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An iterative method for solving nonlinear functional equations

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Abstract

An iterative method for solving nonlinear functional equations, viz. nonlinear Volterra integral equations, algebraic equations and systems of ordinary differential equation, nonlinear algebraic equations and fractional differential equations has been discussed.

Keywords: Nonlinear Volterra integral equations; System of ordinary differential equations; Iterative method; Contraction; Banach fixed point theorem; Fractional differential equation

1. Introduction

A variety of problems in physics, chemistry and biology have their mathematical setting as integral equations [10]. Therefore, developing methods to solve integral equations (especially nonlinear), is receiving increasing attention in recent years [1–8]. In the present work we describe an iterative method which can be utilized to obtain solutions of nonlinear functional equations. The method when combined with algebraic computing software (Mathematica, e.g.) turns out to be powerful.

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The present paper has been organized as follows. In Section 2 the iterative method is described and in Section 3 existence of solutions for nonlinear Volterra integral equations has been proved using this method. Illustrative examples have been presented in Section 4 followed by the conclusions in Section 5.

2. An iterative method

Consider the following general functional equation:

$$y = N(y) + f, \tag{1}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function. We are looking for a solution y of Eq. (1) having the series form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{2}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{3}$$

From Eqs. (2) and (3), Eq. (1) is equivalent to

$$\sum_{i=0}^{\infty} y_i = f + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{4}$$

We define the recurrence relation:

$$\begin{cases} y_0 = f, \\ y_1 = N(y_0), \\ y_{m+1} = N(y_0 + \cdots + y_m) - N(y_0 + \cdots + y_{m-1}), \quad m = 1, 2, \ldots. \end{cases} \tag{5}$$

Then

$$(y_1 + \cdots + y_{m+1}) = N(y_0 + \cdots + y_m), \quad m = 1, 2, \ldots, \tag{6}$$

and

$$y = f + \sum_{i=1}^{\infty} y_i. \tag{7}$$

If N is a contraction, i.e. $\|N(x) - N(y)\| \leqslant K\|x - y\|$, $0 < K < 1$, then

$$\|y_{m+1}\| = \left\|N(y_0 + \cdots + y_m) - N(y_0 + \cdots + y_{m-1})\right\| \leqslant K\|y_m\| \leqslant K^m\|y_0\|,$$
$$m = 0, 1, 2, \ldots,$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Eq. (1) [5], which is unique, in view of the Banach fixed point theorem [10].

3. Nonlinear Volterra integral equation

Consider the Volterra integral equation

$$y(x) = f(x) + \int_a^x F\big(x, t, y(t)\big)\, dt, \tag{8}$$

where $|x-a| \leqslant \alpha$, $|t-a| \leqslant \alpha$, F is a continuous function of its arguments and satisfies Lipschitz condition, $|F(x,t,\phi) - F(x,t,\psi)| < K|\phi - \psi|$. Let $|F(x,t,\phi)| < M$. Define

$$\begin{aligned}
&y_0(x) = f(x),\\
&y_1(x) = \int_a^x F\big(x, t, y_0(t)\big)\, dt,\\
&y_{m+1}(x) = \int_a^x \big|F(x,t,y_0+\cdots+y_m) - F(x,t,y_0+\cdots+y_{m-1})\big|\, dt,\\
&\quad m = 1, 2, \ldots.
\end{aligned} \tag{9}$$

We prove $\sum_{i=1}^{\infty} y_i(x)$ is uniformly convergent.

$$\begin{aligned}
\big|y_1(x)\big| &\leqslant \int_a^x \big|F\big(x,t,y_0(t)\big)\big|\, dt \leqslant M(x-a) \leqslant M\alpha,\\
\big|y_2(x)\big| &\leqslant \int_a^x \big|F\big(x,t,y_0(t)+y_1(t)\big) - F\big(x,t,y_0(t)\big)\big|\, dt \leqslant K\left|\int_a^x y_1(t)\, dt\right|\\
&\leqslant MK\frac{(x-a)^2}{2!} \leqslant \frac{M}{K}\frac{(K\alpha)^2}{2!},\\
\big|y_3(x)\big| &\leqslant \int_a^x \big|F\big(x,t,y_0(t)+y_1(t)+y_2(t)\big) - F\big(x,t,y_0(t)+y_1(t)\big)\big|\, dt\\
&\leqslant K\left|\int_a^x y_2(t)\, dt\right| \leqslant MK^2\frac{(x-a)^3}{3!} \leqslant \frac{M}{K}\frac{(K\alpha)^3}{3!},\\
&\vdots\\
\big|y_{m+1}(x)\big| &\leqslant \int_a^x \big|F(x,t,y_0+\cdots+y_m) - F(x,t,y_0+\cdots+y_{m-1})\big|\, dt\\
&\leqslant K\left|\int_a^x y_{m-1}(t)\, dt\right| \leqslant MK^m\frac{(x-a)^{m+1}}{(m+1)!} \leqslant \frac{M}{K}\frac{(K\alpha)^{m+1}}{(m+1)!}.
\end{aligned} \tag{10}$$

Hence $\sum_{i=0}^{\infty} y_i(x)$ is absolutely and uniformly convergent and $y(x)$ satisfies Eq. (8). If Eq. (8) does not possess unique solution, then this iterative method will give a solution among many (possible) other solutions.

4. Illustrative examples

(i) Consider the following nonlinear differential equation with exact solution $y = x^{-1}$ for $x > 0$:

$$y' = -y^2, \qquad y(1) = 1. \tag{11}$$

The initial value problem in Eq. (11) is equivalent to the integral equation

$$y(x) = 1 - \int_1^x y^2\,dt. \tag{12}$$

Following the algorithm given in (9):

$$\begin{aligned}
&y_0 = 1,\\
&y_1 = N(y_0) = -\int_1^x y_0^2\,dt = 1 - x,\\
&y_2 = N(y_0 + y_1) - N(y_0) = \frac{4}{3} - 3x + 2x^2 - \frac{x^3}{3},\\
&y_3 = \frac{113}{63} - \frac{64x}{9} + \frac{34x^2}{3} - \frac{85x^3}{9} + \frac{41x^4}{9} - \frac{4x^5}{3} + \frac{2x^6}{9} - \frac{x^7}{63},\\
&\quad\vdots
\end{aligned}$$

In Fig. 1 we have plotted $\sum_{i=0}^{5} y_i(x)$, which is almost equal to the exact solution $y = x^{-1}$.

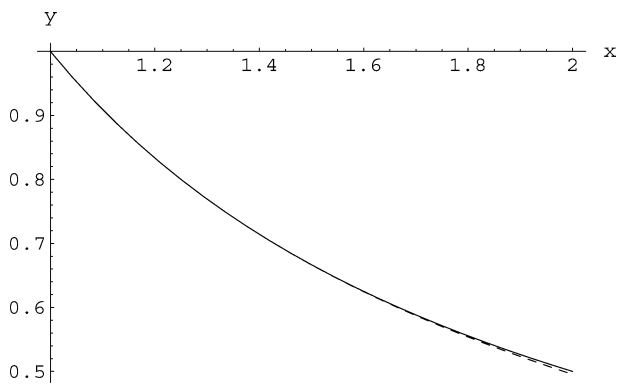

Fig. 1.

(ii) Consider the following linear Volterra integral equation:

$$y(x) = \int_0^x \frac{1+y(t)}{1+t}\,dt.$$

Following (9) we get

$$y_0 = 0, \qquad y_1(x) = N(y_0) = \int_0^x \frac{1}{1+t}\,dt = \log(1+x),$$

$$y_2(x) = N(y_0+y_1) - N(y_0) = \int_0^x \frac{\log(1+x)}{1+t}\,dt = \frac{(\log(1+x))^2}{2!},$$

$$y_{m+1}(x) = N(y_0+\cdots+y_m) - N(y_0+\cdots+y_{m-1}) = \frac{(\log(1+x))^m}{m!},$$
$$m = 1, 2, \ldots.$$

Hence $y(x) = \sum_0^\infty y_i(x) = \exp(\log(1+x)) - 1 = x$.

(iii) Consider the following nonlinear Fredholm integral equation:

$$y(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 xt\,y^2(t)\,dt.$$

Then from (9), first few terms of $y(x)$ are:

$$y_0 = \frac{7}{8}x = 0.875x, \qquad y_1(x) = N(y_0) = \frac{1}{2}\int_0^1 xty_0^2(t)\,dt = \frac{49}{512}x,$$

$$y_2(x) = \frac{1}{2}x\int_0^1 t\big[\big(y_0(t)+y_1(t)\big)^2 - y_0^2(t)\big]\,dt = \frac{46305}{2097152}x.$$

The solution in a series form is given by

$$y(x) = 0.875x + 0.0957031x + 0.0220799x + 0.00541921x + \cdots \approx x.$$

This equation has two exact solutions, x and $7x$ [13].

(iv) Consider the following nonlinear Volterra integral equation:

$$y(x) = e^x - \frac{1}{3}xe^{3x} + \frac{1}{3}x + \int_0^x xy^3(t)\,dt.$$

We use here modified iterative scheme [14], in which we take:

$$y_0 = e^x, \qquad y_1(x) = -\frac{1}{3}xe^{3x} + \frac{1}{3}x + N(y_0),$$
$$y_1(x) = -\frac{1}{3}xe^{3x} + \frac{1}{3}x + \int_0^x xy_0^3(t)\,dt = 0,$$
$$y_2(x) = N(y_0 + y_1) - N(y_0) = 0, \quad \text{and}$$
$$y_{m+1}(x) = 0, \quad m \geqslant 0.$$

Thus we find that the solution is $y(x) = e^x$, which is the exact solution.

(v) Consider the following system of nonlinear ordinary differential equations:

$$\begin{aligned} y_1'(x) &= 2y_2^2, & y_1(0) &= 1, \\ y_2'(x) &= e^{-x}y_1, & y_2(0) &= 1, \\ y_3'(x) &= y_2 + y_3, & y_3(0) &= 0. \end{aligned} \tag{13}$$

System (13) is equivalent to the following system of integral equations:

$$\begin{aligned} y_1(x) &= 1 + \int_0^x 2y_2^2\,dt, \\ y_2(x) &= 1 + \int_0^x e^{-t}y_1\,dt, \\ y_3(x) &= \int_0^x (y_2 + y_3)\,dt. \end{aligned} \tag{14}$$

Equation (9) leads to

$$y_{10}(x) = 1, \qquad y_{11}(x) = N_1(y_{10}, y_{20}, y_{30}) = \int_0^x 2y_{20}^2\,dt,$$
$$y_{20}(x) = 1, \qquad y_{21}(x) = N_2(y_{10}, y_{20}, y_{30}) = \int_0^x e^{-t}y_{10}\,dt,$$
$$y_{30}(x) = 0, \qquad y_{31}(x) = N_3(y_{10}, y_{20}, y_{30}) = \int_0^x (y_{20} + y_{30})\,dt,$$
$$\begin{aligned} y_{i,m}(x) = {} & N_i(y_{10} + \cdots + y_{1,m}, y_{20} + \cdots + y_{2,m}, y_{30} + \cdots + y_{3,m}) \\ & - N_i(y_{10} + \cdots + y_{1,m-1}, y_{20} + \cdots + y_{2,m-1}, y_{30} + \cdots + y_{3,m-1}), \\ & i = 1, 2, 3,\ m = 1, 2, \ldots. \end{aligned}$$

In Figs. 2–4, the approximation solutions $y_1 = \sum_{m=0}^{5} y_{1m}$, $y_2 = \sum_{m=0}^{5} y_{2m}$ and $y_3 = \sum_{m=0}^{5} y_{3m}$ have been plotted, which coincide with the exact solution.

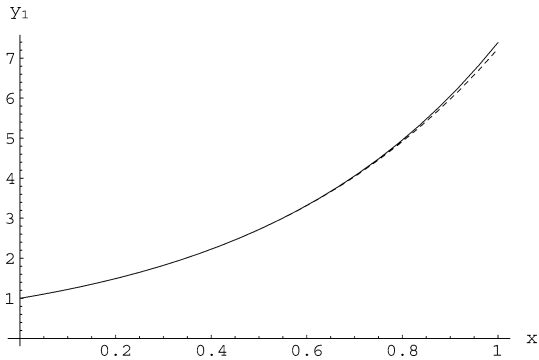

Fig. 2. Exact solution is e^{2x}.

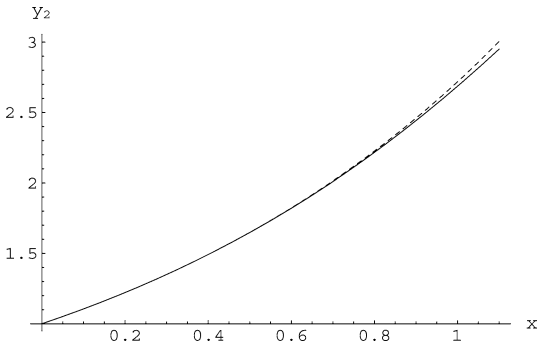

Fig. 3. Exact solution is e^{x}.

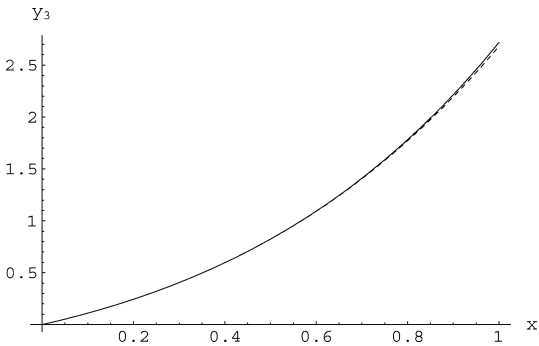

Fig. 4. Exact solution is xe^{x}.

Comment. Examples (iii) and (iv) have been solved by Wazwaz [13] and example (v) has been solved by Biazar et al. [3] using Adomian decomposition method (ADM) [1]. The method presented here is easier compared to ADM, and gives the answers to the same accuracy.

(vi) Consider the nonlinear algebraic equation $x^6 - 5x^5 + 3x^4 + x^3 + 2x^2 - 8x - 0.5 = 0$. We rewrite this equation as

$$x = -\frac{1}{16} + \frac{1}{8}x^6 - \frac{5}{8}x^5 + \frac{3}{8}x^4 + \frac{1}{8}x^3 + \frac{1}{4}x^2 = x_0 + N(x),$$

where $x_0 = -\frac{1}{16}$. As $N'(x_0) < 1$, we can employ the algorithm given in (5) to get one solution as follows:

$$x_0 = -\frac{1}{16}, \qquad x_1 = N(x_0) = \frac{1}{8}x_0^6 - \frac{5}{8}x_0^5 + \frac{3}{8}x_0^4 + \frac{1}{8}x_0^3 + \frac{1}{4}x_0^2,$$
$$x_{m+1} = N(x_0 + \cdots + x_m) - N(x_0 + \cdots + x_{m-1}), \quad m = 1, 2, \ldots.$$

The first five terms are:

$$x_1 = 0.0009523704648, \qquad x_2 = N(x_0 + x_1) - N(x_0) = -0.00002854648344,$$
$$x_3 = N(x_0 + x_1 + x_2) - N(x_0 + x_1) = 8.4940820811044 \times 10^{-7},$$
$$x_4 = N(x_0 + \cdots + x_3) - N(x_0 + x_1 + x_2) = -2.527994113539 \times 10^{-8},$$
$$x_5 = N(x_0 + \cdots + x_4) - N(x_0 + \cdots + x_3) = 7.5237232810 \times 10^{-10}.$$

The sum of first five terms is

$$x \approx x_0 + \cdots + x_5 = -0.06157535114,$$

which matches with the approximate solution given by Ouedraogo et al. [11]. Their answer is $x = -0.06157535132499062$. The answer given by Mathematica software is $x = -0.06157535115974503$ [7].

(vii) Consider the following system of nonlinear algebraic equations:

$$x_1^2 - 10x_1 + x_2^2 + 8 = 0,$$
$$x_1x_2^2 + x_1 - 10x_2 + 8 = 0.$$

We rewrite this system as

$$x_1 = \frac{8}{10} + \frac{1}{10}x_1^2 + \frac{1}{10}x_2^2 = x_{10} + N_1(x_1, x_2),$$
$$x_2 = \frac{8}{10} + \frac{1}{10}x_1 + \frac{1}{10}x_1x_2^2 = x_{20} + N_2(x_1, x_2).$$

We can employ (5) to get solutions as follows:

$$x_{10} = \frac{8}{10}, \qquad x_{20} = \frac{8}{10},$$
$$x_{11} = N_1(x_{10}, x_{20}) = \frac{1}{10}x_{10}^2 + \frac{1}{10}x_{20}^2,$$
$$x_{21} = N_2(x_{10}, x_{20}) = \frac{1}{10}x_{10} + \frac{1}{10}x_{10}x_{20}^2,$$
$$\begin{aligned} x_{1m+1} &= N_1(x_{10} + \cdots + x_{1m}, x_{20} + \cdots + x_{2m}) \\ &\quad - N_1(x_{10} + \cdots + x_{1m-1}, x_{20} + \cdots + x_{2m-1}), \\ x_{2m+1} &= N_2(x_{10} + \cdots + x_{1m}, x_{20} + \cdots + x_{2m}) \\ &\quad - N_2(x_{10} + \cdots + x_{1m-1}, x_{20} + \cdots + x_{2m-1}), \end{aligned}$$

$m = 1, 2, \ldots$. The first five terms are:

$$\begin{aligned}
x_{10} &= 0.80000000, & x_{20} &= 0.80000000,\\
x_{11} &= 0.12800000, & x_{21} &= 0.13120000,\\
x_{12} &= 0.044831744, & x_{22} &= 0.042069983,\\
x_{13} &= 0.016533862, & x_{23} &= 0.016165112,\\
x_{14} &= 0.0064170048, & x_{24} &= 0.0063585583,\\
x_{15} &= 0.0025361898, & x_{25} &= 0.0025269092.
\end{aligned}$$

The sum of first five terms is

$$x_1 \approx x_{10} + \cdots + x_{15} = 0.99831880, \qquad x_2 \approx x_{20} + \cdots + x_{25} = 0.99832056.$$

The exact solution is $x = (1, 1)^t$. Revised ADM [8] (using five iterations) gives the answer $(0.99778, 0.997853)^t$, whereas the standard ADM (using five iterations) gives the answer $(0.99607593, 0.99556077)^t$ [2].

(viii) We apply this method to solve a system of nonlinear fractional differential equations. Consider the system of nonlinear fractional differential equations:

$$\begin{aligned}
D^{0.5} y_1 &= 2y_2^2, & y_1(0) &= 0,\\
D^{0.4} y_2 &= x y_1, & y_2(0) &= 1,\\
D^{0.3} y_2 &= y_2 y_3, & y_3(0) &= 1,
\end{aligned}$$

where D^α denotes Caputo fractional derivative of order α [12]. We apply new iterative method for solving this system. For definitions of Riemann–Liouville fractional integral and Caputo fractional derivative, we refer the reader to [6,12]. The above system is equivalent to the following system of integral equations:

$$y_1 = 2I^{0.5} y_2^2(x) = \frac{2}{\Gamma(0.5)} \int_0^x \frac{y_2^2(t)}{(x-t)^{0.5}}\, dt,$$

$$y_2 = y_2(0) + I^{0.4}\, x\, y_1(x) = 1 + \frac{1}{\Gamma(0.4)} \int_0^x \frac{t\, y_1(t)}{(x-t)^{0.6}}\, dt,$$

$$y_3 = y_3(0) + I^{0.3}\big(y_2(x) y_3(x)\big) = 1 + \frac{1}{\Gamma(0.3)} \int_0^x \frac{y_2(t)\, y_3(t)}{(x-t)^{0.7}}\, dt,$$

where I^α denotes Riemann–Liouville fractional integral of order α [12]. First few terms of the iteration are given below:

$$\begin{aligned}
y_{10} &= 0, & y_{11} &= 2.25676x^{0.5}, & y_{12} &= 0,\\
y_{20} &= 1, & y_{21} &= 0, & y_{22} &= 1.64172x^{1.9},\\
y_{30} &= 1, & y_{31} &= 1.11424x^{0.3}, & y_{32} &= 1.11917x^{0.6}.
\end{aligned}$$

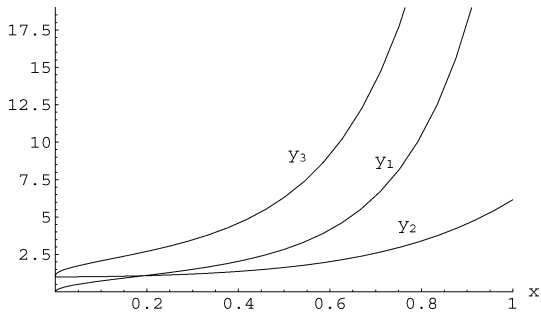

Fig. 5. Using new iterative method.

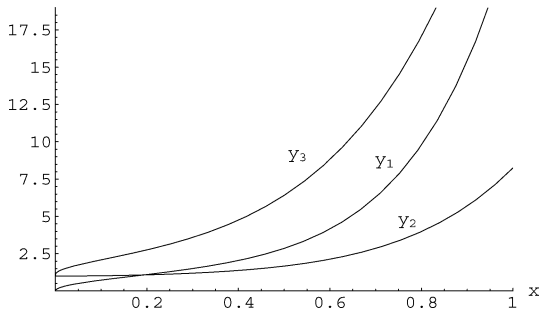

Fig. 6. Using ADM.

This example has been solved using ADM as well [9]. In Figs. 5 and 6, the approximate solutions using the new iterative method and the ADM have been plotted, respectively.

It should be noted that the solution plotted in Fig. 5 has been obtained by summing first five terms of the new iterative method, whereas the solution plotted in Fig. 6 corresponds to the sum of first 7 terms of ADM.

5. Conclusions

An iterative method for solving functional equations has been discussed. The proof of existence of solution for nonlinear Volterra integral equations is presented. Illustrative examples dealing with algebraic equation, Volterra integral equations, systems of ordinary differential equation, nonlinear algebraic equations and fractional differential equations have been given. The method proves to be simple in its principles and convenient for computer algorithms.

Mathematica has been used for computations in this paper.

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