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An Absorption Probability Problem

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INTRODUCTION AND RESULTS

This paper is devoted to a discussion of an absorption probability problem originating in waiting-line theory, and having interpretation in the theory of risk. The problem is: $\{X(t), t \geqslant 0\}$ is a stochastic process whose states are nonnegative real numbers, and for which the time parameter, t, also takes on nonnegative real values. Beginning at some positive state value, the process evolves by decreasing linearly between random instants, determined by a stationary Poisson process, at which positive jumps occur. When jumps occur, their magnitudes are *proportional* to the process value immediately before the jump.

For this process we shall compute the probability, $f(x)$, that the process $X(t)$ is ultimately absorbed at 0 (reaches the state 0 in finite time), *given* that initially $X(t) = x > 0$. An explicit expression is supplied for $f(x)$, from which it is shown that $f(x) < 1$ for all positive x. Hence there is always a chance that the process will grow indefinitely, no matter how small the initial value x is made. If x is large, the chance of ultimate absorption becomes small as follows: $f(x) \sim A_0 e^{-x}$, where A_0 is a constant > 1. An explicit expression is also obtained for $f_n(x)$, the probability that the process is absorbed after exactly n jumps. Somewhat surprisingly, $f_n(x)/f(x)$, the *conditional* probability of ultimate absorption after n jumps, given that absorption ultimately occurs, tends to a definite limit, f_n^*, as x tends to infinite. Properties of this limiting probability are discussed. Finally, some generalizations are briefly mentioned.

Processes similar to the one discussed here, in which jump magnitudes are independently distributed random variables, have been considered by many authors, cf. Cramér [1], Bartlett [2], Takács [3], and Beneš [4]. Processes of the latter type ultimately reach 0 with probability one from any initial value if the product of expected jump magnitude and jump occurrence rate is not greater than unity.

I. Mathematical Formulation

We consider a random process $\{X(t), t \geqslant 0\}$ defined as follows:

(a) $X(0) = x > 0$;

(b) $X(t) = \text{Max}\,[X(t_n + 0) - (t - t_n), 0]$ for $t_n \leqslant t < t_{n+1}$; in particular

(c) $X(t_{n+1} - 0) = X(t_n + 0) - (t_{n+1} - t_n)$ for $X(t_{n+1} - 0) > 0$;

(d) $X(t_{n+1} + 0) = \gamma X(t_{n+1} - 0)$, γ a constant > 1;

(e) $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots$, where $t_{n+1} - t_n = \tau_n$, and $\{\tau_n, n = 1, 2, \cdots\}$ is a sequence of independently and exponentially distributed random variables:

$$P\{\tau_n > z\} = e^{-z}, \qquad z > 0. \tag{1.1}$$

A sample function, $X(\cdot)$, from the process described declines from any positive value (initially, from $X(0) = x$) with slope -1 until an event in a stationary Poisson process occurs. If the function assumes the value $y > 0$ immediately before such an event occurs, its value jumps to level $\gamma y > y$ immediately after the event. After the jump it again declines at rate -1 until either another jump occurs or X reaches 0, whichever occurs first. The discussion that follows will be of the probability, $f(x)$, that a sample function that starts at x reaches 0 (is absorbed) after *finitely many* Poisson events (in finite time). As a first step we shall compute the probability of absorption after exactly n Poisson events:

$$f_n(x) = P\{X(t_i - 0) > 0\ (i = 1, 2, \cdots, n),\ X(t_{n+1} - 0) = 0 \mid X(0) = x\}. \tag{1.2}$$

It then follows that

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \tag{1.3}$$

II. Preliminary Discussion

The present problem is an idealization of certain waiting-line or storage problems. Thus $X(t)$ may be interpreted as the time backlog of unfinished work at a facility. This backlog has the property of ***multiplying itself*** by γ, $\gamma > 1$, when a new arrival occurs. The situation in which ***additions*** are made to backlog has been much discussed, cf. [1-4]; in these discussions (d) above is replaced by

(d′) $X(t_{n+1} + 0) = X(t_{n+1} - 0) + Y_n$

where $\{Y_n\}$ is a sequence of independently and identically distributed random variables, being in particular independent of $X(t_{n+1} - 0)$. In certain problems, (d) is more reasonable than (d') because the contribution to backlog made by an arrival is approximately proportional to the waiting time (backlog) encountered by the arrival when he reaches the facility, perhaps because new requirements for service arise during the waiting time.

Another interpretation of the process is that X represents the economic fortune of an enterprise. Normally, this fortune declines at a constant rate, drained by routine expenditures. However, occasionally (at the times t_n) new business possibilities arise, leading (quickly) to profits proportional to the fortune at the moment opportunity knocks. Interest is probably focussed upon the chance, $1 - f(x)$, that the fortune will grow, rather than vanish, as a function of its initial value x. Of course, the classical ruin problem in its simplest form cf. [1, 2], investigates the question of growth under the assumption (d').

III. Functional Equation

It follows directly from the definition of the process that

$$f_0(x) = e^{-x}$$

$$f_n(x) = \int_0^x e^{-t} f_{n-1}(\gamma(x-t))\, dt; \qquad \gamma > 1, x \geqslant 0, n = 1, 2, \cdots . \tag{3.1}$$

For, since X decreases linearly towards zero from x, the event of absorption with no additions occurs if and only if no Poisson events occur in time x. Absorption occurs after exactly n events if the first Poisson event occurs in $(t, t + dt)$, where $0 < t < x$, and then (independently) absorption follows after $n - 1$ more Poisson events, the process starting afresh from $\gamma(x - t)$ after the first jump.

Solution of (3.1) may be carried out by Laplace transforms. Put

$$\tilde{f}_n(s) = \int_0^\infty e^{-sx} f_n(x)\, dx, \tag{3.2}$$

which exists at least for Re $(s) > 0$. By the convolution property of the transform there results the recursive system

$$\tilde{f}_n(s) = \frac{1}{1+s} \tilde{f}_{n-1}\left(\frac{s}{\gamma}\right) \cdot \frac{1}{\gamma}$$

$$\tilde{f}_0(s) = \frac{1}{1+s} \tag{3.3}$$

and successive substitution yields

$$\tilde{f}_n(s) = \gamma^{-n} \prod_{j=0}^{n} \frac{\gamma^j}{\gamma^j + s}. \tag{3.4}$$

Inversion is accomplished by writing (3.4) in partial fractions:

$$\tilde{f}_n(s) = \sum_{i=0}^{n} \frac{a_{in}}{\gamma^i + s}. \tag{3.5}$$

Use of (3.4) shows that

$$a_{in} = \gamma^{n(n-1)/2} \prod_{j=0}^{n}{}' \frac{1}{\gamma^j - \gamma^i} = \gamma^{-n+i} \prod_{j=0}^{n}{}' \frac{\gamma^j}{\gamma^j - \gamma^i}; \tag{3.6}$$

the prime indicates that the factor in which $j = i$ is to be replaced by unity. Term-by-term inversion of (3.5) then gives

$$f_n(x) = \sum_{i=0}^{n} a_{in}\, e^{-\gamma^i x} \tag{3.7}$$

and

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{i=0}^{\infty} e^{-\gamma^i x} \sum_{n=i}^{\infty} a_{in} = \sum_{i=0}^{\infty} A_i e^{-\gamma^i x}, \tag{3.8}$$

where the rearrangement is justified by the absolute and uniform convergence of the double series. Absolute convergence follows by bounding $\{a_{in}\}$ by a convergent geometric series obtained from (3.6); this is conveniently done using the relation (4.1). We state

THEOREM 1. *Let*

$$\bar{f}_n(x) = \sum_{k=0}^{n} f_k(x) \qquad \textit{and} \qquad f(x) = \lim_{n\to\infty} \bar{f}_n(x).$$

Then $\{\bar{f}_n(x)\}$ converges uniformly, $f(x)$ is continuous, and is the minimal solution of the functional equation

$$p(x) = e^{-x} + \int_0^x e^{-t}\, p(\gamma(x - t))\, dt; \tag{3.9}$$

$f(x)$ is the unique solution of (3.9) with the property that $f(0) = 1$ and $f(x) < 1$ $(x > 0)$.

PROOF. From (3.4), $f_n(x) < \gamma^{-n}$. This follows since the product $\Pi_{j=0}^{n} \gamma^j/(\gamma^j + s)$ represents the convolution of n exponential densities, the maximum of which is less than unity. Since $\gamma > 1$, $f_n(x)$ can be majorized by an absolutely convergent series over any finite interval; hence this sequence converges uniformly and monotonically to a continuous function $f(x)$.

Summation of Eq. (3.1) yields

$$f_n(x) = e^{-x} + \int_0^x e^{-t} f_{n-1}(\gamma(x - t))\, dt. \tag{3.10}$$

Uniform convergence then provides that

$$f(x) = e^{-x} + \int_0^x e^{-t} f(\gamma(x - t))\, dt. \tag{3.11}$$

Suppose that $g(x)$ is another positive solution of (3.9). Then

$$f_0(x) = e^{-x} < e^{-x} + \int_0^x e^{-t} g(\gamma(x - t))\, dt = g(x)$$

and if $f_n(x) < g(x)$,

$$f_{n+1}(x) = e^{-x} + \int_0^x e^{-t} f_n(\gamma(x - t))\, dt < e^{-x} + \int_0^x e^{-t} g(\gamma(x - t))\, dt = g(x), \tag{3.12}$$

so by induction $f(x) \leqslant g(x)$ and $f(x)$ is minimal. Since $g(x) \equiv 1$ solves (3.9), $f(x) \leqslant 1$. From (3.6) and (3.8) it may be seen that $f(x) < 1$ $(x > 0)$. Suppose there exists a solution $g(x)$ such that $f(x) < g(x) < 1$ for $x > 0$, and $g(0) = 1$. Put $h(x) = g(x) - f(x)$; then

$$h(x) = \int_0^x e^{-t} h(\gamma(x - t))\, dt. \tag{3.13}$$

If

$$M = \max_{x \geqslant 0} h(x),$$

then (3.13) implies

$$M \leqslant M(1 - e^{-x})$$

and hence $h(x) \equiv 0$.

Further information concerning our probabilities is contained in

THEOREM 2. *All derivatives of the functions* $f_n(x)$, $\bar{f}_n(x)$, *and* $f(x)$ *exist and are continuous. In particular the first derivatives satisfy the following equations:*

$$\frac{df_n}{dx} = -f_n(x) + f_{n-1}(\gamma x) \tag{3.14a}$$

$$\frac{d\bar{f}_n(x)}{dx} = -\bar{f}_n(x) + \bar{f}_{n-1}(\gamma x) \tag{3.14b}$$

$$\frac{df(x)}{dx} = -f(x) + f(\gamma x). \tag{3.14c}$$

The functions $\bar{f}_n(x)$ and $f(x)$ are monotonically decreasing, and $f(x)$ cannot be analytic throughout any region containing $x = 0$.

PROOF. Differentiation of (3.1) and (3.10) and induction establish (3.14a) and (3.14b), and (3.14c) follows either from (3.14b) and uniform convergence of $\{\bar{f}_n(x)\}$, or by differentiating (3.11). Existence and continuity of higher derivatives are shown by induction in (3.14). Induction also shows that $\bar{f}_n(x)$ decreases, for, assuming $\bar{f}_{n-1}(x)$ decreases, we have

$$\frac{d\bar{f}_n}{dx} = -\bar{f}_n(x) + \bar{f}_{n-1}(\gamma x) < -\bar{f}_n(x) + \bar{f}_{n-1}(x) = -f_n(x) < 0.$$

Letting $n \to \infty$ shows that $df(x)/dx \leqslant 0$. Suppose the derivative equals zero for some finite $x^1 > 0$; (3.14c) then implies

$$f(x^1) = f(\gamma x^1) = f(\gamma^2 x^1) = \cdots = f(\gamma^k x^1) = \cdots$$

and, since $\gamma > 1$, this contradicts the fact that $\lim_{x\to\infty} f(x) = 0$, apparent from (3.8). Finally, it can be seen from (3.14c) that all derivatives of $f(x)$ vanish at $x = 0$, and hence $f(x)$, as given by (3.8), cannot be analytic.

Alternatively, expressions (3.14) may be written down from a direct probability argument. In a time interval of length δ following an instant at which $X = x$, either (i) no Poisson event occurs, an event of probability $1 - \delta + o(\delta)$, or (ii) exactly one Poisson event occurs leading to a jump to level γx, an event of probability $\delta + o(\delta)$; the probabilities of all other events are negligible. We are led to write, for example,

$$f_n(x) = (1 - \delta) f_n(x - \delta) + \delta f_{n-1}(x - \delta + \gamma(x - \delta)) + o(\delta)$$

and division by δ and allowing $\delta \to 0$ leads to (3.14a).

IV. Alternative Formulas

If the expression (3.7) for $f_n(x)$ is substituted into (3.14a), and the coefficients of $e^{-\gamma^i x}$ are identified, the following relations are obtained:

$$a_{in} = \frac{a_{i-1,n-1}}{1-\gamma^i}, \qquad i = 1, 2, \cdots, n \tag{4.1}$$

$$f_n(0) = 0 = \sum_{i=0}^{n} a_{in}; \tag{4.2}$$

These may also be deduced from (3.6). Thus, knowing that $a_{\infty} = 1$, successive coefficients may be recursively computed with ease.

Similarly, substitution of the expression

$$f(x) = \sum_{i=0}^{\infty} A_i e^{-\gamma^i x} \tag{4.3}$$

from (3.8) into (3.14c) and identification of coefficients yields

$$A_i = \frac{A_{i-1}}{1-\gamma^i} \qquad i = 1, 2, \cdots . \tag{4.4}$$

From (3.6) it is clear that

$$0 < A_0 = \sum_{n=0}^{\infty} a_{0n} = 1 + \sum_{n=1}^{\infty} \gamma^{-n} \prod_{j=1}^{n} \frac{\gamma^j}{\gamma^j - 1} < \infty. \tag{4.5}$$

If we now substitute (4.4) into (4.3), we obtain the expression

$$f(x) = A_0 \left[e^{-x} + \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \frac{1}{1-\gamma^j} \right) e^{-\gamma^i x} \right]. \tag{4.6}$$

Since $\lim_{x \to 0} f(x) = 1$, we find an alternative formula for A_0:

$$A_0 = \left[1 + \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \frac{1}{1-\gamma^j} \right) \right]^{-1}. \tag{4.7}$$

The summed terms alternative in sign in (4.6), facilitating error estimation, while expression (4.5) involves a sum of positive terms. The equality of the two expressions (4.5) and (4.7) is recorded as an algebraic

Lemma. *For $\gamma > 1$,*

$$0 < 1 + \sum_{j=1}^{\infty} \gamma^{-n} \prod_{j=1}^{n} \frac{\gamma^j}{\gamma^j - 1} = \left[1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{1}{1-\gamma^j} \right]^{-1} < \infty. \tag{4.8}$$

V. Limit Results

It is apparent from (3.7) and (3.8) that both $f_n(x)$ and $f(x)$ tend to zero as x becomes large, but at the same exponential rate. A consequence of this fact is given in

Theorem 3. *Let $f_n^*(x)$ represent the conditional probability of absorption after exactly n Poisson events, given that absorption ultimately occurs. Then $f_n^*(x)$ tends to a nonzero limit, f_n^*, as x tends to infinity:*

$$f_n^* = \frac{a_{0n}}{A_0}. \tag{5.1}$$

Furthermore,

$$\frac{f_{n+1}^*}{f_n^*} = \frac{\gamma^n}{\gamma^{n+1} - 1} \tag{5.2}$$

which may be used to show that, if $1 < \gamma < 2$, f_n^* at first increases with n but eventually tends to zero like γ^{-n}, while if $\gamma \geqslant 2$, f_n^* is a nonincreasing function of n.

Proof. Equation (5.1) follows immediately by examining the ratio of (3.7) to (3.8). To obtain (5.2), utilize (5.1) and (3.6):

$$\frac{f_{n-1}^*}{f_n^*} = \frac{a_{0,n+1}}{a_{0n}} = \frac{\gamma^{-n-1} \prod_{j=1}^{n+1} \gamma^j/(\gamma^j - 1)}{\gamma^{-n} \prod_{j=1}^{n} \gamma^j/(\gamma^j - 1)} = \frac{\gamma^n}{\gamma^{n+1} - 1} \tag{5.3}$$

The remaining properties of $\{f_n^*\}$ follow from (5.3).

VI. Generalizations

The problem discussed may be generalized in various ways. Two possibilities are (i) to alter process specification (d), allowing jump magnitudes to be a somewhat arbitrary function of the state value immediately prior to jumps, or (ii) to assume a more general stochastic process for determining the sequence $\{t_n\}$ of jump times. We shall first give an example of a result in the category (i).

Theorem 4. *Suppose the X process is defined as in Section I, except that (d) is replaced by*

(d″) $X(t_{n+1} + 0) = X(t_{n+1} - 0) + h(X(t_{n+1} - 0))$

where $h(\cdot)$ is a continuous, positive, and monotonically increasing function such *that* $\lim_{x\to\infty} h(x) = \bar{h}$. *If* $\bar{h} \leqslant 1$, *then* $f(x) = 1$ *is the unique solution of the equation*

$$f(x) = e^{-x} + \int_0^x e^{-\tau} \, d\tau \, f(x - \tau + h(x - \tau)) \tag{6.1}$$

such that $f(0) = 1$ *and* $0 < f(x) \leqslant 1$.

PROOF. Consider the equation

$$\varphi(x) = e^{-x} + \int_0^x e^{-\tau} \, d\tau \, \varphi(x - \tau + \bar{h}); \tag{6.2}$$

note that this describes the absorption probability for the X process with (d′) in force, and $Y_n \equiv \bar{h}$. Successive approximations are

$$\begin{aligned} \bar{\varphi}_0(x) &= e^{-x} \\ \bar{\varphi}_n(x) &= e^{-x} + \int_0^x e^{-\tau} \, d\tau \, \bar{\varphi}_{n-1}(x - \tau + \bar{h}). \end{aligned} \tag{6.3}$$

It can be shown the $\{\bar{\varphi}_n(x)\}$ converges uniformly to the minimal solution of (6.2), and that this solution is

$$\begin{aligned} \varphi(x) &= e^{-\alpha x} \qquad \text{if} \quad \bar{h} > 1 \\ &= 1 \qquad\quad \text{if} \quad \bar{h} \leqslant 1 \end{aligned} \tag{6.4}$$

where α is the positive real root of the equation

$$\alpha = 1 - e^{-\alpha \bar{h}}. \tag{6.5}$$

These facts will be used to show that the minimal solution of (6.1) is $f(x) = 1$, provided $\bar{h} \leqslant 1$. For, introduce the successive approximations

$$f_n(x) = e^{-x} + \int_0^x e^{-\tau} \, d\tau \, f_{n-1}(x - \tau + h(x - \tau)). \tag{6.6}$$

Induction shows that $f_n(x)$ decreases with x, so if we assume that

$$f_{n-1}(x) \geqslant \bar{\varphi}_{n-1}(x),$$

we have

$$\begin{aligned} f_n(x) &= e^{-x} + \int_0^x e^{-\tau} \, d\tau \, f_{n-1}(x - \tau + h(x - \tau)) \\ &\geqslant e^{-x} + \int_0^x e^{-\tau} \, d\tau \, f_{n-1}(x - \tau + \bar{h}) \\ &\geqslant e^{-x} + \int_0^x e^{-\tau} \, d\tau \, \bar{\varphi}_{n-1}(x - \tau + \bar{h}) = \bar{\varphi}_n(x). \end{aligned} \tag{6.7}$$

Since by (6.4) $\lim_{n\to\infty} \bar{\varphi}_n(x) = 1$, we have $\lim_{x\to\infty} f_n(x) = f(x) = 1$, and the latter is the minimal solution of (6.1) obeying the required initial condition $f(0) = 1$.

A generalization of type (ii) is given by assuming that $\{t_n\}$ is a renewal process, i.e., the interjump times $\{\tau_n\}$ are independently distributed, with

$$P\{\tau_n \leqslant x\} = U(x)$$

and $U(0) = 0$. The argument leading to (3.1) may be repeated, giving

$$\begin{aligned} f_0(x) &= 1 - U(x) \\ f_n(x) &= \int_0^x f_{n-1}(\gamma(x - \tau)\, dU(\tau) \end{aligned} \tag{6.8}$$

A formal solution is available by successive substitution:

$$f_n(x) = U_{\gamma^{n-1}} * U_{\gamma^{n-2}} * \cdots * U(x) - U_{\gamma^n} * U_{\gamma^{n-1}} * \cdots * U(x) \tag{6.9}$$

where $U_{\gamma^k}(x) = U(\gamma^k x)$, and $*$ denotes convolution, e.g.,

$$U_\gamma * U(x) = \int_0^x U(\gamma(x - \tau))\, dU(\tau).$$

Then

$$f(x) = 1 - U * U_\gamma * U_{\gamma^2} * \cdots \qquad \cdots (x), \tag{6.10}$$

providing the latter expression has meaning. In general, $f(x)$ need not be continuous. If, for example, $\tau_n = \Delta$, a positive constant, then it can be seen that

$$\begin{aligned} f(x) &= 1 \qquad \text{for} \qquad x \leqslant \gamma\Delta/(\gamma - 1) \\ &= 0 \qquad \text{for} \qquad x > \gamma\Delta/(\gamma - 1). \end{aligned}$$

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