$ is a stationary subset of $s(x)\}$ for all x in $p_\kappa\lambda$. q depends on the function A_λ, which is the reason that we assumed that an A_λ has been fixed for every regular cardinal λ. In addition q depends almost everywhere with respect to μ on the function s.

Solovay defined the function q (he does not use the term "minimal cover") and used it to show that $\lambda^{<\kappa} = \lambda$, which he later proved using a simpler technique [19]. Our interest in the function revived when we noted its potential use in connection with the conjecture $A(\kappa, \lambda)$.

The next theorem and its corollaries are also due to Solovay.

2.11. Theorem (Solovay). *Let λ be a regular cardinal and μ a fine meas-*

ure on $p_\kappa\lambda$. *If* q *is the minimal cover for* μ, $q_*(\mu)$ *is a fine measure on* $p_\kappa\lambda$. *If in addition* $\operatorname{clos}(\mu) > \lambda$, $q_*(\mu)$ *is normal.*

Proof. Let $j : V \to M \simeq V^{p_\kappa\lambda}/\mu$ and s a function from $p_\kappa\lambda$ into λ so that $\ulcorner s \urcorner = \sup(j[\lambda])$.

Suppose there is an α in λ so that $\mu(\{x \in p_\kappa\lambda\colon \alpha \in q(x)\}) = 0$. Then

$$\mu(\{x \in p_\kappa\lambda\colon A_\lambda(\alpha) \cap s(x) \text{ is not stationary in } s(x)\}) = 1\ ,$$

$$M \models (j(A_\lambda)(j(\alpha)) \cap \ulcorner s \urcorner \text{ is not a stationary subset of } \ulcorner s \urcorner)\ .$$

Let U be in M so that

$$M \models (U \text{ is a closed unbounded subset of } \ulcorner s \urcorner \text{ and } U \cap j(A_\lambda)(j(\alpha)) = \emptyset)$$

Define $U^* = \{\alpha < \lambda\colon j(\alpha) \in U\}$.

We show that U^* is an ω-closed unbounded subset of λ. To show that U^* is unbounded, let $\gamma < \lambda$ and let $\langle \gamma_n ; n < \omega\rangle$ be an increasing sequence of ordinals greater than γ and less than λ so that for all $n < \omega$, $[j(\gamma_n), j(\gamma_{n+1})] \cap U = \emptyset$. Set $\gamma_\omega = \bigcup_{n<\omega} \gamma_n$ and $\delta = \bigcup_{n<\omega} j(\gamma_n)$. Then $\langle j(\gamma_n); n < \omega\rangle$ is in M and δ is in E. Since

$$j(\gamma_\omega) = j(\textstyle\bigcup_{n<\omega} \gamma_n) = \quad _{n<\omega}\, j(\gamma_n) = \delta\ ,$$

δ is in U^*. The ω-closure of U^* is gotten by a similar and simpler argument.

Since U^* is an ω-closed unbounded subset of λ and $A_\lambda(\alpha)$ is a stationary subset of $\{\alpha < \lambda\colon \operatorname{cf}(\alpha) = \omega\}$, $U^* \cap A_\lambda(\alpha) \neq \emptyset$. But if β is in $A_\lambda(\alpha) \cap U^*$, then $j(\beta)$ is in $j(A_\lambda)(j(\alpha)) \cap U$, which is a contradiction. It follows that $q_*(\mu)$ is a fine measure on $p_\kappa\lambda$.

Now suppose that $\operatorname{clos}(\mu) > \lambda$. Let $t : p_\kappa\lambda \to p_\kappa\lambda$ be such that $\ulcorner t \urcorner = j[\lambda]$ and $f : p_\kappa\lambda \to \lambda$ so that $f(x) \in q(x)$ a.e. Then $M \models (\ulcorner t \urcorner$ is an ω-closed unbounded subset of $\ulcorner s \urcorner$ and $j(A_\lambda)(\ulcorner f \urcorner) \cap \ulcorner s \urcorner$ is a stationary subset of $\{\alpha < \ulcorner s \urcorner\colon \operatorname{cf}(\alpha) = \omega\})$. If $j(\alpha) \in j(A_\lambda)(\ulcorner f \urcorner) \cap \ulcorner t \urcorner$, then $\alpha \in A_\lambda(f(x))$ a.e. and $f(x) = \beta$ a.e., where $\alpha \in A_\beta$. Thus $\ulcorner q \urcorner \subseteq j[\lambda]$. The first part of the proof shows that $j[\lambda] \subseteq \ulcorner q \urcorner$. It follows that $\ulcorner q \urcorner = j[\lambda]$ and that $q_*(\mu)$ is normal.

Corollary (Solovay [19]). *Let* κ *be strongly compact. Then*

$$\lambda^{<\kappa} = \begin{cases} \lambda & \text{if } \operatorname{cf}(\lambda) \geq \kappa, \\ \lambda^+ & \text{if } \operatorname{cf}(\lambda) < \kappa. \end{cases}$$

For a proof, see Solovay [19].

Corollary (Solovay). *Suppose that in addition to the hypotheses of the theorem, μ is normal on $p_\kappa\lambda$. Then $q(x) = x$ a.e. and s is injective on a set of measure one.*

Proof. The theorem shows that $\ulcorner q \urcorner = j[\lambda]$. Since $\ulcorner \langle x : x \in p_\kappa\lambda\rangle \urcorner = j[\lambda]$, $q(x) = x$ a.e. If x and y are in $p_\kappa\lambda$ so that $x \neq y$, $q(x) = x$ and $q(y) = y$, then $s(x) \neq s(y)$.

2.14. Theorem. *Let λ be a regular cardinal and μ a fine measure on $p_\kappa\lambda$. If q is the minimal cover for μ, $q_*(\mu)$ is a minimal fine measure on $p_\kappa\lambda$.*

Proof. Let $j_0 : V \to M_0 \simeq V^{p_\kappa\lambda}/\mu$, $j_1 : V \to M_1 \simeq V^{p_\kappa\lambda}/q_*(\mu)$, $\ulcorner s_0 \urcorner^\mu = \sup(j_0[\lambda])$, $\ulcorner s_1 \urcorner^{q_*(\mu)} = \sup(j_1[\lambda])$, and $q_1 : p_\kappa\lambda \to p_\kappa\lambda$ the minimal cover for $q_*(\mu)$. The theorem will follow from Proposition 2.10 if we show that $q_1(x) = x$ a.e. with respect to $q_*(\mu)$.

We first show that $s_1(x) = \sup(x)$ a.e. with respect to $q_*(\mu)$. That $q_*(\mu)(\{x \in p_\kappa\lambda : s_1(x) \sup(x)\}) = 1$, follows from the fact that $q_*(\mu)$ is a fine measure on $p_\kappa\lambda$. Suppose $s_1(x) < \sup(x)$ a.e. with respect to $q_*(\mu)$. Then $s_1(q(x)) < \sup(q(x))$ a.e. with respect to μ. Since $j_0[\lambda] \subseteq \ulcorner q \urcorner \subseteq \ulcorner s_0 \urcorner^\mu$, $s_0(x) = \sup(q(x))$ a.e. with respect to μ. Thus there is an $\alpha < \lambda$ so that $s_1(q(x)) < \alpha$ a.e. with respect to μ. But then

$$q_*(\mu)(\{x \in p_\kappa\lambda : s_1(x) < \alpha\}) = 1,$$

which contradicts the definition of s_1.

Now $q(x) = \{\alpha < s_0(x) : A_\lambda(\alpha) \cap s_0(x) \text{ is stationary in } s_0(x)\} = \{\alpha < \sup(q(x)) : A_\lambda(\alpha) \cap \sup(q(x)) \text{ is stationary in } \sup(q(x))\}$ almost everywhere with respect to μ. Translating we have $x = \{\alpha < \sup(x) : A_\lambda(\alpha) \cap \sup(x) \text{ is stationary in } \sup(x)\}$ a.e. with respect to $q_*(\mu)$, i.e., $q_1(x) = x$ a.e. with respect to $q_*(\mu)$, which was to be shown.

2.15. Corollary. *Suppose λ is a regular cardinal so that $A(\kappa, \lambda)$ holds. Then for any fine measure μ on $p_\kappa\lambda$, if q is the minimal cover for μ, $q_*(\mu)$ is a normal measure on $p_\kappa\lambda$.*

The term "minimal cover" may be a bit misleading. For although we will not prove it, under certain conditions on the cardinal κ and the regular cardinal λ, there is a fine measure μ on $p_\kappa\lambda$ so that $q_*(\mu)$

is not a normal measure on $p_\kappa\lambda$ (where q is the minimal cover for μ), but there is a function q' from $p_\kappa\lambda$ into $p_\kappa\lambda$ so that $q'_*(\mu)$ is a normal measure on $p_\kappa\lambda$.

2.16. Proposition. *Suppose that either λ is a regular cardinal or the cofinality of λ is less than κ. Then every normal measure on $p_\kappa\lambda$ is minimal.*

Proof. For λ regular, the theorem follows from Proposition 2.10 and the corollary to 2.11.

Suppose that $\mathrm{cf}(\lambda) < \kappa$, that μ is a normal measure on $p_\kappa\lambda$, and that $j: V \to M \simeq V^{p_\kappa\lambda}/\mu$. Since $\lambda^{<\kappa} = \lambda^+$ and $\mathrm{clos}(\mu) = \lambda^{++}$, there is a function r from $p_\kappa\lambda$ to $p_\kappa\lambda^+$ so that $\ulcorner r \urcorner = j[\lambda^+]$. $M \models (j(\lambda) \cap j[\lambda^+] = j[\lambda])$. Hence there is subset E of $p_\kappa\lambda$ of measure one so that for all x in E, $r(x) \cap \lambda = x$ r is injective on E and $r_*(\mu)$ is a normal measure on $p_\kappa\lambda^+$ isomorphic to μ.

The minimality of μ will follow if we prove that for every $q : p_\kappa\lambda^+ \to p_\kappa\lambda$ so that $q_*(r_*(\mu))$ is a fine measure on $p_\kappa\lambda$, q is injective on a set of measure one with respect to $r_*(\mu)$. The argument needed is the one used in Proposition 2.10 and uses the fact that for every subset E of $p_\kappa\lambda^+$ so that $r_*(\mu)(E) = 1$, $|q[E]| = \lambda^{<\kappa} = \lambda^+$.

We do not know whether Proposition 2.16 is true for singular cardinals λ with cofinality greater than or equal to κ.

§2.2. *The failure of the conjecture $B(\kappa, \lambda)$*

2.17. Theorem. *Let κ be λ-supercompact and $\lambda \geqslant \nu \geqslant \kappa$. There is a fine measure μ on $p_\kappa\lambda$ so that* $\mathrm{clos}(\mu) = (\nu^{<\kappa})^+$.

Corollary. *Let κ be λ-strongly compact and $\lambda > \kappa$. Then $B(\kappa, \lambda)$ fails.*

To prove the theorem we first recall a technique of Keisler and prove a lemma.

2.18. Definition (Keisler). Let D and U be measures on the sets X and Y, respectively. The *product of D with U* is the measure $D \times U$ on $X \times Y$

defined by:

$$(D \times U)(A) = 1 \quad \text{iff} \quad U(\{y \in Y: D(\{x \in X: (x,y) \in A\}) = 1\}) = 1,$$

for all subsets A of $X \times Y$.

Keisler showed [5] that for any first-order structure $\mathfrak{A}$, $\mathfrak{A}^{X\times Y}/D \times U \simeq (\mathfrak{A}^X/D)^Y/U$.

A special case of the following lemma was proved independently by Ketonen in his dissertation [10]. His proof is much different.

2.19. Lemma. *Suppose* λ *and* ν *are cardinals* $\geqslant \kappa$, *and* μ_0 *and* μ_1 *are fine measures on* $p_\kappa\lambda$ *and* $p_\kappa\nu$, *respectively. Then* $\operatorname{clos}(\mu_0 \times \mu_1) = \operatorname{clos}(\mu_1)$.

Proof. Let

$$j_0 : V \to M_0 \simeq V^{p_\kappa\lambda}/\mu_0\,, \qquad j_1 : V \to M_1 \simeq V^{p_\kappa\nu}/\mu_1\,,$$

$$j : V \to M \simeq V^{p_\kappa\lambda \times p_\kappa\nu}/\mu_0 \times \mu_1\,, \qquad k : M_0 \to M \simeq M_0^{p_\kappa\nu}/\mu_1$$

be the usual canonical embeddings.

Suppose that $\operatorname{clos}(\mu_1) > \rho$, for some cardinal ρ. Let $p : p_\kappa\nu \to p_\kappa\rho$ be so that $j_1[\rho] = \ulcorner p \urcorner^{\mu_1}$, and define $q : p_\kappa\lambda \times p_\kappa\nu \to p_\kappa\rho$ by $q(x,y) = p(y)$ for all (x,y) in $p_\kappa\lambda \times p_\kappa\nu$.

If we show that $\ulcorner q \urcorner = j[\rho]$, it will follow by the usual ultrapower arguments that $\operatorname{clos}(\mu_0 \times \mu_1) > \rho$. It is clear that $j[\rho] \subseteq \ulcorner q \urcorner$.

To show that $\ulcorner q \urcorner \subseteq j[\rho]$, let f be a function from $p_\kappa\lambda \times p_\kappa\nu$ into ρ so that $f(x,y)$ is in $p(y)$ for almost all (x,y) in $p_\kappa\lambda \times p_\kappa\nu$. There is an A, subset of $p_\kappa\nu$, so that $\mu_1(A) = 1$ and so that for all y in A there is an A_y a subset of $p_\kappa\lambda$ with $\mu_0(A_y) = 1$ and $f(x,y) \in p(y)$ for all x in A_y. It follows that for every y in A there is an α_y in $p(y)$ and a subset A'_y of A_y so that $\mu_0(A'_y) = 1$ and $f(x,y) = \alpha_y$ for all x in A'_y. Then there is an A', a subset of A, and an α in ρ so that $\mu_1(A') = 1$ and $\alpha_y = \alpha$ for all y in A'. Then $f(x,y) = \alpha$ for almost all (x,y) in $p_\kappa\lambda \times p_\kappa\nu$.

Now suppose that $\operatorname{clos}(\mu_0 \times \mu_1)$ is greater than the cardinal ρ. Let q be a function from $p_\kappa\nu$ into $p_\kappa\rho$ so that $k[\rho] = \ulcorner q \urcorner^{\mu_1}$. An easy check shows that $j_1[\rho] = \ulcorner q \urcorner^{\mu_1}$ and that $\operatorname{clos}(\mu_1) > \rho$.

Proof of Theorem 2.17. Let μ_0 be a fine measure on $p_\kappa\lambda$, μ_1 a normal measure on $p_\kappa\nu$, and let $p_\kappa\lambda = \bigcup_{y\in p_\kappa\lambda} A_y$ so that for all x and y in $p_\kappa\lambda$, A_y is unbounded and if $x \neq y$, $A_x \cap A_y = \emptyset$. Define a function q from $p_\kappa\lambda \times p_\kappa\nu$ into $p_\kappa\lambda$ so that q is injective and for all x and y in $p_\kappa\nu$ and $p_\kappa\lambda$ respectively, $x \subseteq q(x,y) \in A_y$. The measure $\mu = q_*(\mu_0 \times \mu_1)$ is a fine measure on $p_\kappa\lambda$, and by the lemma, $\mathrm{clos}(\mu) = \mathrm{clos}(\mu_0 \times \mu_1) = \mathrm{clos}(\mu_1) = (\nu^{<\kappa})^+$.

§2.3. *The failure of the conjecture* $A(\kappa, \lambda)$

2.20. Theorem. *Suppose λ is a singular strong limit cardinal of cofinality greater than or equal to κ, and κ is λ-strongly compact. Then $A(\kappa,\lambda)$ fails.*

Proof. We may assume that κ is λ-supercompact. By Proposition 2.7, there is a normal measure μ on $p_\kappa\lambda$ so that if $j: V \to M \simeq V^{p_\kappa\lambda}/\mu$ is the canonical embedding associated with μ, $M \models$ (κ is not λ-supercompact).

By our assumptions on λ, $M \models$ (κ is ν-supercompact for every cardinal ν greater than κ and less than λ). It follows from work of Ketonen and others [7] that if α is a singular cardinal of cofinality $\geqslant \kappa$ and κ is β-strongly compact for all cardinals β greater than κ and less than α, then κ is α-strongly compact. Hence $M \models$ (κ is λ-strongly compact). Let U be a fine measure on $p_\kappa\lambda$ in M. Since M is $\lambda^{<\kappa}$-closed, the existence of a function q from $p_\kappa\lambda$ into $p_\kappa\lambda$ such that $q_*(U)$ is a normal measure on $p_\kappa\lambda$, would imply that κ is λ-supercompact in M which is a contradiction.

2.21. Theorem. *Suppose κ is a measurable cardinal which is a limit of strongly compact cardinals. Then κ is strongly compact. If the cofinality of λ is less than κ or λ is regular, $A(\kappa,\lambda)$ fails.*

Proof. Suppose λ is a regular cardinal greater than κ and U is a fine measure on $p_\kappa\kappa$ so that $U(\{\gamma < \kappa: \gamma \text{ is strongly compact}\}) = 1$. For every strongly compact cardinal γ less than κ, select a minimal fine measure μ_γ on $p_\gamma\lambda$ so that if $j_\gamma : V \to M_\gamma$ is the canonical embedding associated with μ_γ and q_γ is the minimal cover for μ_γ, then $\ulcorner\langle \sup(x); x \in p_\gamma\lambda\rangle\urcorner^{\mu_\gamma} = \sup(j_\gamma[\lambda])$ and $q_\gamma(x) = x$ a.e.

The measure μ on $p_\kappa\lambda$, defined for all subsets A of $p_\kappa\lambda$ so that

$\mu(A) = 1$ iff $U(\{\gamma < \kappa: \mu_\gamma(A \cap p_\gamma\lambda) = 1\}) = 1$, is a fine measure on $p_\kappa\lambda$. Let $j : V \to M \simeq V^{p_\kappa\lambda}/\mu$ be the usual embedding.

Suppose f is a function from $p_\kappa\lambda$ into λ so that $f(x) < \sup(x)$ for all x in $p_\kappa\lambda$. Then for almost all γ in κ (with respect to U), there is an α_γ in λ so that $\mu_\gamma(\{x \in p_\gamma\lambda: f(x) < \alpha_\gamma\}) = 1$. Then

$$\mu(\{x \in p_\kappa\lambda: f(x) \leqslant \bigcup_{\gamma<\kappa} \alpha_\gamma < \lambda\}) = 1 .$$

It follows that $\ulcorner\langle \sup(x); x \in p_\kappa\lambda\rangle\urcorner^\mu = \sup(j[\lambda])$.

Now let q be the minimal cover for μ. By our choice of the μ_γ's and the preceding observation, $q(x) = x$ a.e. with respect to μ. Since $\ulcorner\langle x \cap \kappa; x \in p_\kappa\lambda\rangle\urcorner^\mu$ is not an ordinal, μ is not a normal measure on $p_\kappa\lambda$. By Theorems 2.20 and 2.21, μ is a minimal fine measure on $p_\kappa\lambda$ and $\operatorname{clos}(\mu) \leqslant \lambda$. Then $A(\kappa, \lambda)$ fails.

Suppose $\operatorname{cf}(\lambda) < \kappa$. Define $q : p_\kappa\lambda^+ \to p_\kappa\lambda$ by $q(y) = y \cap \lambda$ for all y in $p_\kappa\lambda^+$, and let μ be a fine measure on $p_\kappa\lambda^+$ so that μ is minimal and $\operatorname{clos}(\mu) \leqslant \lambda$. The existence of a function r from $p_\kappa\lambda$ into $p_\kappa\lambda$ such that $(r \circ q)_*(\mu)$ is a normal measure of $p_\kappa\lambda$ implies, by the argument used in 2.16, the existence of a function s from $p_\kappa\lambda$ into $p_\kappa\lambda^+$ such that $(s \circ r \circ q)_*(\mu)$ is a normal measure on $p_\kappa\lambda^+$, which contradicts our assumptions on μ.

The observation that a cardinal is strongly compact if it is both a measurable cardinal and a limit of strongly compact cardinals is the most important argument of this section.

2.22. Theorem. *Suppose there is a measurable cardinal that is a limit of strongly compacts. Then there is a strongly compact cardinal κ such that the set of measurable cardinals less than κ is a non-stationary subset of κ. A fortiori, κ is not 2^κ-supercompact.*

Proof. Let κ be the least measurable cardinal that is a limit of strongly compacts, and let A be the set of measurable cardinals less than κ. By Theorem 2.21, κ is strongly compact.

Define a function f from A into κ so that for every α in A, $f(\alpha) = \sup\{\beta < \alpha: \beta$ is strongly compact$\}$. By our assumption on κ, $f(\alpha) < \alpha$ for all α in A, and there is no unbounded subset B of A such that f is constant on B. Thus A is not stationary.

κ is not 2^κ-supercompact, because by a theorem of Solovay, every cardinal δ that is 2^δ-supercompact has a normal measure concentrating on measurable cardinals [15].

The existence of a measurable cardinal that is a limit of strongly compacts is a consequence of the axiom of extendibility.

2.23. Definition. A cardinal κ is extendible if for every $\alpha > \kappa$ there is a $\beta > \kappa$ and an elementary embedding $j : \langle R(\alpha); \in \rangle \to \langle R(\beta); \in \rangle$ such that κ is the least ordinal moved by j and $j(\kappa) > \alpha$.

The concept of extendibility is due to Reinhardt and Solovay [15].

2.24. Theorem. *If κ is an extendible cardinal, the set $\{\gamma < \kappa : \gamma$ is strongly compact and γ is not 2^γ-supercompact$\}$ is an unbounded subset of κ.*

Lemma. *Suppose α is supercompact and β is a cardinal less than α such that for all γ greater than β and less than α, β is γ-supercompact. Then β is supercompact.*

Proof. Let γ_0 be a cardinal greater than α and γ_1 a cardinal greater than $2^{\gamma_0^{<\alpha}}$. Suppose μ is a normal measure on $p_\alpha\gamma_1$ and $j : V \to M \simeq V^{p_\alpha \gamma_1}/\mu$, the canonical embedding. Then $M \models (j(\beta)$ is ζ-supercompact for all cardinals ζ between $j(\beta)$ and $j(\alpha))$. Since $j(\beta) = \beta$ and $j(\alpha) > \gamma_1$, there is a set U such that $M \models (U$ is a normal measure on $p_\beta\gamma_0)$. Since $\operatorname{clos}(\mu) > \gamma_1$, U is a normal measure on $p_\beta\gamma_0$.

Lemma. *Suppose κ is extendible. Then κ has a normal measure μ concentrating on supercompact cardinals.*

Proof. κ is supercompact by a theorem of Reinhardt and Solovay [15]. Let α be a limit ordinal greater than κ and j an elementary embedding from $\langle R(\alpha); \in \rangle$ into $\langle R(\beta); \in \rangle$ so that κ is the least ordinal moved by j and $j(\kappa) > \alpha$. Define a normal measure μ on κ by $\mu(A) = 1$ iff $\kappa \in j(A)$, for all subsets A of κ. Let $A = \{\gamma < \kappa : \gamma$ is δ-supercompact for all δ greater than γ and less than $\kappa\}$. Since κ is supercompact, $\kappa \in j(A)$. Also by the previous lemma, if γ is in A, γ is supercompact.

To complete the proof of the theorem, note that if ζ is an ordinal less than κ and γ_ζ is the least measurable cardinal greater than ζ that is the limit of strongly compact cardinals, then by the arguments of 2.21 and 2.22, γ_ζ is strongly compact and γ_ζ is not 2^{γ_ζ}-supercompact.

We note without proof that if there is an elementary embedding $j : \langle V; \in\rangle \to \langle M; \in\rangle$ such that κ is the least ordinal moved by j and M is a transitive class closed under ν-sequences for every cardinal ν less than $j(\kappa)$, then $\langle R(\kappa); \in\rangle \models$ (there is a class of extendible cardinals).

We close this subsection with a theorem that was of greater interest before we discovered 2.21 and 2.22. Magidor has shown in his dissertation [9] that if κ is strongly compact and there is a cardinal λ greater than κ such that λ is the least cardinal greater than κ with a non-normal fine measure μ on $p_\kappa\lambda$ so that $\ulcorner\langle x \cap \kappa; x \in p_\kappa\lambda\rangle\urcorner^\mu = \kappa$ and μ assigns measure one to the closed unbounded subsets of $p_\kappa\lambda$, then λ is ineffable (improved to "λ is measurable" by Solovay). In order to motivate our theorem we state a special case of Magidor's result in our terminology: Suppose $2^\kappa = \kappa^+$. There is a function w from $\kappa^+ \times \kappa^+$ into $p_\kappa\kappa^+$ so that for every fine measure μ on $p_\kappa\kappa^+$, if $\mu(\mathcal{C}(\{w\})) = 1$ and $\ulcorner\langle x \cap \kappa; x \in p_\kappa\kappa^+\rangle\urcorner^\mu = \kappa$, then μ is normal.

2.25. Theorem. *Let κ be strongly compact, $\lambda > \kappa$, and $w_\alpha : \lambda \to p_\kappa\lambda$ for every $\alpha < \kappa$. There is a fine measure μ on $p_\kappa\lambda$ so that $\ulcorner\langle x \cap \kappa; x \in p_\kappa\lambda\rangle\urcorner^\mu = \kappa$ and $\mu(\mathcal{C}(\{w_\alpha\})) = 1$ for every $\alpha < \kappa$.*

Proof. Select a fine measure μ_0 on $p_\kappa\lambda$ so that $\mu_0(\mathcal{C}(\{w_\alpha\})) = 1$ for all $\alpha < \kappa$. [By the strong compactness of κ, every κ-additive filter on a set X can be extended to a κ-additive ultrafilter on X]. Let $j_0 : V \to M_0 \simeq V^{p_\kappa\lambda}/\mu_0$ be the canonical embedding associated with μ_0 and $c = \ulcorner\langle x; x \in p_\kappa\lambda\rangle\urcorner^{\mu_0}$. Since $\operatorname{clos}(\mu_0) > \kappa$, $\langle j(w_\alpha); \alpha < \kappa\rangle$ is in M_0 and $M_0 \models (c \in \bigcap_{\alpha<\kappa} \mathcal{C}(\{j(w_\alpha)\}))$.

Working in M_0, we define "β is a descendant of α" iff there is an $n < \omega$, $\alpha_0, ..., \alpha_n < j_0(\lambda)$, and $i_0, ..., i_{n-1} < \kappa$, such that $\alpha_0 = \alpha$, $\alpha_n = \beta$, and α_{k+1} is in $j_0(w_{i_k})(\alpha_k)$ for all $k < n$. Define $c_0 = \{\alpha \in c$: there is a β greater than or equal to κ and less than $j(\kappa)$, such that β is a descendant of $\alpha\}$ and let $\ulcorner q\urcorner^{\mu_0} = c - c_0$, for some function q from $p_\kappa\lambda$ into $p_\kappa\lambda$.

For every $\alpha < \lambda$ and $\beta < \kappa$, $j_0(\alpha) \in c - c_0$ and $c - c_0$ is in $\mathcal{C}(\{j_0(w_\alpha)\})$. The former assertion follows from the fact that for every function w from λ into $p_\kappa \lambda$ and for every $\gamma < \lambda$, $j_0(w)(j_0(\gamma)) = j_0(w(\gamma)) = j_0[w(\gamma)]$, since $|w(\gamma)| < \kappa$; so the only descendants of $j_0(\alpha)$ are in $j_0[\lambda]$. The latter assertion follows from the fact if δ is in c and γ is in $j(w_\beta)(\delta) \cap c_0$, then δ is in c_0.

Now let $\mu = q_*(\mu_0)$, $j : V \to M \simeq V^{p_\kappa \lambda}/\mu$, and let k be the elementary embedding from M into M_0 induced by q. Since $k(\ulcorner\langle x \cap \kappa; x \in p_\kappa \lambda\rangle\urcorner^\mu) = \ulcorner\langle q(x) \cap \kappa; x \in p_\kappa \lambda\rangle\urcorner^{\mu_0} = (c - c_0) \cap j_0(\kappa) = \kappa$, $\ulcorner\langle x \cap \kappa; x \in p_\kappa \lambda\rangle\urcorner^\mu = \kappa$. To show that $\mu(\mathcal{C}(\{w_\alpha\})) = 1$ for every $\alpha < \kappa$, we need only prove that $\ulcorner\langle x; x \in p_\kappa \lambda\rangle\urcorner^\mu$ is in $j(\mathcal{C}(\{w_\alpha\}))$ for every $\alpha < \kappa$. This follows from the fact that $\ulcorner q \urcorner^{\mu_0}$ is in $j_0(\mathcal{C}(\{w_\alpha\}))$ for every $\alpha < \kappa$.

§ 3. A consistency result

Assume throughout this section that M is a countable standard model of ZFC. We will develop a method of preserving in suitable Cohen extensions of M the strong compactness of certain cardinals in M. Using this method, we will then outline a proof that the consistency of "ZFC + there is a measurable cardinal that is the limit of strongly compact cardinals" implies the consistency of "ZFC + there is exactly one strongly compact cardinal and no supercompacts".

Following the announcement of Theorem 2.22 and shortly after we proved our results of this section, Magidor announced the much stronger result that the consistency of "ZFC + there is a strongly compact cardinal" implies the consistency of "ZFC + the first strongly compact cardinal is the first measurable". In addition, using the techniques of this section, Jacques Stern has shown that the existence of an extendible cardinal implies the consistency of "ZFC + there are exactly two strongly compact cardinals + the first strongly compact is supercompact and the second is not" and the consistency of "ZFC + there exist at least two strongly compact cardinals + the first two strongly compacts are not supercompact".

We will assume that the reader is well acquainted with the theory of Boolean valued models in Scott [16], Solovay and Tennenbaum [20], and Jech [2]. Unless otherwise noted, we adopt the notation of the latter paper. The last part of this section requires the Silver forcing techniques developed in [13].

§ 3.1. *Preserving strong compactness in Cohen extensions*

We now work in M. Let $\mathcal{P} = \langle P; \leqslant \rangle$ be a non-empty partially ordered set and for $p \in P$, define $[p] = \{q \in P: q \leqslant p\}$. Let τ be the topology on P generated by the family $\{[p] : p \in P\}$, and define $B^M(\mathcal{P})$ to be the complete Boolean algebra of all regular open subsets of P with respect to the topology τ. If $\phi(v_0, ..., v_{n-1})$ is a formula of ZF and $x_0, ..., x_{n-1}$ are elements of $M^{(B^M(\mathcal{P}))}$, $\|\phi(x_0, ..., x_{n-1})\|^{(B^M(\mathcal{P}))}$ is the Boolean value of the statement $\phi(x_0, ..., x_{n-1})$. We generally omit the superscript "$(B^M(\mathcal{P}))$".

2.26. Definition. Let $\psi(v_0, v_1, v_2, v_3)$ be a formula of ZF so that $\psi(\mathcal{P}, \mathcal{R}, i, \nu)$ iff $\mathcal{P}$ and $\mathcal{R}$ are partially ordered sets, ν is an ordinal, i is a complete embedding of $B(\mathcal{P})$ into $B(\mathcal{R})$ so that for all x in $V^{(B(\mathcal{R}))}$ with $\|x \subseteq \nu\| = 1$, $\sup\{\|i_*(y) = x\| : y \in V^{(B(\mathcal{P}))}\} = 1$, where i_* is the mapping from $V^{(B(\mathcal{P}))}$ into $V^{(B(\mathcal{R}))}$ induced by i [27].

Suppose $\mathcal{P}$, $\mathcal{R}$, i and ν are in M so that $M \models \psi(\mathcal{P}, \mathcal{R}, i, \nu)$. If H is an M-generic ultrafilter on $B^M(\mathcal{R})$, then $i^{-1}[H]$ is an M-generic ultrafilter on $B^M(\mathcal{P})$ and all subsets of ν in $M[H]$ are in $M[i^{-1}[H]]$.

The context of Definition 2.26 is found in many situations. For example, if $\mathcal{P}_0$ and $\mathcal{P}_1$ are partially ordered sets so that $\mathcal{P}_0$ has the ν^+-chain condition and $\mathcal{P}_1$ is ν-closed, a lemma due to Easton [1] shows that there is an i so that $\psi(\mathcal{P}_0, \mathcal{P}_0 \otimes \mathcal{P}_1, i, \nu)$ is true, where $\mathcal{P}_0 \otimes \mathcal{P}_1$ is the cartesian product of $\mathcal{P}_0$ and $\mathcal{P}_1$. [$\mathcal{R} = \langle R; \leqslant \rangle$ is γ-closed iff for every function f from γ into R with $f(\beta) \leqslant f(\alpha)$ for all $\alpha \leqslant \beta < \gamma$, there is a q in R, so that $q \leqslant f(\alpha)$ for all $\alpha < \gamma$.] Also, if $\mathcal{P}_0$ is a partially ordered set and $\mathcal{P}_1$ is a set so that $V^{B(\mathcal{P}_0)} \models [\mathcal{P}_1$ is a non-empty partially ordered set that is $\check{\nu}$-closed], then there is an i so that $\psi(\mathcal{P}_0, \mathcal{P}_0 \tilde{\otimes} \mathcal{P}_1, i, \nu)$ is true, where $\mathcal{P}_0 \tilde{\otimes} \mathcal{P}_1$ is defined in Chapter 4.

Suppose that in M, κ, λ and ν are cardinals with $\lambda \geqslant 2^{2^\nu} \geqslant \kappa$, μ_0 is a fine measure on $p_\kappa\lambda$, $j : M \to j(M) \simeq M^{p_\kappa\lambda}/\mu_0$ is the usual elementary embedding, and $\mathcal{P}$ is a partially ordered set in $j(M)$ so that $|B^M(\mathcal{P})| \leqslant 2^\kappa$. Suppose also that in $j(M)$ there is an i so that $j(M) \models \psi(\mathcal{P}, j(\mathcal{P}), i, |\ulcorner\langle x; x \in p_\kappa\lambda\rangle\urcorner^{\mu_0}|)$ and that $B^M(\mathcal{P}) = B^{j(M)}(\mathcal{P})$.

Let G be an M-generic ultrafilter on $B^M(\mathcal{P})$ and H an M-generic ultrafilter on $B^M(j(\mathcal{P}))$ so that $i^{-1}[H \mid B^{j(M)}(j(\mathcal{P}))] = G$ and $j[G] \subseteq H$.

2.27. Theorem. *In $M[G]$, κ is ν-strongly compact.*

Proof. In M let μ_1 be a fine measure on $p_\kappa \nu$ defined so that for every subset A of $p_\kappa \nu$, $\mu_1(A) = 1$ iff $\mu_0(\{y \in p_\kappa \lambda : y \cap \nu \in A\}) = 1$. Let $c_0 = \ulcorner\langle x; x \in p_\kappa \lambda\rangle\urcorner^{\mu_0}$ and $c_1 = \ulcorner\langle x \cap \nu; x \in p_\kappa \lambda\rangle\urcorner^{\mu_0}$. Then for every subset A of $p_\kappa \nu$, $\mu_1(A) = 1$ iff $c_1 \in j(A)$. We will show that in $M[G]$, there is a fine measure on $p_\kappa \nu$ extending μ_1.

Let $H^* = H \restriction B^{j(M)}(j(\mathcal{P}))$. Since $j(M) \models \psi(\mathcal{P}, j(\mathcal{P}), i, |c_0|)$, every subset of $|c_0|$ in $j(M)[H^*]$ is in $j(M)[G]$. Since $j(M)$ is a class of M definable from μ_0, $j(M)[G]$ is contained in $M[G]$. Then every subset of c_0 in $j(M)[H^*]$ is in $M[G]$.

In $M[H]$, define a map $k : M[G] \to j(M)[H^*]$ as follows. Let x be in $M[G]$. Then there is an $\underset{\sim}{x}$ in $M^{B^M(\mathcal{P})}$ so that $i_G(\underset{\sim}{x}) = x$, where i_G is as in [2] the interpretation of $M^{(B^M(\mathcal{P}))}$ by G. Let $k(x) = i_{H^*}(j(\underset{\sim}{x}))$, where i_{H^*} is the interpretation of $j(M)^{(B^{j(M)}(j(\mathcal{P})))}$ by H^*. To see that k is well defined, let $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ be in $M^{(B^M(\mathcal{P}))}$ so that $i_G(\underset{\sim}{x}) = i_G(\underset{\sim}{x}') = x$. Then there is a p in P so that $[p]$ is in G and $[p] \leqslant \|\underset{\sim}{x} = \underset{\sim}{x}'\|$. So in $j(M)$, $[j(p)] \leqslant \|j(\underset{\sim}{x}) = j(\underset{\sim}{x}')\|$. Since $[j(p)]$ is in H by assumption, $i_{H^*}(j(\underset{\sim}{x})) = i_{H^*}(j(\underset{\sim}{x}'))$.

Similarly, since j is elementary, we can show that k is an elementary embedding. An easy check establishes that k extends j.

In $M[H]$, define a measure μ_1^* on the family of all subsets of $p_\kappa \nu$ in $M[G]$ so that for every subset A of $p_\kappa \nu$ in $M[G]$, $\mu_1^*(A) = 1$ iff $c_1 \in k(A)$ The theorem will follow if we show that μ_1^* is in $M[G]$.

In M let e be a map from λ onto $r = {}^{(\nu_{B^M(\mathcal{P})})} B^M(\mathcal{P})$, and let g be a function with domain $p_\kappa \lambda$ so that $g(y) = e \restriction y$ for all y in $p_\kappa \lambda$. [Note that $i_G(r) = p(p(\nu)) \cap M|G|$.] Then $j(M) = (\ulcorner g \urcorner^{\mu_0}$ is a function from c_0 into $j(r)$). In $j(M)[H^*]$, define a function $f : c_0 \to 2$ so that

$$f(\alpha) = \begin{cases} 1 & \text{if } c_1 \in i_{H^*}(\ulcorner g \urcorner^{\mu_0}(\alpha)), \\ 0 & \text{otherwise} \end{cases}$$

for all α in c_0.

$j \restriction \lambda$ is in M, and since all subsets of c_0 in $j(M)[H^*]$ are in $M[G]$, f is also in $M[G]$. For every α in λ, $\mu_1^*(i_G(e(\alpha))) = 1$ iff there is a $[p]$ in G

so that $[p] \leqslant \|e(\alpha)$ is a subset of the family of all subsets of $\check{\nu}$ of cardinality less than $\check{\kappa}\|$ and $f(j(\alpha)) = 1$. Hence μ_1^* is in $M[G]$.

We mention without proof that in $M[G]$, every mess on $p_\kappa\lambda$ has a solution. [The notion of a mess on $p_\kappa\lambda$ is defined and studied by Jech in [3].]

Suppose that κ is a strongly compact cardinal in M, $\mathcal{P}$ a partially ordered subset of $R(\kappa)$, and G an M-generic ultrafilter on $B^M(\mathcal{P})$ so that for every $\nu > \kappa$ there is a cardinal $\lambda > \nu$ and a fine measure μ on $p_\kappa\lambda$ with the following properties:

(i) There is a function $f : \kappa \to \kappa$ and an i in $j(M)$ so that

$$j(M) \models [\psi(\mathcal{P}, j(\mathcal{P}), i, j(f)(\kappa)), j(f)(\kappa) \geqslant |\ulcorner\langle x; x \in p_\kappa\lambda\rangle\urcorner^{\mu}|] .$$

(ii) There is an M-generic ultrafilter H on $B^M(j(\mathcal{P}))$ so that $i^{-1}[H] = G$ and $j[G] \subseteq H$. Then by Theorem 2.27, κ is strongly compact in $M[G]$.

The stipulations on κ, $\mathcal{P}$, and G are generally easily met. It is the existence of the function f that poses a problem.

2.28. Definition. A fine measure μ on $p_\kappa\lambda$ has the *σ-property* iff there is a function $f : \kappa \to \kappa$ so that $\mu(\{x \in p_\kappa\lambda: f(\kappa_x) \geqslant |x|\}) = 1$. For such μ and f we write $\sigma(\mu, f)$.

Let $\lambda > \kappa$. Suppose that μ_0' is fine on $p_\kappa\lambda$ and that f is a function from κ into κ so that $\sigma(\mu, f)$. Let $\ulcorner\langle\kappa_x; x \in p_\kappa\lambda\rangle\urcorner^{\mu_0'} = \kappa$ and define q on $p_\kappa\lambda$ so that $q(x) = (x - \kappa) \cup \kappa_x$ for all $x \in p_\kappa\lambda$. Then $\mu_0 = q_*(\mu_0')$ is fine on $p_\kappa\lambda$, $\ulcorner\langle x \cap \kappa; x \in p_\kappa\lambda\rangle\urcorner^{\mu_0} = \kappa$, and $\sigma(\mu_0, f)$. If $\kappa < \nu < \lambda$ and μ_1 is the projection of μ_0 on $p_\kappa\nu$ (i.e., for $A \subseteq p_\kappa\nu$, $\mu_1(A) = 1$ iff $\mu_0(\{x \in p_\kappa\lambda: x \cap \nu \in A\}) = 1$), then $\ulcorner\langle x \cap \kappa; x \in p_\kappa\nu\rangle\urcorner^{\mu_1} = \kappa$ and $\sigma(\mu_1, f)$. It is now easy to check that if the preceding discussion took place in the M of Theorem 2.27 and if in addition μ_0, μ_1, and μ_1^* are related as in the proof of that theorem, then in $M[G]$, $\ulcorner\langle x \cap \kappa; x \in p_\kappa\nu\rangle\urcorner^{\mu_1^*} = \kappa$ and $\sigma(\mu_1^*, f)$.

§3.2. *The σ-property*

We digress to briefly consider the property σ.

We do not know whether for every strongly compact cardinal κ and

$\lambda > \kappa$, there is a fine measure μ on $p_\kappa\lambda$ with the σ-property. In fact we conjecture that this is not always the case. However if κ is supercompact, we prove in [14] that for every cardinal $\lambda \geq \kappa$ there is a normal measure μ on $p_\kappa\lambda$ and a function $f : \kappa \to \kappa$ so that $\mu(\{x \in p_\kappa\lambda : f(\kappa_x) = |x|\}) = 1$.

The following proposition due to Solovay shows that under certain conditions on κ, there is a normal measure μ on $p_\kappa\lambda$ without the σ-property.

2.29. Proposition. *Suppose $j : \langle V; \in\rangle \to \langle M; \in\rangle$ is an elementary embedding so that M is a transitive class closed under α-sequences for every cardinal $\alpha < j(\kappa)$, and κ is the least ordinal moved by j. Then there is a cardinal $\lambda < j(\kappa)$ and a normal measure μ on $p_\kappa\lambda$ without the σ-property.*

Proof. Let λ be any cardinal less than $j(\kappa)$ and greater than the supremum of the set $\{j(f)(\kappa): f : \kappa \to \kappa\}$ (By the closure properties of M, $j(\kappa)$ is inaccessible and λ exists.) Define a normal measure μ on $p_\kappa\lambda$ so that for every subset A of $p_\kappa\lambda$, $\mu(A) = 1$ iff $j[\lambda] \in j(A)$. Then note that if $f : \kappa \to \kappa$, $\mu(\{x \in p_\kappa\lambda : f(|\kappa \cap x|) < |x|\}) = 1$.

We note that the existence of a supercompact cardinal κ and a cardinal $\lambda > \kappa$ so that there is a normal measure on $p_\kappa\lambda$ without the σ-property constitutes a much stronger cardinal axiom than that of supercompactness. For it can be shown, for example, that under these assumptions, there is a normal measure on κ concentrating on supercompact cardinals.

It turns out however that fine measures on $p_\kappa\lambda$ without the σ-property are easy to come by.

2.30. Proposition. *If κ is strongly compact and $\lambda > \kappa$, there is a fine measure μ on $p_\kappa\lambda$ without the σ-property.*

Proof. Let μ_0 be a fine measure on $p_\kappa\lambda$, μ_1 a normal measure on κ and q an injective function from $p_\kappa\lambda \times \kappa$ into $p_\kappa\lambda$ so that $x \subseteq q(x, \alpha)$ for every (x, α) in $p_\kappa\lambda \times \kappa$. Then as in the proof of Theorem 2.17, $\mu = q_*(\mu_0 \times \mu_1)$ is a fine measure on $p_\kappa\lambda$.

Define $k : p_\kappa\lambda \to \kappa$ so that $k(q(x,\alpha)) = \alpha$ for every (x, α) in $p_\kappa\lambda \times \kappa$. By the normality of μ_1, $\ulcorner k \urcorner^\mu = \kappa$.

Let $f: \kappa \to \kappa$. Since

$$(\mu_0 \times \mu_1)(\{(x,\alpha) \in p_\kappa\lambda \times \kappa : |x| > f(\alpha)\}) = 1,$$

then $\mu(\{x \in p_\kappa\lambda : f(k(x)) < |x|\}) = 1$.

We finally note that for certain strongly compact cardinals κ there is always a fine measure on $p_\kappa\lambda$ with the σ-property.

2.31. Proposition. *Let κ be a measurable cardinal, that is, a limit of strongly compacts. There is a function f from κ into κ so that for every $\lambda > \kappa$ there is a fine measure μ on $p_\kappa\lambda$ so that* $\mu(\{x \in p_\kappa\lambda : f(\kappa_x) \geq |x|\}) = 1$

Proof. Let U be a normal measure on κ and define f for every $\alpha < \kappa$ by $f(\alpha)$ = (least strongly compact cardinal greater than α). By the normality of U, there is a subset E of κ so that $U(E) = 1$ and for all $\alpha < \beta$ in E, $f(\alpha) < \beta$.

For every γ in E, let μ_γ be a fine measure on $p_{f(\gamma)}\lambda$, and define a fine measure μ on $p_\kappa\lambda$ so that for every subset A of $p_\kappa\lambda$,

$$\mu(A) = 1 \quad \text{iff} \quad U(\{\alpha \in E : \mu_\alpha(A \cap p_{f(\alpha)}\lambda) = 1\}) = 1.$$

For every γ in E, let $D_\gamma = \{x \in p_{f(\gamma)}\lambda : |x| > \gamma\}$. Then $\mu(\bigcup_{\gamma\in E} D_\gamma) = 1$. For every x in $\bigcup_{\gamma\in E} D_\gamma$, let $s(x)$ be the unique γ in E so that $x \in D_\gamma$. Then $\ulcorner s \urcorner^\mu = \kappa$ and $\mu(\{x \bigcup_{\gamma\in E} D_\gamma : |x| < f(s(x))\}) = 1$.

§3.3. *A model in which the first strongly compact is not supercompact*

We are now ready to give a sketch of a model of ZFC in which there is exactly one strongly compact cardinal and the unique strongly compact κ is not κ^+-supercompact.

Let M be a countable standard model of ZFC in which κ is the least measurable cardinal that is the limit of supercompacts. We may assume that the G.C.H. is true in M (A Silver forcing argument for the satisfaction of the G.C.H. in a Cohen extension preserves supercompactness and measurability. Silver forcing is discussed in [13].)

Let M_1 be a Cohen extension of M using Silver forcing to ensure that $2^\nu = \nu^{++}$ for those cardinals ν in M that are 2^ν-supercompact. Supercompactness is preserved in the extension and the Kunen–Paris method

[8] shows that κ is measurable and that $2^\kappa = \kappa^+$ in M_1.

In M_1, there is a normal measure U^* on κ so that if $E = \{\gamma < \kappa: \gamma$ is an inaccessible limit of supercompacts and γ is not measurable$\}$, then $U^*(E) = 1$.

Let M_2 be a Cohen extension of M_1 using Silver forcing to ensure that $2^{\nu^+} = \nu^{+++}$ for precisely those ν that are measurable in M_1 and for which $2^\nu = \nu^+$ in M_1. Supercompactness is again preserved, and the Kunen–Paris method shows that in M_2 there is a normal measure U on κ so that $U(E) = 1$. Also in M_2, $2^\kappa = \kappa^+$, $2^{\kappa^+} = \kappa^{+++}$, and $2^{\gamma^+} = \gamma^{++}$ for all γ in E.

For γ in E let $f(\gamma)$ be the least supercompact greater than γ, and let A be a cofinal subset of κ so that no point in A is a limit point of A, $A \cap [\gamma, f(\gamma)] = \emptyset$ for all γ in E, and for every ν in A, ν is measurable and $2^\nu = \nu^{++}$.

For every ν in A select a normal measure U_ν on ν. Let $m(A)$ be the set of all functions σ with domain A so that for every ν in A, $\sigma(\nu) = \langle r, B\rangle$ where r is a finite sequence of elements of ν and $U_\nu(B) = 1$, and the first coordinate of $\sigma(\nu)$ is the empty sequence for all but a finite number of ν's. Define a partial ordering $\leqslant^*$ on these functions so that $\sigma_0 \leqslant^* \sigma_1$ iff for all ν in A if $\sigma_0(\nu) = \langle r_0, B_0\rangle$ and $\sigma_1(\nu) = \langle r_1, B_1\rangle$, then $B_1 \subseteq B_0$, r_1 extends r_0, and the ordinals in r_1 but not in r_0 are in B_0. $\mathcal{P} = \langle m(A); \leqslant^*\rangle$ is the set of Magidor conditions with respect to A [20].

Magidor shows that if $\alpha < \nu$ and ν is in A, then in every generic extension M_3 of M_2, every subset of α in M_3 is in M_3^* which is the Cohen extension of M_2 obtained by forcing with the conditions $m(A \cap \nu) = \{\rho$: there is a σ in $m(A)$ so that $\rho = \sigma \restriction A \cap \nu\}$. It follows from the discussion in subsection 3.2, that in M_3, κ is strongly compact and that for every $\lambda > \kappa$ there is a fine measure μ on $p_\kappa\lambda$ so that $\mu(\{x \in p_\kappa\lambda: f(\kappa_x) > |x|\}) = 1$.

Magidor shows that in M_3 if ν is in A, then $\mathrm{cof}(\nu) = \omega$ and ν is a strong limit cardinal so that $2^\nu = \nu^{++}$. But a theorem of Solovay [32] shows that if α is strongly compact and $\beta > \alpha$ is a singular strong limit cardinal, then $2^\beta = \beta^+$. It follows that in M_3 there are no strongly compact cardinals less than κ.

Now let M_4 be a Cohen extension of M_3 using Easton forcing [1] to ensure that $2^\alpha = \alpha^{++}$ for all inaccessible cardinals α less than κ and not in $\bigcup_{\gamma \in E} [\gamma, f(\gamma)]$.

Again by the results of subsection 2, κ is strongly compact in M_4. Since $\{\beta < \kappa: \beta$ is a singular strong limit cardinal and $2^\beta = \beta^{++}\}$ is cofinal in κ in M_4, κ is the first strongly compact.

Now suppose that κ is κ^+-supercompact in M_4 and let μ be a normal measure on $p_\kappa \kappa^+$ and $j : M_4 \to j(M_4) \simeq V^{p_\kappa \kappa^+}/\mu$. Then $j(M_4) \models 2^\kappa = \kappa^+$ and $2^{\kappa^+} \geqslant \kappa^{+++}$. Then $\mu(\{x \in p_\kappa \lambda: 2^{\kappa_x} = \kappa_x^+, \kappa_x$ is inaccessible, and $2^{\kappa_x^+} \geqslant \kappa_x^{+++}\}) = 1$. But if κ_x is in E, $2^{\kappa_x^+} = \kappa_x^{++}$ and if κ_x is not in E, $2^{\kappa_x} = \kappa_x^{++}$, which is a contradiction.

Note that the above gives an example of a strongly compact cardinal κ so that $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{+++}$ but for every normal measure μ on κ, $\mu(\{\alpha < \kappa: 2^\alpha \geqslant \alpha^{++}$ or $2^{\alpha^+} = \alpha^{++}\}) = 1$.

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