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Journal of Algebra 281 (2004) 303–305
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The group of fractions of a torsion free lcm monoid is torsion free

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Received 3 December 2003
Available online 24 August 2004
Communicated by Michel Broué

Abstract

We improve and shorten the argument given in [J. Algebra 210 (1998) 291–297]. In particular, the fact that Artin braid groups are torsion free now follows from Garside's results almost immediately.

An algebraic proof of the fact that Artin braid groups are torsion free was given in [3]. The aim of this note is to observe that the proof of [3], which uses group presentations and words, is unnecessarily complicated and requires needless hypotheses. A shorter and better argument can be given that uses elementary properties of the lcm operation only.

For x, y in a monoid M, we say that y is a *right multiple* of x if $y = xz$ holds for some z in M; we say that z is a right least common multiple, or *right lcm*, of x and y if z is a right multiple of x and y and any common right multiple of x and y is a right multiple of z. The result we prove here is:

Proposition 1. *Assume that G is a group and M is a submonoid of G such that M generates G and, in M, any two elements admit a right lcm. Then the torsion elements of G are the elements xtx^{-1} with x in M and t a torsion element of M.*

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.03.031

Corollary 2. *Under the previous hypotheses, G is torsion free if and only if M is torsion free. In particular, a sufficient condition for G to be torsion free is that M contains no invertible element but* 1.

In comparison with [3], the current result eliminates a useless Noetherianity hypothesis. In this way, the result directly extends the standard result that a left-orderable group is torsion free. Indeed, if $<$ is a linear ordering on G that is compatible with left multiplication, the submonoid defined by $M = \{x \in G;\ x \geqslant 1\}$ is eligible for Corollary 2, as, in M, the element $\sup(x, y)$ is a right lcm of x and y.

The above results apply to Artin's braid group B_n, as, according to Garside's theory [5], the submonoid B_n^+ admits unique right lcm's. Alternatively, one could also use the dual monoid of [1], or some more exotic monoids. Artin–Tits groups of finite Coxeter type, and, more generally, all Garside groups [4] are eligible, as well as, for instance, Richard Thompson's group F [2]. All lattice-ordered groups of [6] also are eligible, but, then, the result is trivial: the point is that, here, we only assume one-sided compatibility between multiplication and ordering.

In order to prove Proposition 1, we begin with two simple observations about lcm's. First, a right lcm need not be unique, but the set, here denoted $\mathrm{LCM}(x, y)$, of all right lcm's of two elements x, y is easily described:

Lemma 3. *Assume that M is a left cancellative monoid, and z is a right lcm of two elements x, y of M. Then $\mathrm{LCM}(x, y)$ consists of all elements of the form zu with u an invertible element of M.*

Proof. If u is invertible in M, the element zu is a right multiple of x and y, and z is a right multiple of zu, so zu is a right lcm of x and y. Conversely, let z' be an arbitrary right lcm of x and y. There must exist u, u' satisfying $z' = zu$ and $z = z'u'$, hence $z = zuu'$ and $z' = z'u'u$, and we deduce $uu' = u'u = 1$. □

Lemma 4. *Assume that M is a left cancellative monoid, and that, in M, we have $xy'_1 = y_1x' \in \mathrm{LCM}(x, y_1)$ and $x'y'_2 = y_2x'' \in \mathrm{LCM}(x', y_2)$. Then we have $xy'_1y'_2 = y_1y_2x'' \in \mathrm{LCM}(x, y_1y_2)$.*

Proof. First we have $xy'_1y'_2 = y_1x'y'_2 = y_1y_2x''$, so this element is a common right multiple of x and y_1y_2. Assume that z is a right multiple of x and of y_1y_2, say $z = y_1y_2z'$. Then z is a right multiple of x and y_1, hence of y_1x', say $z = y_1x'z''$. Cancelling y_1 on the left, we obtain $y_2z' = x'z''$, so y_2z' is a right multiple of y_2x'', and z is a right multiple of y_1y_2x''. □

Proof of Proposition 1. The condition is obviously sufficient and the only problem is to prove that it is necessary. As M is a submonoid of a group, it admits cancellation, and, as any two elements of M admit a common right multiple, M satisfies the Ore conditions on the right, and G is a group of right fractions of M. Let z be an arbitrary element of G. Write $z = x_1y_1^{-1}$ with x_1, y_1 in M, and, inductively, choose $x_2, y_2, x_3, y_3, \ldots$ in M satisfying $x_iy_{i+1} = y_ix_{i+1} \in \mathrm{LCM}(x_i, y_i)$. We claim that, for all positive k, ℓ, we have

$$x_1 \dots x_k y_{k+1} \dots y_{k+\ell} = y_1 \dots y_\ell x_{\ell+1} \dots x_{\ell+k} \in \mathrm{LCM}(x_1 \dots x_k, y_1 \dots y_\ell), \tag{1}$$

$$z = (x_1 \dots x_k)\big(x_{k+1} y_{k+1}^{-1}\big)(x_1 \dots x_k)^{-1}, \tag{2}$$

$$z^k = (x_1 \dots x_k)(y_1 \dots y_k)^{-1}. \tag{3}$$

Indeed, (1) follows from Lemma 4 inductively; for (2) and (3), for each i, we have $y_i^{-1} x_i = x_{i+1} y_{i+1}^{-1}$ by construction, and we deduce

$$zx_1 \dots x_k = \big(x_1 y_1^{-1}\big)x_1 \dots x_k = x_1\big(x_2 y_2^{-1}\big)x_2 \dots x_k = \dots = x_1 \dots x_k\big(x_{k+1} y_{k+1}^{-1}\big),$$

$$z^k y_1 \dots y_k = \big(x_1 y_1^{-1}\big)^k y_1 \dots y_k = x_1\big(y_1^{-1} x_1\big)^{k-1} y_2 \dots y_k = x_1\big(x_2 y_2^{-1}\big)^{k-1} y_2 \dots y_k$$
$$= x_1 x_2\big(y_2^{-1} x_2\big)^{k-2} y_3 \dots y_k = x_1 x_2\big(x_3 y_3^{-1}\big)^{k-2} y_3 \dots y_k = \dots = x_1 \dots x_k.$$

Now assume $z^p = 1$, and let $t = x_{p+1} y_{p+1}^{-1}$. By (2), we have $z = xtx^{-1}$ with $x = x_1 \dots x_p \in M$. Relation (3) implies

$$x_1 \dots x_p = y_1 \dots y_p \in \mathrm{LCM}(x_1 \dots x_p, y_1 \dots y_p). \tag{4}$$

Comparing relations (1)—with $k = \ell = p$—and (4), we deduce from Lemma 3 that $y_{p+1} \dots y_{2p}$, hence y_{p+1} as well, is invertible in M. Therefore t belongs to M, and, as z and t are conjugates, $z^p = 1$ implies $t^p = 1$. □

In lattice-ordered groups, the next result after torsion freeness is that $x^p = y^p$ implies that x and y are conjugate. This result does *not* extend to our current framework: the group $\langle x, y;\ x^2 = y^2 \rangle$ satisfies all hypotheses of Proposition 1 but x and y are not conjugate there.

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