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### SHORT COMMUNICATION

# A note on discontinuous problem with a free boundary

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KEYWORDS	Abstract We study a nonlinear elliptic problem with discontinuous nonlinearity
Variational inequalities;	$(P) \begin{cases} -\Delta u = f(u)H(\mu - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$
Uniqueness; Free boundary	$\partial \Omega,$ on $\partial \Omega,$
The boundary	where <i>H</i> is the Heaviside's unit function, <i>f</i> , <i>h</i> are given functions and $\mu$ is a positive real parameter. We prove the existence of a unique solution and characterize the corresponding free boundary. Our methods relies on variational inequality approach combining with fixed point arguments.
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In this note, we are interested to study the following free boundary problem

$$(P) \begin{cases} -\Delta u = f(u)H(\mu - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

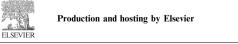
where  $\Omega$  is the unit ball of  $\mathbb{R}^n$ , *H* is the Heaviside function, *f* is a given function and  $\mu$  is a positive real parameter.

Introduce the following assumptions. Let  $\lambda_1$  be the first eigenvalue of  $-\Delta$  in  $\Omega$  under homogeneous Dirichlet boundary conditions with the corresponding eigenfunction  $\varphi_1 > 0$  in  $\Omega$ .

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(*H*<sub>1</sub>) The function *f* is *k* – lipstchitzian, non-decreasing, positive and there exist two strictly positive constants *k*,  $\beta > 0$  such that  $f(s) \leq ks + \beta$  with  $k < \min{\{\lambda_1, 1\}}$ .

(*H*<sub>2</sub>) For  $\mu > 0$ , we suppose

$$\delta := \frac{f(\mu)}{\mu} > \lambda_1.$$

(*H*<sub>3</sub>) The function f(s)/s is non increasing.

By a solution of problem (P), we mean a function  $u \in H_0^1(\Omega)$  satisfying

$$\int_{\Omega} \nabla u \nabla \xi dx = \int_{\Omega} f(u) H(\mu - u) \xi dx,$$

for  $\xi \in C_0^1(\Omega)$ . The main result of this note is the following theorem

**Theorem 1.** Under the above assumptions, the problem (P) have a unique positive solution. Moreover, the set  $\{x \in \Omega \mid u(x) = \mu\}$  is a ball of radius  $\rho \in (0, 1)$  centered at the origin.

Remark 1. In [1], the authors study the following problem

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$$(P_0) \begin{cases} -\Delta u = f(u)H(u-\mu) & \text{ in } \Omega, \\ u = 0 & \text{ on } \partial\Omega, \end{cases}$$

and prove that the problem  $(P_0)$  have a multiple solutions. On the other hand, we state here that the problem (P) have a unique positive solution for the same functions f. For example,

$$f(s) = ks + \beta$$
,  $0 < k < \lambda_1$  and  $\beta > 0$ ,

satisfying

$$\frac{f(\mu)}{\mu} > \max(\lambda_1, \lambda^*)$$

where

$$\lambda^* = 2\lambda_1 \frac{||\varphi_1||_{L^1}}{||\varphi_1||_{L^2}^2}.$$

The proof of Theorem 1 will be given in several steps.

$$K := \{ v \in H_0^1(\Omega) : v \leq \mu \text{ in } \Omega \}.$$

Define

$$T(\widetilde{f}) = -(\Delta)^{-1}\widetilde{f}$$
 if  $\widetilde{f} \ge 0$   
and

and

 $-(\Delta)^{-1}\widetilde{f}\leqslant \mu \quad \text{for} \quad \mu>0,$ 

where  $-(\Delta)^{-1}$  denote the inverse of Laplacian under Dirichlet conditions. Now, consider the operator  $T_{\delta}: L^2(\Omega) \to L^2(\Omega)$  defined by

$$T_{\delta}(u) = T(\delta f(u)).$$

The proof consists to give the existence of a fixed point of  $T_{\delta}$ . The fixed point of  $T_{\delta}$  is equivalent to the solution of problem (*P*) by using the variational inequality approach. It easy to prove the uniqueness of solutions of (*P*). It remains give the characterization of the set  $\{x \in \Omega, u(x) = \mu\}$ . For that, we prove the symmetry of solution via the moving plane method [2] and we conclude by the following result.

**Lemma 1.** Let  $\gamma$  be a Jordan curve in  $E_{\mu}$  and w the interior of contour  $\gamma$ . Then  $w \subset E_{\mu}$ .

#### Proof of lemma.

Suppose that the contrary hold, then there exists  $x_0 \in w$  satisfying  $u(x_0) < \mu$ . Consider the set

$$w_1 := \{ x \in \Omega/u(x) < \mu \}$$

which is not empty and is open since u is continuous. The function u verifies

$$\begin{cases}
-\Delta u = f(u) & \text{in } w_1, \\
u = \mu & \text{on } \partial w_1
\end{cases}$$

Hence, the maximum principle implies that  $u \ge \mu$  in  $w_1$  which is a contradiction.

**Proposition 1.** Let u be a solution of problem (P), then the set  $E_{\mu}$  is a ball of radius  $r_0 \in (0, 1)$ .

*Proof of proposition.* Since the solution is radial and non increasing, then there exists a unique  $r_0$  such that  $u(r_0) = \mu$ . Hence Lemma 1 implies that

 $E_{\mu} = \{ x \in \Omega / |x| \leqslant r_0 \}.$ 

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