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Polar Spaces, Generalized Hexagons and Perfect Codes

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1. OVOIDS AND SPREADS OF FINITE CLASSICAL POLAR SPACES

Let P be a finite classical polar space of rank (or index) r, $r \geqslant 2$ [4]. An ovoid O of P is a pointset of P, which has exactly one point in common with every totally isotropic subspace of rank r [12]. A spread S of P is a set of maximal totally isotropic subspaces, which constitutes a partition of the pointset [12].

We shall use the following notation:

$W_n(q)$: the polar space arising from a symplectic polarity of $PG(n, q)$, n odd;

$Q(2n, q)$: the polar space arising from a non-singular quadric Q in $PG(2n, q)$;

$Q^+(2n + 1, q)$: the polar space arising from a non-singular hyperbolic quadric Q^+ [8] in $PG(2n + 1, q)$;

$Q^-(2n + 1, q)$: the polar space arising from a non-singular elliptic quadric Q^- [8] in $PG(2n + 1, q)$;

$H(n, q^2)$: the polar space arising from a non-singular hermitian variety H [8] in $PG(n, q^2)$.

The following results are known:

(a) $W_n(q)$, n odd, has always a (regular) spread ([10], [9]). $W_3(q)$ has an ovoid iff q is even [11]; every ovoid of $W_3(q)$, q even, is an ordinary ovoid of $PG(3, q)$ and every ordinary ovoid of $PG(3, q)$, q even, is an ovoid of some $W_3(q)$ [13]. $W_n(q)$, n odd and $n > 3$, has no ovoid [12].

(b) $Q^+(4n + 1, q)$ has no spreads [9]. $Q^+(m, q)$, q even, m odd and $m \neq 4n + 1$, has always a spread [9]. $Q^+(3, q)$ has spreads and ovoids (trivial). $Q^+(5, q)$ has always ovoids (if we consider Q^+ as the Klein quadric, these ovoids correspond to the ordinary spreads of $PG(3, q)$). $Q^+(7, q)$ has a spread, and consequently $Q^+(7, q)$ has an ovoid [15].

$Q(2n, q)$, q even, has always a spread [9]. $Q(2n, q)$, $n > 2$ and q even, has no ovoids [12]. Since $Q(4, q)$ is the dual of $W_3(q)$ [14], the polar space $Q(4, q)$

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has always an ovoid, and has a spread iff q is even. Since $Q^+(7, q)$ has a spread, we see by intersection that also $Q(6, q)$ has a spread. Finally $Q(6, q)$, $q = 3^{2h+1}$, has an ovoid [12].

$Q^-(2n + 1, q)$, $n > 1$, has no ovoid [12]. $Q^-(m, q)$, q even and m odd, has always a spread [9]. Since $Q(6, q)$ has a spread, we see by intersection that also $Q^-(5, q)$ has a spread. Moreover $Q^-(5, q)$ is the dual of $H(3, q^2)$ ([2], [14]), hence the existence of spreads of $Q^-(5, q)$ also follows from the existence of ovoids of $H(3, q^2)$ (see (c)).

(c) $H(n, q^2)$, n even and $n > 2$, has no ovoid [12]. $H(3, q^2)$ has ovoids (any hermitian curve on H is an ovoid of $H(3, q^2)$). Since $H(3, q^2)$ is the dual of $Q^-(5, q)$ ([2], [14]), the polar space $H(3, q^2)$ has no spread.

2. Ovoids and Spreads of Generalized Hexagons of Order n

A generalized hexagon [6] of order n ($\geqslant 1$) is an incidence structure $S = (P, B, \mathrm{I})$, with an incidence relation satisfying the following axioms:

(i) each point (resp. line) is incident with $n + 1$ lines (resp. points);

(ii) $|P| = |B| = 1 + n + n^2 + n^3 + n^4 + n^5 = v$;

(iii) 6 is the smallest positive integer k such that S has a circuit consisting of k points and k lines.

As usual the distance of two elements $\alpha, \beta \in P \cup B$ is denoted by $\lambda(\alpha, \beta)$ or $\lambda(\beta, \alpha)$ [6].

If V is a set of points (resp. lines) such that $\lambda(x, y) = 6$ (resp. $\lambda(L, M) = 6$) for all distinct $x, y \in V$ (resp. $L, M \in V$), then $|V| \leqslant v/(n^3 + n + 1)$ or $|V| \leqslant n^3 + 1$. If $|V| = n^3 + 1$, then we say that V is an ovoid (resp. spread) of the hexagon S [6].

In [6] we remarked that the classical generalized hexagon $H(q)$ (of order q) arising from $G_2(q)$ has always a spread. This followed from a result of P. Fong, who proved that $G_2(q)$ has a subgroup isomorphic to $SU_3(q)$ which has an orbit of length $q^3 + 1$ on the lines of $H(q)$ and which is doubly transitive on this orbit. We also proved [6] that a generalized hexagon S of order q has an ovoid (resp. spread) if S admits a polarity. J. Tits informed us that it is possible to prove that the generalized hexagon $H(q)$ of order q, $q = 3^{2h+1}$, admits a polarity. Since $H(q)$ has an ovoid iff $Q(6, q)$ has an ovoid [12], there follows that $H(q)$, q even, has no ovoid.

In the presentation of J. Tits of the classical generalized hexagon of order q, the set P is the pointset of $Q(6, q)$ and the set B is a subset of the lineset of $Q(6, q)$ [15]. For $x, y \in P$, $x \neq y$, we have $\lambda(x, y) \leqslant 4$ iff x and y are on a line of the polar space $Q(6, q)$ (see also [17]). Now it is easy to show that all the lines of $H(q)$ containing the point x are in a plane of the quadric Q. Indeed, let us consider the lines L, M of $H(q)$ containing x. If $y \in L$, $z \in M$, $x \neq y$ and $x \neq z$, then $\lambda(y, z) = 4$ and so yz is a line of Q. Consequently the plane LM is on Q. Let u be a point of the plane LM, $u \notin L$, $u \notin M$. Then xu is a line of Q, and so $\lambda(x, u) \leqslant 4$. Let us assume that $\lambda(x, u) = 4$ and call w the point defined by $\lambda(x, w) = \lambda(u, w) = 2$. If w is in the plane LM, then $w \in L$ or $w \in M$, say $w \in M$. Then, if $y \in L - \{x\}$, we have $\lambda(y, u) = 6$, a contradiction. If w is not in the plane LM, then the planes LM, Lw, Mw, uxw are on Q, and so the threespace LMw is on Q, a contradiction. Hence $\lambda(x, u) = 2$, i.e., the $q + 1$ lines of $H(q)$ containing x are the $q + 1$ lines through x of the plane LM of Q. So with the $q^5 + q^4 + q^3 + q^2 + q + 1$ points of $H(q)$ there correspond $q^5 + q^4 + q^3 + q^2 + q + 1$ planes of Q. These planes will be called the $H(q)$-planes.

We consider a $PG(5, q)$ in $PG(6, q)$, for which $PG(5, q) \cap Q$ is an elliptic quadric Q^-. Now we count in two ways the number of ordered pairs (line L of $H(q)$, point of Q^- on L). There results $\alpha(q + 1) + (q^5 + q^4 + q^3 + q^2 + q + 1 - \alpha) = (q + 1)^2(q^3 + 1)$, where α is the number of lines of $H(q)$ on Q^-. Hence $\alpha = q^3 + 1$. No two of these $q^3 + 1$ lines intersect, since otherwise their plane (which is an $H(q)$-plane) is on Q^-, a contradiction. Consequently these lines constitute a spread of $Q^-(5, q)$. Moreover any two of these lines are at distance 6 in $H(q)$. So they also constitute a spread of $H(q)$. This provides an elementary proof that $H(q)$ has always a spread.

Let us suppose that $H(q)$ admits a polarity (then $3 \mid q$) and call O (resp. T) the corresponding ovoid (resp. spread). We remark that the $q^3 + 1$ $H(q)$-planes which correspond to the points of O constitute a spread of $Q(6, q)$ (this provides a proof that $Q(6, q)$ has a spread for $q = 3^{2h+1}$). Let us assume that the spread T is of the type described in the preceding paragraph. Then O is contained in a $PG(5, q)$, where $PG(5, q) \cap Q$ is an elliptic quadric Q^-. Since O is an ovoid of $H(q)$, it is an ovoid of $Q(6, q)$ [12], and consequently also of $Q^-(5, q)$, a contradiction. Hence for $q = 3^{2h+1}$ the hexagon $H(q)$ has at least two types of spreads.

3. A Model of the Classical Generalized Quadrangle with $s = t^2$ $(t > 1)$

Consider the polar space $Q(6, q)$, and let $PG(5, q)$ be such that $PG(5, q) \cap Q$ is an elliptic quadric Q^- of $PG(5, q)$. Let T be a spread of $Q^-(5, q)$ and suppose that the regulus (on Q^-) defined by each two elements of T is contained in T (then T is a spread corresponding to an hermitian curve on a non-singular hermitian variety H in $PG(3, q^2)$). Define as follows the incidence structure $S = (P, B, \mathrm{I})$: points of type (i) are the points of $Q - Q^-$, and points of type (ii) are the lines of T; lines are the planes of Q containing an element of T; I is the natural incidence.

THEOREM. *The incidence structure S is isomorphic to the generalized quadrangle $H(3, q^2)$.*

Proof. Any point of type (ii) of S is incident with $q + 1$ lines of S. Now we consider a point x of type (i). If V is the tangent hyperplane of Q at x, then the number of lines of S incident with x equals the number of lines of T in V. So we are looking for the number of elements of T in a $PG(4, q) \subset PG(5, q)$. Now any spread of $Q^-(5, q)$ has exactly $q + 1$ elements in a $PG(4, q) \subset PG(5, q)$. Hence any point of S is incident with $q + 1$ lines of S. Also any two points of S are incident with at most one line of S. Next we remark that any line of S is incident with $q^2 + 1$ points of S, and that any two lines of S are incident with at most one point of S.

Next we consider a point $L \in T$ of type (ii) and a line $PG(2, q)$ of S, where $L \not\subset PG(2, q)$. If $PG(4, q)$ is the polar space of L with respect to Q and if $\{y\} = PG(4, q) \cap PG(2, q)$, then Ly is the unique line of S which is incident with the point L of S and concurrent with the line $PG(2, q)$ of S. Now we consider a point x of type (i) and a line $PG(2, q)$ of S, where $L = PG(2, q) \cap Q^-$ and $x \notin PG(2, q)$. Suppose that $L_1x,..., L_{q+1}x$, $L_i \in T$, are the $q + 1$ lines of S containing x. Then $L_1,..., L_{q+1}$ constitute a regulus, i.e., are contained in a threespace. If $L \in \{L_1,..., L_{q+1}\}$, then Lx is the unique line of S which is incident with the point x of S and concurrent with the line $PG(2, q)$ of S. Consider now the case $L \notin \{L_1,..., L_{q+1}\}$. If $PG(3, q)$ is the threespace containing $L_1,..., L_{q+1}$, and if $\{z\} = xPG(3, q) \cap PG(2, q)$, then z is the unique point of S which is incident with the line $PG(2, q)$ and collinear with the point x.

Hence S is a generalized quadrangle with parameters $s = q^2$, $t = q$.

Finally we show that S is of classical type. Let $Q(6, q)$ be embedded in the polar space $Q^+(7, q)$ and let V_3^1 be one of the families of generating threespaces of the quadric Q^+. If L is a point of type (ii) of S and if $\pi_1, \pi_2,..., \pi_{q+1}$ are the $q + 1$ elements of B incident with L, then the $q + 1$ elements of V_3^1 containing $\pi_1, \pi_2,..., \pi_{q+1}$ evidently are the $q + 1$ elements of V_3^1 containing L. Now we consider a point x of type (i) of S and the $q + 1$ elements ξ_1, $\xi_2,..., \xi_{q+1}$ of B which are incident with x (remark that the $q + 1$ lines of T defined by $\xi_1, \xi_2,..., \xi_{q+1}$ are the elements of a regulus, i.e., are contained in a $PG(3, q)$). We show that the $q + 1$ elements $\alpha_1^1, \alpha_2^1,..., \alpha_{q+1}^1$ of V_3^1 containing $\xi_1, \xi_2,..., \xi_{q+1}$ are the $q + 1$ elements of V_3^1 containing some line M, $x \in M$, of $Q^+(7, q)$. Let $PG^{(i)}(4, q)$ be the polar space of ξ_i with respect to the quadric Q^+. Then $PG^{(i)}(4, q) \cap Q^+ = \alpha_i^1 \cup \alpha_i^2$, where α_i^j belongs to the family V_3^j of generating threespaces of Q^+. Moreover $PG^{(i)}(4, q)$ contains a plane π, the polar plane of the fourdimensional space $xPG(3, q)$ with respect to Q^+, and $\pi \cap Q^+$ consists of two lines M and N. One of these lines, say M, is contained in the spaces α_i^1, and the other line N is contained in the spaces α_i^2. Hence the $q + 1$ elements of V_3^1 containing $\xi_1, \xi_2,..., \xi_{q+1}$ are the

$q + 1$ elements of V_3^1 containing some line M of $Q^+(7, q)$. Consequently to every point of type (i) of S there corresponds a line M of $Q^+(7, q)$.

The set of all elements of V_3^1 which contain an element of B, is denoted by B'. And P' is the union of T and the set of the lines M of $Q^+(7, q)$ which correspond to the points of type (i) of S. If I$'$ is the natural incidence, then $S' = (P', B', \mathrm{I}')$ evidently is a generalized quadrangle isomorphic to S. By triality [15] S' is the dual of a generalized quadrangle S'' with parameters $s = q$, $t = q^2$, whose points are points of a projective space of order q and whose lines are lines of that projective space. By the theorem of Buekenhout–Lefèvre [5] S'' is isomorphic to $Q^-(5, q)$, and consequently S', and thus S, is isomorphic to $H(3, q^2)$.

4. Spreads of the Polar Space $Q^-(5, q)$

To an hermitian curve on a non-singular hermitian variety H of $PG(3, q^2)$ there corresponds a spread of $Q^-(5, q)$. Such a spread will be called regular. In 1 we also remarked that any spread of $Q(6, q)$ defines by intersection a spread of $Q^-(5, q)$.

THEOREM. *A spread T of $Q^-(5, q)$ corresponding to a spread of $Q(6, q)$, is never regular.*

Proof. Let T^* be a spread of $Q(6, q)$ and T the corresponding spread of $Q^-(5, q)$. Suppose that T is regular, i.e., suppose that the regulus (on Q^-) defined by each two elements of T is contained in T. Then by 3 the spread T defines a classical generalized quadrangle $S = (P, B, \mathrm{I})$ with parameters $s = q^2$, $t = q$. Evidently T^* is a spread of S, a contradiction since $H(3, q^2)$ has no spread. We conclude that T is not regular.

COROLLARY. *$Q^-(5, q)$ has at least two types of spreads, and dually $H(3, q^2)$ has at least two types of ovoids.*

5. Perfect Codes

In a graph (assumed to be connected, undirected, and without loops or multiple edges), the distance $d(\alpha, \beta)$ between two vertices α and β is the length of the shortest path joining them. For a positive integer e, a perfect e-code is a nonempty subset C of the vertex set with the property that any vertex lies at distance at most e from a unique vertex in C. Biggs [1] has shown that necessary conditions for the existence of a perfect 1-code in a regular graph of valency k on v vertices are that $k + 1$ divides v and that -1 is an eigenvalue of the adjacency matrix of the graph. These results have been extended to perfect e-codes in metrically regular graphs by Delsarte [7] (see also Biggs [1]). (Metrically regular graphs are defined in [7]).

The point graph (resp. line graph) of a generalized hexagon $S = (P, B, \mathrm{I})$ is defined to have vertex set P (resp. B), vertices α and β being adjacent whenever $\lambda(\alpha, \beta) = 2$. These graphs are metrically regular and satisfy the necessary conditions for the existence of a perfect 1-code. Indeed, it is easy to see that a perfect 1-code is the same thing as an ovoid (resp. spread). These perfect 1-codes were discovered by Cameron, Thas and Payne [6] and to their knowledge this was the first infinite class of perfect e-codes, apart from the Hamming codes, the classical repetition codes, and more generally the perfect e-codes in the metrically regular antipodal graphs of diameter $2e + 1$ [1]. In Thas [10] a new such infinite class of perfect 1-codes is described in the following way. Consider the polar space $W_5(q)$. The vertices of the graph Γ are the totally isotropic subspaces of rank 3, two vertices α and β ($\alpha \neq \beta$) being adjacent whenever $\alpha \cap \beta$ is a line. The graph is metrically regular and satisfies the necessary conditions for the existence of a perfect 1-code. Now it is easy to see that a perfect 1-code is the same thing as a spread of $W_5(q)$.

Define as follows the incidence structure $S^* = (P^*, B^*, \mathrm{I}^*)$: P^* is the set of the $H(q)$-planes (see 2); B^* is the set of all lines of $H(q)$; if $\pi \in P^*$ and $L \in B^*$, then $\pi \mathrm{I}^* L$ iff $L \subset \pi$. Evidently S^* is a generalized hexagon isomorphic to $H(q)$. The point graph of S^* is denoted by Γ^*. If O is an ovoid of $H(q)$, then the corresponding ovoid O^* of S^* is a spread of $Q(6, q)$. Next, let Γ' be the graph with vertices the totally isotropic subspaces of rank 3 of $Q(6, q)$, two vertices α and β ($\alpha \neq \beta$) being adjacent iff $\alpha \cap \beta$ is a line. Then Γ' is metrically regular with the same parameters as the graph Γ arising from $W_5(q)$ and described in the preceding paragraph. Since $Q(6, q)$ is isomorphic to $W_5(q)$ iff q is even [12], it follows easily that Γ is isomorphic to Γ' iff q is even. Again a perfect 1-code of Γ' is the same thing as a spread of $Q(6, q)$. So for any q there arises a perfect 1-code of Γ', and hence we have a new infinite class of perfect 1-codes. Finally we remark that Γ^* is a subgraph of $\Gamma'(|\Gamma^*| = (q^3 + 1)(q^2 + q + 1), |\Gamma'| = (q^3 + 1)(q^2 + 1)(q + 1))$ and that a set consisting of $q^3 + 1$ vertices of Γ^* is a perfect 1-code of Γ^* iff it is a perfect 1-code of Γ'.

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