ELSEVIER
Topology and its Applications 110 (2001) 133–161
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Se^x: Dynamics, topology, and bifurcations of complex exponentials

Robert L. Devaney

Boston University, Department of Mathematics, Boston, MA 02215-2411, USA

Received 16 August 1999; received in revised form 8 July 2000

1. Introduction

Our goal in this paper is to describe some of the interesting topology that arises in the dynamics of entire functions such as the complex exponential family $E_\lambda(z) = \lambda e^z$. We will see that the important invariant sets for this family possesses a extremely rich topological structure, including such objects as Cantor bouquets, Knaster continua, hair transplants, and explosion points.

For a complex analytic function E, the interesting orbits lie in the *Julia set*, which we denote by $J(E)$. This is the set on which the map is chaotic. For the exponential family, the Julia set of E_λ has three characterizations:

(1) $J(E_\lambda)$ is the set of points at which the family of iterates of E_λ, $\{E_\lambda^n\}$ is not a normal family in the sense of Montel. This is the characterization that is most useful to prove theorems.

(2) $J(E_\lambda)$ is the closure of the set of repelling periodic points of E_λ. This is the dynamical definition of the Julia set.

(3) $J(E_\lambda)$ is the closure of the set of points whose orbits tend to ∞. This is the characterization that is most useful to compute the Julia set.

We remark that characterization 3 differs from the case of polynomial iterations, where the Julia set is the *boundary* of the set of escaping orbits. The reason for the difference is E_λ has an essential singularity at ∞, while polynomials have superattracting fixed points at ∞. The equivalence of (1) and (2) was shown by Baker, see [6]. The equivalence of (1) and (3) is shown in [19].

In this paper we will concentrate on the dynamics of E_λ where λ is real. For λ positive, the Julia set for E_λ undergoes a remarkable transformation as λ passes through $1/e$. We will show below that E_λ possesses an attracting fixed point when $0 < \lambda < 1/e$. All points in the left half plane have orbits that tend to this fixed point. Indeed, the full basin of attraction of this fixed point is open and dense in the plane.

E-mail address: bob@math.bu.edu (R.L. Devaney).

PII: S0166-8641(00)00099-7

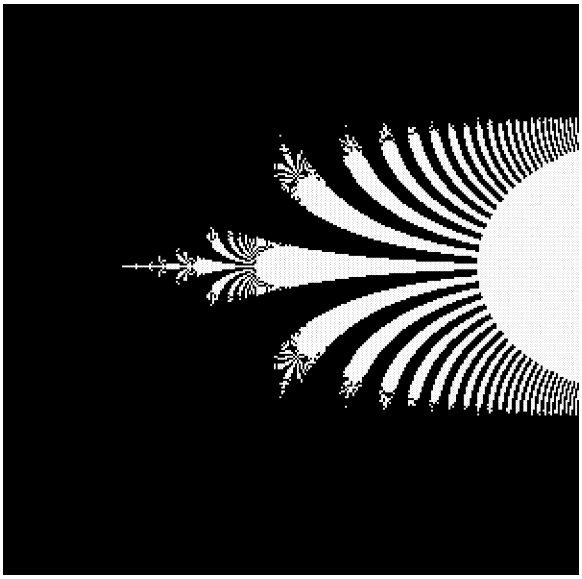

Fig. 1. A Julia set for $\lambda < 1/\mathrm{e}$.

We will show that the complement of the basin, $J(E_\lambda)$, is a *Cantor bouquet* for $0 < \lambda \leqslant 1/\mathrm{e}$. Roughly speaking, a Cantor bouquet has the property that all points in the set lie on a curve (or "hair") homeomorphic to a closed half line. Each of these curves in $J(E_\lambda)$ extend to ∞ in the right half-plane. All repelling periodic points and points with bounded orbits lie at the endpoints of the curves, while points that do not lie at the endpoints have unbounded orbits. Since repelling periodic points are dense in $J(E_\lambda)$, the endpoints of the Cantor bouquet must be dense in $J(E_\lambda)$. Indeed, we will show that the set of endpoints is a totally disconnected set, but that the set of endpoints together with the point at infinity forms a connected set, a result due to Mayer [24].

In Figs. 1 and 2 we display a computer graphics rendering of the Julia set of E_λ for a particular λ with $0 < \lambda < 1/\mathrm{e}$. This image was computed using characterization (3) of the Julia set: Points are shaded in white and grey if their orbits ever enter the region $\mathrm{Re}\, z > 50$. The complement of the Julia set is displayed in black. It appears that this Julia set contains large open sets, but this in fact is not the case. The Julia set actually consists of uncountably many curves lying in the Cantor bouquet and extending to ∞ in the right half plane. These curves are packed together so tightly that the resulting set has Hausdorff dimension 2, thus giving the appearance of an open set. See Fig. 2.

At $\lambda = 1/\mathrm{e}$, E_λ undergoes a simple saddle-node bifurcation. The attracting fixed point merges with a repelling fixed point at this λ-value, producing a neutral fixed point. When $\lambda > 1/\mathrm{e}$, this neutral fixed point gives way to a pair of repelling fixed points.

This apparently simple bifurcation has profound global ramifications. When $\lambda \leqslant 1/\mathrm{e}$, the Julia set is a nowhere dense subset of the right half plane. However, when $\lambda > 1/\mathrm{e}$,

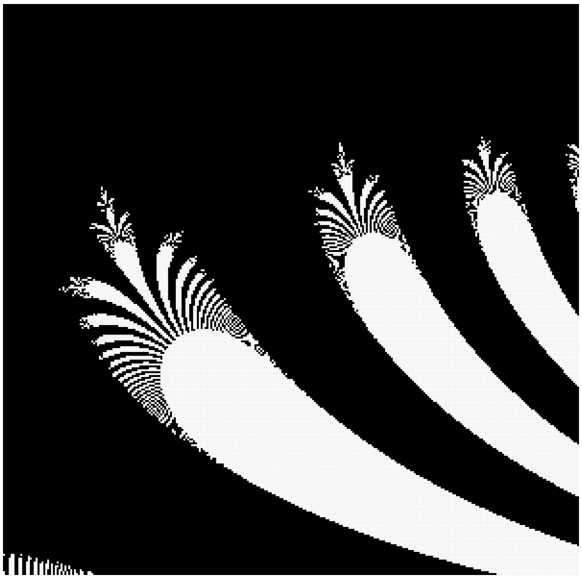

Fig. 2. Magnification of a Julia set for $\lambda < 1/e$.

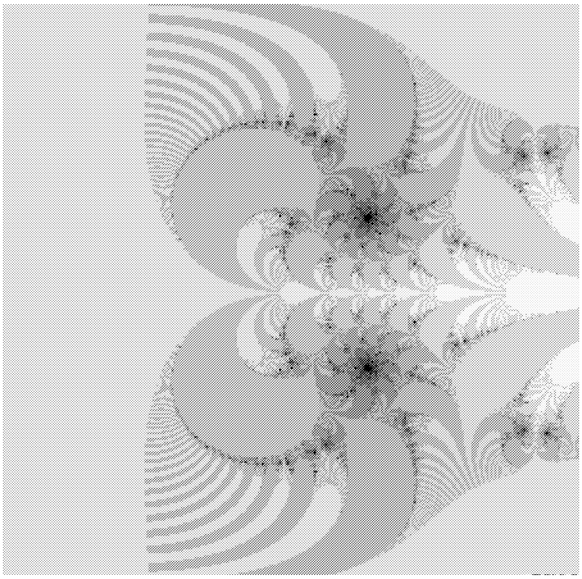

Fig. 3. The Julia set for $\lambda > 1/e$.

$J(E_\lambda)$ suddenly becomes the whole plane. No new repelling periodic points (except the two fixed points involved in the saddle-node) are born in this bifurcation; all simply move smoothly as λ crosses through $1/e$. Yet somehow, as soon as λ exceeds $1/e$, the repelling periodic points become dense in $\mathbb{C}$.

In Fig. 3 we display the Julia set for E_λ for a particular $\lambda > 1/\mathrm{e}$. Note the striking difference between this image and that in Fig. 1.

At this bifurcation, the attracting fixed point and its entire basin of attraction disappear. Most of the points in the Cantor bouquet remain in the Julia set. However, a new and interesting topological invariant set arises. We will show that this set is an indecomposable continuum on which most orbits cycle toward the orbit of 0 and ∞.

When λ is complex but close to the real axis ($\mathrm{Re}\,\lambda > 1/\mathrm{e}$), the dynamics of E_λ undergoes remarkable changes. We will show that in any neighborhood U of $\lambda > 1/\mathrm{e}$ in the parameter plane, there is a positive integer N such that for any $n \geqslant N$ there is a $\lambda_n \in U$ for which the corresponding exponential has an attracting cycle of period n. Hence E_λ is not structurally stable for any $\lambda > 1/\mathrm{e}$.

This paper is a summary of a series of lectures delivered by the author at the Summer Conference on Topology and Its Applications held at the C.W. Post Campus of Long Island University in August, 1999.

2. Exponential dynamics

As in the often-studied quadratic family $Q_c(z) = z^2 + c$, it is the orbit of 0 that plays a crucial role in determining the dynamics of E_λ. For the exponential family, 0 is an asymptotic value (an omitted value) rather than a critical point. Nevertheless, any stable domain in the complement of the Julia set of E_λ must be associated with the orbit of 0 in the following sense:

Theorem 2.1. *Suppose E_λ has an attracting or rationally neutral* (*parabolic*) *periodic point. Then $E_\lambda^n(0)$ must tend to the attracting or neutral cycle. If, on the other hand, $E_\lambda^n(0) \to \infty$, then $J(E_\lambda) = \mathbb{C}$.*

The proof of the first statement in this theorem is a classical fact that goes back to Fatou. The second follows from the Sullivan No Wandering Domains Theorem [29], as extended to the case of the exponential by Goldberg and Keen [21] and Eremenko and Lyubich [20]. Rather than rely on this big machinery, we will give a bare-hands approach due to Misiurewicz [26] to show that $J(E_\lambda) = \mathbb{C}$ when $\lambda > 1/\mathrm{e}$ in Section 5.

Consider for the moment the restriction of E_λ to the real line. The exponential family undergoes a saddle node bifurcation at $\lambda = 1/\mathrm{e}$ since, when $\lambda = 1/\mathrm{e}$, the graph of $E_{1/\mathrm{e}}$ is tangent to the diagonal at 1. See Fig. 4. We have $E_{1/\mathrm{e}}(1) = 1$ and $E'_{1/\mathrm{e}}(1) = 1$. When $\lambda < 1/\mathrm{e}$, the graph of E_λ lies above the diagonal and all orbits (including 0) tend to ∞. When $\lambda < 1/\mathrm{e}$, the graph of E_λ crosses the diagonal twice, at an attracting fixed point a_λ and a repealing fixed point r_λ. For later use note that $0 < a_\lambda < 1 < r_\lambda$. Note also that the orbit of 0 tends to a_λ, as it must by Fatou's theorem. See Fig. 4.

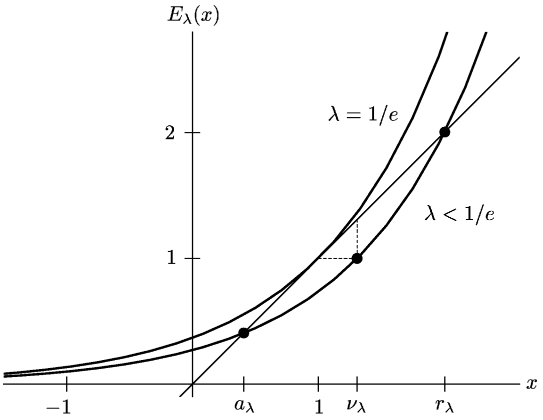

Fig. 4. The graphs of E_λ for $\lambda = 1/\mathrm{e}$ and $\lambda < 1/\mathrm{e}$.

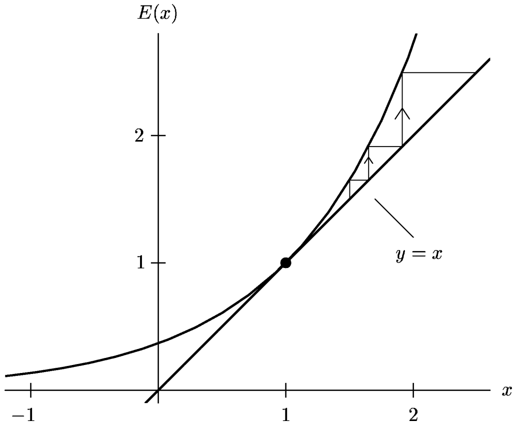

Fig. 5. The graph of $E(x) = (1/\mathrm{e})\mathrm{e}^x$.

3. Cantor bouquets

In this section, we begin the study of the dynamics of E_λ by considering the case where $\lambda \leqslant 1/\mathrm{e}$. We show here that $J(E_\lambda)$ is a *Cantor bouquet*.

Here is a rough idea of the construction of a Cantor bouquet. We will "tighten up" these ideas in following sections.

Let $E(z) = (1/\mathrm{e})\mathrm{e}^z$. We have $E(1) = 1$ and $E'(1) = 1$. If $x_0 \in \mathbb{R}$ and $x_0 < 1$, then $E^n(x_0)$ tends to the fixed point at 1. If $x_0 > 1$, then $E^n(x_0) \to \infty$ as $n \to \infty$. This can be shown using the web diagram as shown in Fig. 5.

The vertical line $\operatorname{Re} z = 1$ is mapped to the circle of radius 1 centered at the origin. In fact, E is a contraction in the half plane H to the left of this line, since

$$\left|E'(z)\right| = \frac{1}{\mathrm{e}}\exp(\operatorname{Re} z) < 1$$

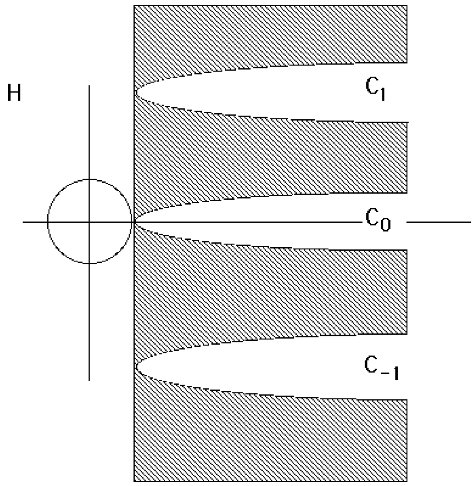

Fig. 6. The preimage of H consists of H and the shaded region.

if $z \in H$. Consequently, all points in H have orbits that tend to 1. Hence this half plane lies in the stable set, i.e., in the complement of the Julia set. We will try to paint the picture of the Julia set of E by painting instead its complement.

Since the half plane H is forward invariant under E, we can obtain the entire stable set by considering all preimages of this half plane. Now the first preimage of H certainly contains the horizontal lines $\operatorname{Im} z = (2k+1)\pi$, $\operatorname{Re} z \geqslant 1$, for each integer k, since E maps these lines to the negative real axis which lies in H. Hence there are open neighborhoods of each of these lines that lie in the stable set. The first preimage of H is shown in Fig. 6. The complement of $E^{-1}(H)$ consists of infinitely many "fingers". The fingers are $2k\pi \mathrm{i}$ translates of each other, and each is mapped onto the complementary half plane $\operatorname{Re} z \geqslant 1$.

We denote the fingers in the complement of $E^{-1}(H)$ by C_j with $j \in \mathbb{Z}$, where C_j contains the half line $\operatorname{Im} z = 2j\pi$, $\operatorname{Re} z \geqslant 1$, which is mapped into the positive real axis. That is, the C_j are indexed by the integers in order of increasing imaginary part. Note that C_j is contained within the strip

$$-\frac{\pi}{2} + 2j\pi \leqslant \operatorname{Im} z \leqslant \frac{\pi}{2} + 2j\pi.$$

Now each C_j is mapped in one-to-one fashion onto the entire half plane $\operatorname{Re} z \geqslant 1$. Consequently each C_j contains a preimage of each other C_k. Each of these preimages forms a subfinger which extends to the right in the half plane H. See Fig. 7. The complement of these subfingers necessarily lies in the stable set.

Now we continue inductively. Each subfinger is mapped onto one of the original fingers by E. Consequently, there are infinitely many sub-subfingers which are mapped to the C_j's by E^2. So at each stage we remove the complement of infinitely many subfingers from each remaining finger.

This process is reminiscent of the construction of the Cantor set in the dynamics of polynomials when all critical points tend to ∞. In that construction, the complements of

Fig. 7. The second preimage of H in one of the fingers C_j.

disks are removed at each stage; here we remove the complement of infinitely many fingers. As a result, after performing this operation infinitely many times, we do not end up with points. Rather, as we will see, the intersection of all of these fingers, if nonempty, is a simple curve extending to ∞.

This collection of curves forms the Julia set. E permutes these curves and each curve consists of a well-defined endpoint together with a "hair" which extends to ∞. It is tempting to think of this structure as a "Cantor set of curves", i.e., a product of the set of endpoints and the half-line. However, this is not the case as the set of endpoints is not closed.

Note that we can assign symbolic sequences to each point on these curves. To do this, we attach an infinite sequence $s_0s_1s_2\ldots$ to each hair in the Julia set via the rule: $s_j \in \mathbb{Z}$ and $s_j = k$ if the jth iterate of the hair lies in C_k. The sequence $s_0s_1s_2\ldots$ is called the *itinerary* of the curve.

For example, the portion of the real line $\{x \mid x \geqslant 1\}$ lies in the Julia set since all points (except 1) tend to ∞ under iteration, not to the fixed point. These points all have itinerary $000\ldots.$

One temptation is to say that there is a hair corresponding to every possible sequence $s_0s_1s_2\ldots.$ This, unfortunately, is not true, as certain sequences simply grow too quickly to correspond to orbits of E.

So this is $J(E)$: a "hairy" object extending toward ∞ in the right-half plane. We call this object a *Cantor bouquet*. We will see that this bouquet has some rather interesting topological properties as we investigate further.

We remark that the same construction works if $0 < \lambda < 1/\mathrm{e}$. We still define the half plane H as the set $\operatorname{Re} z < 1$. As we saw earlier, the point 1 on the real axis sits between the attracting fixed point a_λ and the repelling fixed point r_λ, and so $E_\lambda(1) < 1$ and as a consequence $E_\lambda(H)$ is strictly contained in H. The construction of the fingers now proceeds exactly as above.

3.1. Straight brushes

To describe the structure of a Cantor bouquet in more detail, we need to introduce the notion of a *straight brush*.

To each irrational number ζ, we assign an infinite string of integers $n_0n_1n_2\ldots$ as follows. We will break up the real line into open intervals $I_{n_0n_1\ldots n_k}$ which have the following properties

(1) $I_{n_0\ldots n_k} \supset I_{n_0\ldots n_{k+1}}$.
(2) The endpoints of $I_{n_0\ldots n_k}$ are rational.
(3) $\zeta = \bigcap_{k=1}^{\infty} I_{n_0\ldots n_k}$.

Now there are many ways to do this. We choose the following method based on the Farey tree. Inductively, we first define $I_k = (k, k+1)$. Given $I_{n_0\ldots n_k}$ we define $I_{n_0\ldots n_k j}$ as follows. Let

$$I_{n_0\ldots n_k} = \left(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right).$$

Let $p_0/q_0 = (\alpha+\gamma)/(\beta+\delta)$, the Farey child of α/β and γ/δ. We write

$$\frac{\alpha}{\beta} \oplus \frac{\gamma}{\delta} = \frac{p_0}{q_0}$$

to indicate the Farey child of these fractions. Let p_n/q_n be the Farey child of p_{n-1}/q_{n-1} and γ/δ for $n > 0$, and let p_{n-1}/q_{n-1} be the Farey child for α/β and p_n/q_n for $n \geqslant 0$. We then set $I_{n_0\ldots n_k j}$ to be the open interval $(p_j/q_j, p_{j+1}/q_{j+1})$.

Example. $I_0 = (0, 1)$. The Farey child of $\frac{0}{1}$ and $\frac{1}{1}$ is $\frac{1}{2}$, so $p_0/q_0 = \frac{1}{2}$. Then $p_1/q_1 = \frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$, $p_2/q_2 = \frac{2}{3} \oplus \frac{1}{1} = \frac{3}{4}$, and $p_n/q_n = \frac{n+1}{n+2}$ for $n > 0$.

For negative n we have

$$\begin{aligned}
\frac{p_{-1}}{q_{-1}} &= \frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3},\\
\frac{p_{-2}}{q_{-2}} &= \frac{0}{1} \oplus \frac{1}{3} = \frac{1}{4},\\
\frac{p_{-n}}{q_{-n}} &= \frac{1}{n+2}.
\end{aligned}$$

Therefore, if $n \geqslant 0$,

$$I_{0n} = \left(\frac{n+1}{n+2}, \frac{n+2}{n+3}\right)$$

and if $n < 0$,

$$I_{0n} = \left(\frac{1}{-n+2}, \frac{1}{-n+1}\right).$$

See Fig. 8. Note that we exhaust all of the rationals via this procedure, so each irrational is contained in a unique $I_{n_0n_1\ldots}$.

We now define a straight brush, a notion due to Aarts and Oversteegen [2].

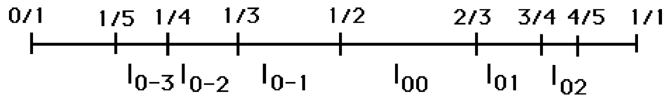

Fig. 8. Construction of I_{0n}.

Definition 3.1. A straight brush B is a subset of $[0, \infty) \times \mathcal{N}$, where $\mathcal{N}$ is a dense subset of $\mathbb{R} - \mathbb{Q}$. B has the following 3 properties.

(1) B is "hairy" in the following sense. If $(y, \alpha) \in B$, then there exists a $y_\alpha \leqslant y$ such that $(t, \alpha) \in B$ iff $t \geqslant y_\alpha$. That is the "hair" (t, α) is contained in B where $t \geqslant y_\alpha$. y_α is called the endpoint of the hair corresponding to α.

(2) Given an endpoint $(y_\alpha, \alpha) \in B$ there are sequences $\beta_n \uparrow \alpha$ and $\gamma_n \downarrow \alpha$ in $\mathcal{N}$ such that $(y_{\beta_n}, \beta_n) \to (y_\alpha, \alpha)$ and $(y_{\gamma_n}, \gamma_n) \to (y_\alpha, \alpha)$. That is, any endpoint of a hair in B is the limit of endpoints of other hairs from both above and below.

(3) B is a closed subset of $\mathbb{R}^2$.

The following facts are easily verified:

(1) For any rational number v and any sequence of irrationals $\alpha_n \in \mathcal{N}$ with $\alpha_n \to v$, it can be shown that the hairs $[y_{\alpha_n}, \alpha_n]$ must tend to (∞, v) in $[0, \infty] \times \mathbb{R}$.

(2) Condition (2) above may be changed to: if (y, α) is any point in B (y need not be the endpoint of the α-hair), then there are sequences $\beta_n \uparrow \alpha$, $\gamma_n \downarrow \alpha$ so that $(y_{\beta_n}, \beta_n) \to (y, \alpha)$ and $(y_{\gamma_n}, \gamma_n) \to (y, \alpha)$ in B.

(3) Let $(y, \alpha) \in B$ and suppose y is not the endpoint y_α. Then (y, α) is inaccessible in $\mathbb{R}^2$ in the sense that there is no continuous curve $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(t) \notin B$ for $0 \leqslant t < 1$ and $\gamma(1) = (y, \alpha)$.

(4) On the other hand, the endpoint (y_α, α) is accessible in $\mathbb{R}^2$.

These facts show that a straight brush is a remarkable object from the topological point of view. We view a straight brush as a subset of the Riemann sphere and set $B^* = B \cup \infty$, i.e., the straight brush with the point at infinity added. Let $\mathcal{E}$ denote the set of endpoints of B, and let $\mathcal{E}^* = \mathcal{E} \cup \infty$. Then we have the following result, due to Mayer [24]:

Theorem 3.2. *The set $\mathcal{E}^*$ is a connected set, but $\mathcal{E}$ is totally disconnected.*

That is, the set $\mathcal{E}^*$ is a connected set, but if we remove just one point form this set, the resulting set is totally disconnected. Topology really is a weird subject!

The reason for this is that, if we draw the straight line in the plane (γ, t) where γ is a fixed rational, and then we adjoin the point at infinity, we find a disconnection of $\mathcal{E}$. This, however, is not a disconnection of $\mathcal{E}^*$. Moreover, the fact that any non-endpoint in B is inaccessible shows that we cannot disconnect $\mathcal{E}^*$ by any other curve.

Remark. Aarts and Oversteegen have shown that any two straight brushes are ambiently homeomorphic, i.e., there is a homeomorphism of $\mathbb{R}^2$ taking one brush onto the other. This leads to a formal definition of a Cantor bouquet.

Definition 3.3. A Cantor bouquet is a subset of $\overline{\mathbb{C}}$ that is homeomorphic to a straight brush (with ∞ mapped to ∞).

Our main goal in this section is to sketch a proof of the following result. For more details, see [2].

Theorem 3.4. *Suppose* $0 < \lambda < 1/\mathrm{e}$. *Then* $J(E_\lambda)$ *is a Cantor bouquet.*

We now describe the construction of the homeomorphism between the brush and $J(E_\lambda)$. To do this, we first introduce symbolic dynamics. Recall that E_λ has a repelling fixed point $r_\lambda > 0$ in $\mathbb{R}$ and that the half plane $\operatorname{Re} z < r_\lambda$ lies in the stable set. Similarly the horizontal strips

$$-\frac{\pi}{2} + 2k\pi < \operatorname{Im} z < \frac{\pi}{2} + (2k)\pi$$

are contained in the stable set since E_λ maps these strips to $\operatorname{Re} z < 0$ which is contained in $\operatorname{Re} z < r_\lambda$.

Let $\nu_\lambda = -\ln\lambda$. So $E_\lambda(\nu_\lambda) = 1$ and so $\nu_\lambda < r_\lambda$. Fix $\omega_\lambda \in (\nu_\lambda, r_\lambda)$. Then the half plane $\operatorname{Re} z < \omega_\lambda$ also lies in the stable set, and we have $|E_\lambda'(z)| > 1$ in the half plane $\operatorname{Re} z \geqslant \omega_\lambda$.

We denote by S_k the closed halfstrip given by

$$\operatorname{Re} z \geqslant \omega_\lambda \quad \text{and} \quad -\frac{\pi}{2} + 2k\pi \leqslant \operatorname{Im} z \leqslant \frac{\pi}{2} + 2k\pi.$$

Note that these strips contain the Julia set since the complement of the strips lies in the stable set.

Given $z \in J(E_\lambda)$, we define the itinerary of z, $S(z)$, as usual by

$$S(z) = s_0 s_1 s_2 \ldots,$$

where $s_j \in \mathbb{Z}$ and $s_j = k$ iff $E_\lambda^j(z) \in S_k$. Note that $S(z)$ is an infinite string of integers that indicates the order in which the orbit of z visits the S_k. We will associate to z the irrational number given by the itinerary of z (in the decomposition of the irrationals described above). This will determine the hair in the straight brush to which z is mapped. See Fig. 9. Thus we need only define the y-value along this hair. This takes a little work.

Given z on a hair, we will construct a sequence of closed rectangles $R_k(z)$ for each $k \geqslant 0$. By construction, the $R_k(z)$ will be nested. Each $R_k(z)$ will have sides parallel to the axes and be contained in some strip S_α. Finally each $R_k(z)$ will have height π. Since the $R_k(z)$ are nested with respect to k, the intersection $R_\infty(z) = \bigcap_{k=0}^{\infty} R_k(z)$ will be a nonempty rectangle of height π that contains z. We then define $h(z)$ to be the real part of the left hand edge of this limiting rectangle.

To begin the construction, for $z \in J(E_\lambda)$, we set $R_0(E_\lambda^j(z))$ to be the square centered around the line $\operatorname{Re} z = \operatorname{Re} E_\lambda^j(z)$ with sidelength π and contained in the strip S_α where $\alpha = s_j$. We assume that $\operatorname{Re} w \geqslant \omega_\lambda$ in $R_0(E_\lambda^j(z))$ for all $w \in R_0(E_\lambda^j(z))$ and for all j; otherwise we choose the rectangle

$$R_0\bigl(E_\lambda^j(z)\bigr) \cap \{\operatorname{Re} z \geqslant \omega_\lambda\}$$

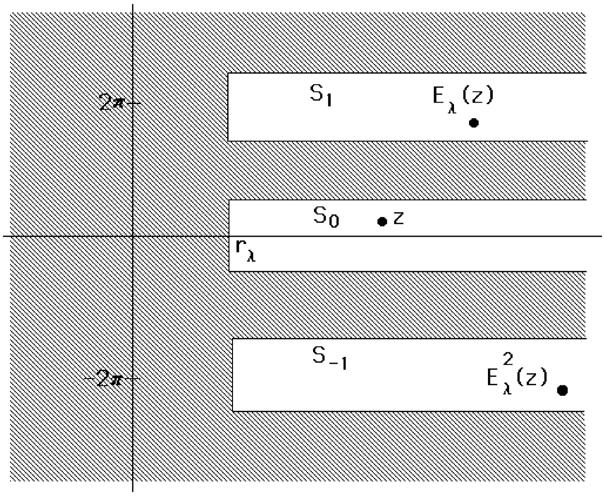

Fig. 9. The itinerary of z is $0, 1, -1, \ldots$.

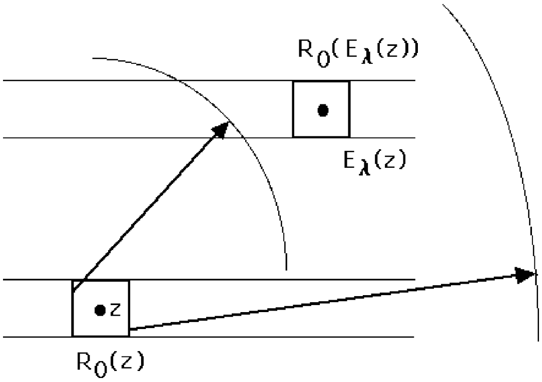

Fig. 10. Construction of $R_0(0)$ and $R_0(1)$.

for the initial box. Observe that $E_\lambda(R_0(E_\lambda^j(z)) \supset R_0(E_\lambda^{j+1}(z))$. Indeed, the image of $R_0(E_\lambda^j(z))$ is an annulus whose inner radius is $\mathrm{e}^{-\pi/2}|E_\lambda^{j+1}(z)|$ and outer radius $\mathrm{e}^{\pi/2}|E_\lambda^{j+1}(z)|$. Now $\mathrm{e}^{\pi/2} > 4$ and $\mathrm{e}^{-\pi/2} < \frac{1}{4}$ so the image annulus is much larger than $R_0(E_\lambda^{j+1})$. See Fig. 10.

It follows that we may find a narrower rectangle $R_1(E_\lambda^j(z))$ strictly contained in $R_0(E_\lambda^j(z))$ having the property that the height of $R_1(E_\lambda^j(z))$ is π and the image $E_\lambda(R_1(E_\lambda^j(z)))$ just covers $R_0(E_\lambda^{j+1}(z))$. That is, $R_1(E_\lambda^j(z))$ is the smallest rectangle in $R_0(E_\lambda^j(z))$ whose image annulus is just wide enough so that $R_0(E_\lambda^{j+1}(z))$ fits inside. See Fig. 11. Note that $E_\lambda^j(z) \in R_1(E_\lambda^j(z))$ for each j.

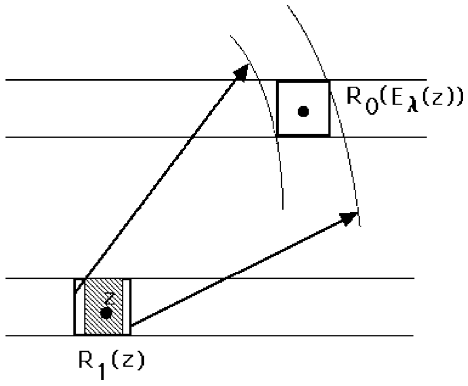

Fig. 11. Construction of $R_1(0)$.

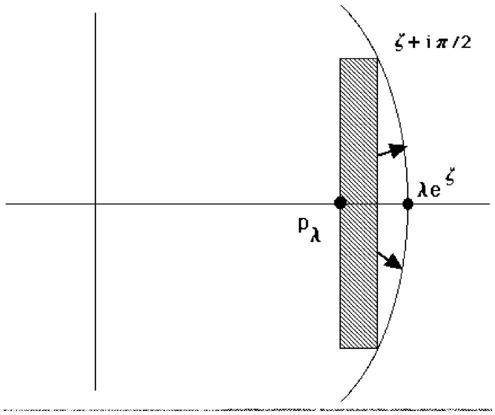

Fig. 12. The intersection of $R_j(0)$.

We now continue inductively by setting $R_k(E_\lambda^j(z))$ to be the subrectangle of $R_{k-1}(E_\lambda^j(z))$ whose image just covers $R_{k-1}(E_\lambda^{j+1}(z))$. The $R_k(E_\lambda^j(z))$ are clearly nested for each fixed j.

Example. Suppose $z = r_\lambda$. We have that $R_0(z)$ is the square bounded by $\operatorname{Re} z = r_\lambda \pm \pi/2$ (or the rectangle bounded by $\operatorname{Re} z = r_\lambda + \pi/2$ and $\operatorname{Re} z = \omega_\lambda$ if r_λ is close to 1) and $\operatorname{Im} z = \pm\pi/2$ for each j. One may check that $\bigcap_{k=0}^{\infty} R_k(z)$ is the strip bounded by $\operatorname{Re} z = r_\lambda$ and $\operatorname{Re} z = \zeta$ where the circle of radius λe^ζ passes through $\zeta \pm i\pi/2$. See Fig. 12.

Suppose z has itinerary $S(z) = s_0 s_1 s_2 \ldots$. Let $I(S(z))$ denote the irrational number determined by the sequence $S(z)$ as above. Then set $\phi(z) = (h(z), I(S(z)))$. We now claim that ϕ is a homeomorphism onto a straight brush. The proof is adapted from [17] and [2].

Let B_λ denote the image of $J(E_\lambda)$ under ϕ. We first show that ϕ is a homeomorphism onto B_λ. We will show later that B_λ is a straight brush.

We need the following lemma.

Lemma 3.5 (Expansion Lemma). *Suppose $z_1, z_2 \in J(E_\lambda)$ and $S(z_1) = S(z_2)$. If $z_1 \neq z_2$ then $|E_\lambda^k(z_1) - E_\lambda^k(z_2)| \to \infty$ as $k \to \infty$.*

Proof. Suppose that $|E_\lambda^k(z_1) - E_\lambda^k(z_2)| \leqslant M$ for all k. There is a well-defined branch $L_{\lambda,j}$ of the inverse of E_λ that maps the strip $S_{s_{j+1}}$ into S_{s_j} and $E_\lambda^{j+1}(z_i)$ to $E_\lambda^j(z_i)$. Clearly, $L_{\lambda,j}$ is a contraction for each j in $\operatorname{Re} z \geqslant r_\lambda$. Thus

$$\left|L_{\lambda,0} \circ \cdots \circ L_{\lambda,j}(E_\lambda^{j+1}(z_1)) - L_{\lambda,0} \circ \cdots \circ L_{\lambda,j}(E_\lambda^{j+1}(z_2))\right|$$

tends to 0 since $|E_\lambda^{j+1}(z_1) - E_\lambda^{j+1}(z_2)|$ is bounded.

We first claim that ϕ is one-to-one. Suppose not. Then there exist $z_1, z_2 \in J(E_\lambda)$ with $z_1 \neq z_2$ but $\phi(z_1) = \phi(z_2) = (t, I(S(z_1)))$. Then the left-hand edges of $R_\infty(E_\lambda^j(z_1))$ and $R_\infty(E_\lambda^j(z_2))$ are the same for all j. But then $|E_\lambda^j(z_1) - E_\lambda^j(z_2)| \leqslant 2\pi$ for all j. This contradicts the Expansion Lemma.

Next we prove continuity of ϕ. If z_0 and z_1 are close, then the first n terms of their itineraries must agree and so $I(S(z_0))$ must be close to $I(S(z_1))$. So suppose there is a sequence $z_i \to z_0$ in $J(E_\lambda)$, but $|h(z_i) - h(z_0)| > \delta$. Then the difference of the inner radii of the annuli covered by $E_\lambda(R_\infty(z_i))$ and $E_\lambda(R_\infty(z_0))$ is at least $c\delta$ for some constant $c > 1$. Some elementary geometry shows that the distance between $h(E_\lambda(z_i))$ and $h(E_\lambda(z_0))$ is also larger than $c\delta > \delta$. Continuing, we see that the left hand edges of $R_k(z_i)$ must eventually differ from $h(E_\lambda^k(z_0))$ by at least π. This contradicts the fact that $E_\lambda^k(z_i) \to E_\lambda^k(z_0)$.

The continuity of ϕ^{-1} follows similarly. Hence ϕ is a homeomorphism onto its image. It remains to show that $\phi(J(E_\lambda))$ is a straight brush.

We first show that the image of $J(E_\lambda)$ under ϕ projects onto a dense set of irrationals. Let $\zeta \in \mathbb{R} - \mathbb{Q}$. Say $\zeta = I_{n_0 n_1 n_2 \ldots}$. Let $\overline{n}_k = (\overline{n_0 n_1 \ldots n_k})$. Then $\zeta_k = I_{\overline{n}_k}$ converges to ζ as $k \to \infty$ with itinerary $\overline{n}_k$. We will produce a point in $J(E_\lambda)$ with this itinerary.

Suppose $m = \max_{0 \leqslant j \leqslant n_k} |n_j|$. Choose x so that $E_\lambda(x) > x + |m|\pi + \pi/2$. Let $S_k(x)$ denote the rectangle in S_k with height π and satisfying $r_\lambda \leqslant \operatorname{Re} z \leqslant x$. Then $E_\lambda(S_{n_i}(x))$ covers $S_{n_j}(x)$ for any pair of indices n_i and n_j since the right border of $S_{n_i}(x)$ is mapped to a semi-circle which lies further from the origin than $x \pm \mathrm{i}m$. Consider

$$L_{\lambda,n_0} \circ \cdots \circ L_{\lambda,n_{k-1}} \circ L_{\lambda,n_k} | S_{n_0}(x).$$

This inverse contracts $S_{n_0}(x)$ strictly inside itself. Hence by the Schwarz Lemma, this map has a unique fixed point $z_{\overline{n}_k}$ and $z_{\overline{n}_k}$ has itinerary $\overline{n}_k$. Note that $z_{\overline{n}_k}$ has period $k+1$ for E_λ and is repelling.

We single this fact out as a corollary.

Corollary 3.6. *Let $s = \overline{s_0 \ldots s_n}$ be a repeating sequence. Then there is a unique repelling periodic point in $J(E_\lambda)$ that has itinerary s.*

The uniqueness part of this result follows from the Expansion Lemma.

The fact that $\phi(J(E_\lambda))$ is hairy follows from the fact that if $\phi(z)=(t,s)$ then for any $t'>t$, $\phi^{-1}(t',s)$ is a point in $J(E_\lambda)$. Indeed, let R_0 denote the square in S_{s_0} with side length π and left hand boundary on the line $\operatorname{Re} z=t'$. $E_\lambda(R_0)$ is an annulus that maps across S_{s_1} since we know that a rectangle with side on $\operatorname{Re} z=t<t'$ has this property. Let R_1 be the square of sidelength π that lies in S_{s_1} and meets the image of $\operatorname{Re} z=t'$ at just one point. Then $E_\lambda(R_0)\supset R_1$. Now continue in this fashion, defining the square R_{k+1} to be the square in $S_{s_{k+1}}$ with side length π just meeting the inner boundary of $E_\lambda(R_k)$. Then using the expansion lemma there is a unique point z' whose orbit travels through the R_i in order. Note that $h(z')=t'$ by construction.

Corollary 3.7. *If $s=(\overline{s_0\ldots s_k})$, then the endpoint of the hair corresponding to $I(s)$ is a repelling periodic point.*

Proof. We have already constructed such a point z_s. If z has itinerary s and $h(z)<h(z_s)$, the above proof shows that the orbit of z must be bounded. This contradicts the Expansion Lemma.

We emphasize that, even though most hairs spiral in to their respective endpoints, the map ϕ is still one-to-one.

Remarks.

(1) If $s=(s_0s_1s_2\ldots)$ is a bounded sequence, then the endpoint of the hair also has a bounded orbit. If s is unbounded then the endpoint has an unbounded orbit. This orbit tends to ∞ but does so at a slower rate than points on the corresponding hair.

(2) It can be shown that the construction above works for any exponential for which there exists an attracting or neutral periodic point. See [19]. However, in the general case, some of the hairs in the Cantor bouquet may be attached to the same point in the crown. We discuss this briefly in Section 6. See [8].

(3) McMullen [25] has shown that the Hausdorff dimension of the Cantor bouquet constructed above is 2 but its Lebesgue measure is zero. This accounts for why Figs. 1 and 2 seem to have open regions in the Julia set.

3.2. Uniformization of the attracting basin

The basin of attraction Ω_λ of E_λ is an open, dense, and simply connected subset of the Riemann sphere. Hence the Riemann Mapping Theorem guarantees the existence of a uniformization $\phi_\lambda : D\to\Omega_\lambda$. Given such a uniformization, it is natural to ask if the uniformizing map extends to the boundary of D.

In order to extend ϕ_λ to the boundary, we need that the image under ϕ_λ of a straight ray $r\mathrm{e}^{\mathrm{i}\theta}$, where θ is constant, to converge to a single point as $r\to 1$. It is known that if the boundary of the uniformizing region is locally connected, then in fact ϕ_λ does extend continuously to D. On the other hand, if the boundary of the region is not locally connected, then not all rays need converge (though a full measure set of them must converge). In our case, the boundary of Ω_λ is nowhere locally connected (except at ∞). However, it is a fact that all rays do converge. Moreover, they land at precisely the endpoints of the Cantor bouquet and ∞. This means that we can induce a map on the set of endpoints, but that map is necessarily nowhere continuous [27].

In the case of a straight brush, it is clear that all rays do land at the crown of the bouquet. A direct proof for E_λ is given in [14].

In fact, it can be shown that, if we normalize the Riemann map ϕ_λ so that 0 is mapped to 0, then the induced map $\phi_\lambda^{-}1 \circ E_\lambda \circ \phi_\lambda$ on the unit disk is given by

$$T_\mu(z) = \exp\left(\mathrm{i}\left(\frac{\mu + \overline{\mu}}{1+z}\right)\right).$$

Here μ is a parameter that lies in the upper half plane and depends upon λ.

4. Indecomposable continua

We now consider the case $\lambda > 1/\mathrm{e}$. Since the orbit of 0 tends to ∞, the Julia set is now the entire plane (we will prove this in the next section). For these λ values, the attracting basin for the attracting fixed point a_λ disappears. What replaces it is a complicated invariant set that is an indecomposable continuum. We describe the construction of this set in this section.

Consider the horizontal strip

$$S = \{z \mid 0 \leqslant \operatorname{Im} z \leqslant \pi\}$$

(or its symmetric image under $z \to \overline{z}$). The exponential map E_λ takes the boundary of S to the real axis and the interior of S to the upper half plane. Thus, E_λ maps certain points outside of S while other points remain in S after one application of E_λ. Our goal is to investigate the set of points whose entire orbit lie in S. Call this set Λ. The set Λ is clearly invariant under E_λ. There is a natural way to compactify this set in the plane to obtain a new set Γ. Moreover, the exponential map extends to Γ in a natural way. Our main results in this section include:

Theorem 4.1. *Γ is an indecomposable continuum.*

Moreover, we will see that Λ is constructed in similar fashion to a family of indecomposable continua known as *Knaster continua*. See [13] for additional details.

As we will show in Section 4.2, the topology of Λ is quite intricate. Despite this, we will show that the dynamics of E_λ on Λ is quite tame. Specifically, we will prove:

Theorem 4.2. *The restriction of E_λ to $\Lambda - \{$orbit of $0\}$ is a homeomorphism. This map has a unique repelling fixed point $w_\lambda \in \Lambda$, and the α-limit set of all points in Λ is w_λ. On the other hand, if $z \in \Lambda$, $z \neq w_\lambda$, then the ω-limit set of z is either*

(1) *The point at ∞, or*

(2) *The orbit of 0 under E_λ together with the point at ∞.*

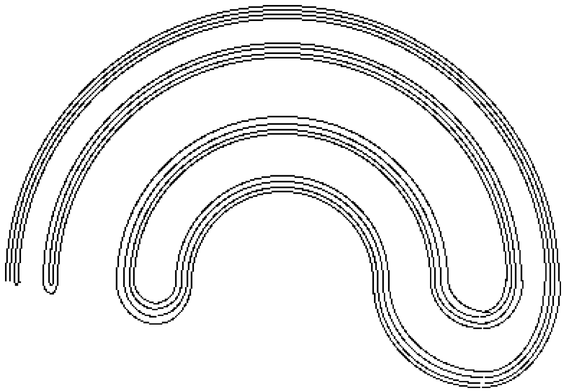

Fig. 13. The Knaster continuum.

Thus we see that E_λ possesses an interesting mixture of topology and dynamics in the case where the Julia set is the whole plane. In the plane the dynamics of E_λ are quite chaotic, but the overall topology is tame. On our invariant set Λ, however, it is the topology that is rich, but the dynamics are tame.

4.1. Topological preliminaries

In this section we review some of the basic topological ideas associated with indecomposable continua. See [22] for a more extensive introduction to these concepts.

Recall that a continuum is a compact, connected space. A continuum is decomposable if it is the union of two proper subcontinua. Otherwise, it is indecomposable. A well-known example of an indecomposable continuum is the Knaster continuum, K. One way to construct this set is to begin with the Cantor middle-thirds set. Then draw the semicircles lying in the upper half plane with center at $(1/2, 0)$ that connect each pair of points in the Cantor set that are equidistant from $1/2$. Next draw all semicircles in the lower half plane which have for each $n \geqslant 1$ centers at $(5/(2 \cdot 3^n), 0)$ and pass through each point in the Cantor set lying in the interval

$$2/3^n \leqslant x \leqslant 1/3^{n-1}.$$

The resulting set is partially depicted in Fig. 13.

For a proof that this set is indecomposable, we refer to [22]. Dynamically, this set appears as the closure of the unstable manifold of Smale's horseshoe map (see [4,28]).

Note that the curve passing through the origin in this set is dense, since it passes through each of the endpoints of the Cantor set. It also accumulates everywhere upon itself. Such a phenomenon gives a criterion for a continuum to be indecomposable, as was shown by Curry.

Theorem 4.3. *Suppose X is a one-dimensional nonseparating plane continuum which is the closure of a ray that limits upon itself. Then X is indecomposable.*

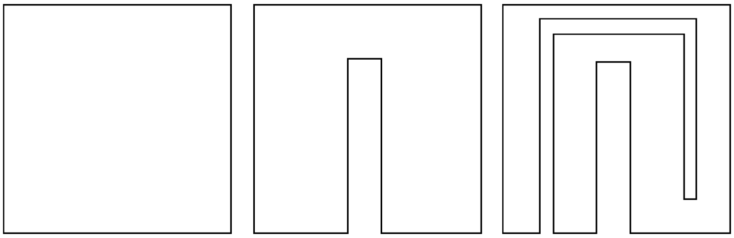

Fig. 14. A different construction of the Knaster continuum.

We refer to [11] for a proof.

Another view of the Knaster continuum which is intimately related to our own construction is as follows. Begin with the unit square S_0 in the plane. Next remove a "canal" C_1 from S_0 whose boundary lies within a distance $1/3$ of each boundary point of S_0 as depicted in Fig. 2. Call this set S_1. Next remove a new canal C_2 from S_1. This time the boundary of C_2 should be within 1/9 of the boundary of S_1 as depicted in Fig. 14. It is possible to continue this construction inductively in such a way that the resulting set is homeomorphic to the Knaster continuum.

4.2. *Construction of Λ*

Recall that the strip S is given by $\{z \mid 0 \leqslant \mathrm{Im}(z) \leqslant \pi\}$. Note that E_λ maps S in one-to-one fashion onto $\{z \mid \mathrm{Im}\, z \geqslant 0\} - \{0\}$. Hence E_λ^{-1} is defined on $S - \{0\}$ and, in fact, E_λ^{-n} is defined for all n on $S - \{\text{orbit of } 0\}$. We will always assume that E_λ^{-n} means E_λ^{-n} restricted to this subset of S.

Define

$$\Lambda = \left\{z \mid E_\lambda^n(z) \in S \text{ for all } n \geqslant 0\right\}.$$

If $z \in \Lambda$ it follows immediately that $E_\lambda^n(z) \in S$ for *all* $n \in \mathbb{Z}$ provided z does not lie on the orbit of 0. Our goal is to understand the structure of Λ.

Toward that end we define L_n to be the set of points in S that leave S at precisely the nth iteration of E_λ. That is,

$$L_n = \left\{z \in S \mid E_\lambda^i(z) \in S \text{ for } 0 \leqslant i < n \text{ but } E_\lambda^n(z) \notin S\right\}.$$

Let B_n be the boundary of L_n.

Recall that E_λ maps a vertical segment in S to a semi-circle in the upper half plane centered at 0 with endpoints in $\mathbb{R}$. Either this semi-circle is completely contained in S or else an open arc lies outside S. As a consequence, L_1 is an open simply connected region which extends to ∞ toward the right in S as shown in Fig. 15. There is a natural parametrization $\gamma_1 : \mathbb{R} \to B_1$ defined by

$$E_\lambda\bigl(\gamma_1(t)\bigr) = t + \mathrm{i}\pi.$$

As a consequence,

$$\lim_{t \to \pm\infty} \mathrm{Re}\, \gamma_1(t) = \infty.$$

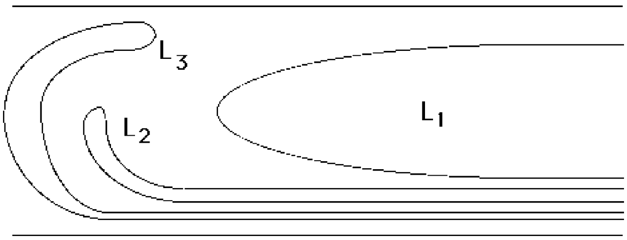

Fig. 15. Construction of the L_n.

If $c > 0$ is large, the segment $\operatorname{Re} z = c$ in S meets $S - L_1$ in two vertical segments v_+ and v_- with $\operatorname{Im} v_- > \operatorname{Im} v_+$. E_λ maps v_- to an arc of a circle in $S \cap \{z \mid \operatorname{Re} z < 0\}$ while E_λ maps v_+ to an arc of a circle in $S \cap \{z \mid \operatorname{Re} z > 0\}$. As a consequence, if c is large, v_+ meets L_2 in an open interval. Since $L_2 = E_\lambda^{-1}(L_1)$, it follows that L_2 is an open simply connected subset of S that extends to ∞ in the right half plane *below* L_1.

Continuing inductively, we see that L_n is an open, simply connected subset of S that extends to ∞ toward the right in S. We may also parametrize the boundary B_n of L_n by $\gamma_n : \mathbb{R} \to B_n$ where

$$E_\lambda^n\bigl(\gamma_n(t)\bigr) = t + \mathrm{i}\pi$$

as before. Again

$$\lim_{t \to \pm\infty} \operatorname{Re} \gamma_n(t) = \infty.$$

Since each L_n is open, it follows that Λ is a closed subset of S.

Proposition 4.4. *Let $J_n = \bigcup_{i=n}^{\infty} B_i$. Then J_n is dense in Λ for each $n > 0$.*

Proof. Let $z \in \Lambda$ and suppose $z \notin B_i$ for any i. Let U be an open connected neighborhood of z. Fix $n > 0$. Since $E_\lambda^i(z) \in S$ for all i, we may choose a connected neighborhood $V \subset U$ of z such that $E_\lambda^i(V) \subset S$ for $i = 0, \ldots, n$.

Now the family of functions $\{E_\lambda^i\}$ is not normal on V, since z belongs to the Julia set of E_λ. Consequently, $\bigcup_{i=0}^{\infty} E_\lambda^i(V)$ covers $\mathbb{C} - \{0\}$. In particular, there is $m > n$ such that $E_\lambda^m(V)$ meets the exterior of S. Since $E_\lambda^m(z) \in S$, it follows that $E_\lambda^m(V)$ meets the boundary of S. Applying E_λ^{-m}, we see that B_m meets V.

In fact, it follows that for any $z \in \Lambda$ and any neighborhood U of z, all but finitely many of the B_m meet V. This follows from the fact that E_λ has fixed points outside of S (in fact one such point in each horizontal strip of width 2π—see [17]), so we may assume that $E_\lambda^m(V)$ contains this fixed point for all sufficiently large m. In particular, we have shown:

Proposition 4.5. *Let $z \in \Lambda$ and suppose that V is any connected neighborhood of z. Then $E_\lambda^m(V)$ meets the boundary of S for all sufficiently large m.*

Proposition 4.6. *Λ is a connected subset of S.*

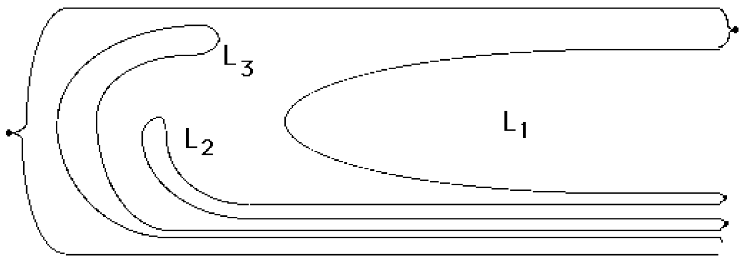

Fig. 16. Embedding Γ in the plane.

Proof. Let G be the union of the boundaries of the L_i for all i. Since Λ is the closure of G, it suffices to show that G is connected. Suppose that this is not true. Then we can write G as the union of two disjoint sets A and B. One of A or B must contain infinitely many of the boundaries of the L_i. Say A does. But then, if $b \in B$, the previous proposition guarantees that infinitely many of these boundaries meet any neighborhood of b. Hence b belongs to the closure of A. This contradiction establishes the result.

We can now prove:

Theorem 4.7. *There is a natural compactification Γ of Λ that makes Γ into an indecomposable continuum.*

Proof. We first compactify Λ by adjoining the backward orbit of 0. To do this we identify the "points" $(-\infty, 0)$ and $(-\infty, \pi)$ in S: this gives $E_\lambda^{-1}(0)$. We then identify the points (∞, π) and $\lim_{t\to-\infty} \gamma_1(t)$. This gives $E_\lambda^{-2}(0)$. For each $n > 1$ we identify

$$\lim_{t\to\infty} \gamma_n(t)$$

and

$$\lim_{t\to-\infty} \gamma_{n+1}(t)$$

to yield $E_\lambda^{-n-1}(0)$. This augmented space Γ may easily be embedded in the plane. See Fig. 16. Moreover, if we extend the B_i and the lines $y = 0$ and $y = \pi$ in the natural way to include these new points, then this yields a curve which accumulates everywhere on itself but does not separate the plane. See the proposition above. By a theorem of Curry [11], it follows that Γ is indecomposable. $\square$

As a consequence of this theorem, Λ must contain uncountably many composants (see [22, p. 213]). In fact, in [17] it is shown that Λ contains uncountably many curves.

4.3. Dynamics on Λ

In this section we describe completely the dynamics of E_λ on Λ.

Proposition 4.8. *There exists a unique fixed point w_λ in S if $\lambda > 1/\mathrm{e}$. Moreover, w_λ is repelling and, if $z \in S -$ orbit of 0, $E_\lambda^{-n}(z) \to w_\lambda$ as $n \to \infty$.*

Proof. First consider the equation

$$\lambda e^{y \cot y} \sin y = y.$$

Since $y \cot y \to 1$ as $y \to 0$ and $\lambda e > 1$, we have $\lambda e^{y \cot y} \sin y > y$ for y small and positive. Since the left-hand side of this equation vanishes when $y = \pi$, it follows that this equation has at least one solution y_λ in the interval $0 < y < \pi$.

Let $x_\lambda = y_\lambda \cot y_\lambda$. Then one may easily check that $w_\lambda = x_\lambda + iy_\lambda$ is a fixed point for E_λ in the interior of S. Since the interior of S is conformally equivalent to a disk and E_λ^{-1} is holomorphic, it follows from the Schwarz Lemma that w_λ is an attracting fixed point for the restriction of E_λ^{-1} to S and that $E_\lambda^{-n}(z) \to w_\lambda$ for all $z \in S$.

Remarks.

(1) Thus the α-limit set of any point in Λ is w_λ.
(2) The bound $\lambda > 1/e$ is necessary for this result, since we know that E_λ has two fixed points on the real axis for any positive $\lambda < 1/e$. These fixed points coalesce at 1 as $\lambda \to 1/e$ and then separate into a pair of conjugate fixed points, one of which lies in S.

We now describe the ω-limit set of any point in Λ. Clearly, if $z \in B_n$ then $E_\lambda^{n+1}(z) \in \mathbb{R}$ and so the ω-limit set of z is infinity. Thus we need only consider points in Λ that do not lie in B_n. We will show:

Theorem 4.9. *Suppose $z \in \Lambda$ and $z \neq w_\lambda$, $z \notin B_n$ for any n. Then the ω-limit set of z is the orbit of 0 under E_λ together with the point at infinity.*

To prove this we first need a lemma.

Lemma 4.10. *Suppose $z \in \Lambda$, $z \neq w_\lambda$. Then $E_\lambda^n(z)$ approaches the boundary of S as $n \to \infty$.*

Proof. Let h be the uniformization of the interior of S taking S to the open unit disk and w_λ to 0. Recall that E_λ^{-1} is well defined on S and takes S inside itself. Then $g = h \circ E_\lambda^{-1} \circ h^{-1}$ is an analytic map of the open disk strictly inside itself with a fixed point at 0. This fixed point is therefore attracting by the Schwarz Lemma. Moreover, if $|z| > 0$ we have $|g(z)| < |z|$. As a consequence, if $\{z_n\}$ is an orbit in Λ, we have $|h(z_{n+1})| > |h(z_n)|$, and so $|h(z_n)| \to 1$ as $n \to \infty$. □

The remainder of the proof is essentially contained in [17] (see pp. 45–49). In that paper it is shown that there is a "quadrilateral" Q containing a neighborhood of 0 in $\mathbb{R}$ as depicted in Fig. 17. The set Q has the following properties:

(1) If $z \in \Lambda - \bigcup_n B_n$ and $z \neq w_\lambda$, then the forward orbit of z meets Q infinitely often.
(2) Q contains infinitely many closed "rectangles" $R_k, R_{k+1}, R_{k+2}, \ldots$ for some $k > 1$ having the property that if $z \in R_j$, then $E_\lambda^j(z) \in Q$ but $E_\lambda^i(z) \notin Q$ for $0 < i < j$.
(3) If $z \in Q$ but $z \notin \bigcup_{j=k}^{\infty} R_j$, then $z \in L_n$ for some n.

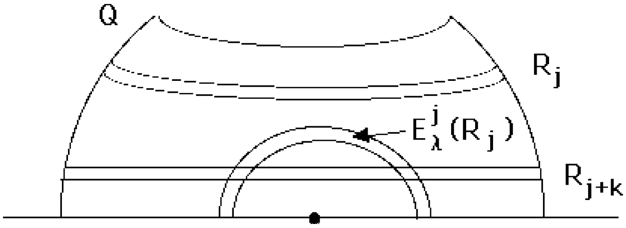

Fig. 17. The return map on Q.

(4) $E^j_\lambda(R_j)$ is a "horseshoe" shaped region lying below R_j in Q as depicted in Fig. 17.
(5) $\lim_{j\to\infty} E^j_\lambda(R_j) = \{0\}$.

As a consequence of these facts, any point in Λ has orbit that meets the $\bigcup R_j$ infinitely often. We may thus define a return map

$$\Phi : \Lambda \cap \left(\bigcup_j R_j\right) \to \Lambda \cap \bigcup_j R_j$$

by

$$\Phi(z) = E^j_\lambda(z)$$

if $z \in R_j$. By item (4), $\Phi(z)$ lies in some R_k with $k > j$. By item (5), it follows that

$$\Phi^n(z) \to 0$$

for any $z \in \Lambda \cap Q$. Consequently, the ω-limit set of z contains the orbit of 0 and infinity.

For the opposite containment, suppose that the forward orbit of z accumulates on a point q. By the lemma, q lies in the boundary of S. Now the orbit of z must also accumulate on the preimages of q. If q does not lie on the orbit of 0, then these preimages form an infinite set, and some points in this set lie on the boundaries of the L_n. But these points lie in the interior of S, and this contradicts the lemma. Thus the orbit of z can only accumulate in the finite plane on points on the orbit of 0. Since the "preimage" of 0 is infinity, the orbit also accumulates at infinity. □

It is known [7] that there are uncountably many curves in the λ-plane having the property that, if λ lies on one of these curves, then $E^n_\lambda(0) \to \infty$. Consequently, for such a λ-value, the Julia set of E_λ is again the complex plane. For these λ-values, a variant of the above construction also yields invariant indecomposable continua in the Julia set. Whether these continua are homeomorphic to any of those constructed above is an open question. We plan to discuss these constructions in a later paper.

Douady and Goldberg [18] have shown that if $\lambda, \mu > 1/e$, then E_λ and E_μ are not topologically conjugate. Each such map possesses invariant indecomposable continua Λ_λ and Λ_μ in S, and the dynamics on each are similar, as shown above. In fact, one can show that each pair of these invariant sets is non-homeomorphic.

As a final remark, Lyubich has shown that each Λ_λ is a set of measure 0 in S. Indeed, it follows from his work [23] that the set of points in $\mathbb{C}$ whose orbits have arguments that are equidistributed on the unit circle have full measure. In Λ_λ, the arguments of all orbits tend to 0 and/or π, and so Λ_λ has measure 0 in S.

5. After the explosion

As we have mentioned, when $\lambda > 1/\mathrm{e}$, the Julia set of E_λ is the entire plane. In 1981, Misiurewicz showed that $J(E_1) = \mathbb{C}$, answering a sixty-year-old question of Fatou. We present his proof of this fact below, generalizing it to the case $\lambda > 1/\mathrm{e}$.

The following proposition highlights one of the differences between $E_\lambda(z)$ and polynomials: points which tend to ∞ under iteration of E_λ need not be in the stable set.

Proposition 5.1. *When $\lambda > 1/\mathrm{e}$, the real line is contained in $J(E_\lambda)$ and hence all preimages of the real line lie in $J(E_\lambda)$.*

Proof. Let S denote the strip $|\operatorname{Im}(z)| \leqslant \pi$. Suppose $E_\lambda^j(z) \in \mathbb{R}$. Hence $E_\lambda^n(z) \to \infty$. Let U be a neighborhood of z. Then $E_\lambda^i(U)$ meets the real line for all $i \geqslant j$. For sufficiently large n, we have that $(E_\lambda)'(E_\lambda^n(z)) > 2$. Let B_δ be a ball of radius δ about $E_\lambda^n(z)$ that is strictly contained in $E_\lambda^n(U)$. Then $E_\lambda^k(B_\delta)$ contains a ball of radius $2^k\delta$ about $E_\lambda^{n+k}(z)$, provided the successive images of this ball lies in S (so that E_λ is one-to-one). This cannot happen for all k, however, since eventually these balls must grow to meet the lines $\operatorname{Im} z = \pm\pi$. Therefore the next iteration of E_λ maps points in this ball to the far left half plane. This happens for all sufficiently large k. Thus there are points arbitrarily close to $E_\lambda^{n+k}(z)$ whose images eventually lie in the far left half plane, and so their next images lie in the unit disk about 0. Thus the family of iterates $\{E_\lambda^n\}$ does not converge uniformly to ∞ on U. Hence the family of iterates of E_λ is not a normal family on U and so $z \in J(E_\lambda)$. $\square$

Thus to show that $J(E_\lambda) = \mathbb{C}$, it suffices to show that inverse images of the real line are dense in $\mathbb{C}$. For this, we need several lemmas.

Lemma 5.2. $|\operatorname{Im}(E_\lambda^n(z))| \leqslant |(E_\lambda^n)'(z)|$.

Proof. If $z = x + \mathrm{i}y$, we have

$$\left|\operatorname{Im}(E_\lambda(z))\right| = \lambda \mathrm{e}^x |\sin y| \leqslant \lambda \mathrm{e}^x |y| = \left|E_\lambda'(z)\right| \left|\operatorname{Im}(z)\right|$$

so that

$$\frac{|\operatorname{Im}(E_\lambda(z))|}{|\operatorname{Im}(z)|} \leqslant |E_\lambda'(z)|$$

if $z \notin \mathbb{R}$. More generally, if $E_\lambda^n(z) \notin \mathbb{R}$, we may apply this inequality repeatedly to find

$$\frac{|\operatorname{Im}(E_\lambda^n(z))|}{|\operatorname{Im}(E_\lambda(z))|} = \prod_{i=1}^{n-1} \frac{|\operatorname{Im} E_\lambda(E_\lambda^i(z))|}{|\operatorname{Im}(E_\lambda^i(z))|} \leqslant \prod_{i=1}^{n-1} \left|E_\lambda'(E_\lambda^i(z))\right|.$$

Since $|\operatorname{Im}(E_\lambda(z))| \leqslant |E_\lambda(z)| = |E'_\lambda(z)|$ we may write

$$\left|\operatorname{Im}(E^n_\lambda(z))\right| \leqslant \prod_{i=0}^{n-1} \left|E'_\lambda(E^i_\lambda(z))\right| = \left|(E^n_\lambda)'(z)\right|. \qquad \square$$

Now let W denote the strip $\{z \mid |\operatorname{Im} z| \leqslant \pi/3\}$ which is contained in S. Since orbits in the right half plane tend to diverge from the real axis, it follows that an open set of points in $W \subset S$ have orbits that leave S and hence W under iteration. The next lemma shows, however, that the orbits of most of these points must eventually return.

Lemma 5.3. *Let U be an open connected set. Then only finitely many of the $E^n_\lambda(U)$ can be disjoint from W.*

Proof. Let us assume that infinitely many of the images of U are disjoint from W. If there is an n for which E^n_λ is not a homeomorphism taking U onto its image, then there exist $z_1, z_2 \in U$, $z_1 \neq z_2$, for which $E^n_\lambda(z_1) = E^n_\lambda(z_2)$. Consequently, there is a j for which $E^j_\lambda(z_1) = E^j_\lambda(z_2) + 2k\pi\mathrm{i}$ for some $k \in \mathbb{Z} - \{0\}$. But then $E^j_\lambda(U)$ must meet a horizontal line of the form $y = 2m\pi$ for $m \in \mathbb{Z}$ and so $E^{j+1}_\lambda(U)$ meets $\mathbb{R}$. Hence $E^{j+\alpha}_\lambda(U)$ meets $\mathbb{R}$ for all $\alpha > 0$ and only finitely many of the images of U can be disjoint from W. We thus conclude that each E^n_λ must be a homeomorphism on U.

Now suppose there is a sequence n_j such that for each j, $E^{n_j}_\lambda(U) \cap W = \emptyset$. By the previous lemma, $|(E^{n_j}_\lambda)'(z)| \geqslant (\pi/3)^j$ for each j and all $z \in U$. It follows that, if U contains a disk of radius $\delta > 0$, then $E^{n_j}_\lambda(U)$ contains a disk of radius $\delta(\pi/3)^j$. Hence for j large enough, $E^{n_j}_\lambda(U)$ must meet a line of the form $y = 2\pi$ and again we are done. $\square$

We can now prove

Theorem 5.4. $J(E_\lambda) = \mathbb{C}$.

Proof. By Proposition 5.1, it suffices to show that any open set in $\mathbb{C}$ contains some preimage of $\mathbb{R}$. To that end, let U be open and connected and suppose $E^n_\lambda(U) \cap \mathbb{R} = \emptyset$ for each n. By Montel's Theorem, $\{E^n_\lambda\}$ is a normal family on U.

By the previous lemma, we have that at most finitely many iterates of U are disjoint from W. Since none of the iterates of U meet the boundary of S, it follows that all but finitely many of the iterates of U lie in S. By replacing U by $E^n_\lambda(U)$, we may assume that all of the iterates of U lie in S.

Now we invoke the results of the previous section. The ω-limit set of any point in U must be the orbit of 0 and ∞. Hence the orbit of U must enter any small neighborhood of 0 infinitely often. But we saw above that, after entering this neighborhood, subsequent iterates of U move along the real axis until suddenly jumping above the exit set L_1. But this image lies outside the strip W. Since this happens infinitely often, we have infinitely many images of U that do not meet W. This contradiction establishes the theorem. $\square$

6. Hair transplants

In this section we will consider λ-values for which λ is negative. We will see that there is a unique bifurcation that occurs at $\lambda = -\mathrm{e}$. On the real line this is a simple period doubling bifurcation. But in the plane this simple bifurcation has global ramifications, though they are not as spectacular as occurred in the saddle-node bifurcation.

When $\lambda = -\mathrm{e}$, E_λ has a fixed point at -1 and $E'_{-\mathrm{e}}(-1) = -1$. For $\lambda < -\mathrm{e}$, E_λ has a single repelling fixed point r_λ on the real axis; when $-\mathrm{e} < \lambda < 0$ this fixed point is attracting.

Proposition 6.1. *Suppose* $\lambda < -\mathrm{e}$. *Then* E_λ *has a unique attracting* 2*-cycle on the negative real axis. Moreover, the orbit of any point on the real axis* (*except* r_λ) *tends to this* 2*-cycle.*

Proof. Consider the graph of

$$y(x) = E_\lambda^2(x) = \lambda \mathrm{e}^{\lambda \mathrm{e}^x}.$$

We have $y'(x) > 0$ and

$$\lim_{x\to\infty} y(x) = 0, \qquad \lim_{x\to-\infty} y(x) = \lambda.$$

Moreover

$$y''(x) = y(x)(\lambda \mathrm{e}^x)(\lambda \mathrm{e}^x + 1).$$

Hence $y''(x) < 0$ if $\lambda \mathrm{e}^x + 1 < 0$ whereas $y''(x) > 0$ if $\lambda \mathrm{e}^x + 1 > 0$. Thus there is a unique inflection point for y.

Now $E_\lambda(-1) = \lambda \mathrm{e}^{-1} < -1$ since $\lambda < -\mathrm{e}$. Hence $r_\lambda < -1$ since the graph of E_λ decreases. Therefore

$$y''(r_\lambda) < 0$$

and so $y'' < 0$ for $x > r_\lambda$. It follows that the graph of y crosses the diagonal exactly once in each of the intervals $(-\infty, r_\lambda)$ and (r_λ, ∞). This yields the 2-cycle, which must be attracting since y' lies between 0 and 1 at each point. The graph of y shows that the intervals (r_λ, ∞) and (λ, r_λ) are mapped inside themselves by E_λ^2, and so all orbits except r_λ tend to the 2-cycle. □

Thus we have a typical period-doubling bifurcation when $\lambda = -\mathrm{e}$. In the complex plane, this bifurcation is accompanied by a “hair transplant”. By this we mean: When $\lambda > -\mathrm{e}$ the Julia set is a Cantor bouquet. As λ approaches $-\mathrm{e}$, a repelling 2-cycle approaches the attracting fixed point, dragging with it the attached hairs. See Fig. 18. When $\lambda = -\mathrm{e}$, the fixed point becomes neutral and it now has a pair of hairs attached. When $\lambda < -\mathrm{e}$, the fixed point becomes repelling, but retains the two hairs—they have been transplanted from the 3-cycle to the fixed point.

For the remainder of this section we will consider only the case $\lambda = -\mathrm{e}$. We will see how two hairs can be attached to a single point. We write $E(z) = E_{-\mathrm{e}}(z)$ for simplicity.

Let H be the half plane $\operatorname{Re} z \leqslant -1$. Then $E_\lambda(H)$ is the disk $0 < |z| \leqslant 1$.

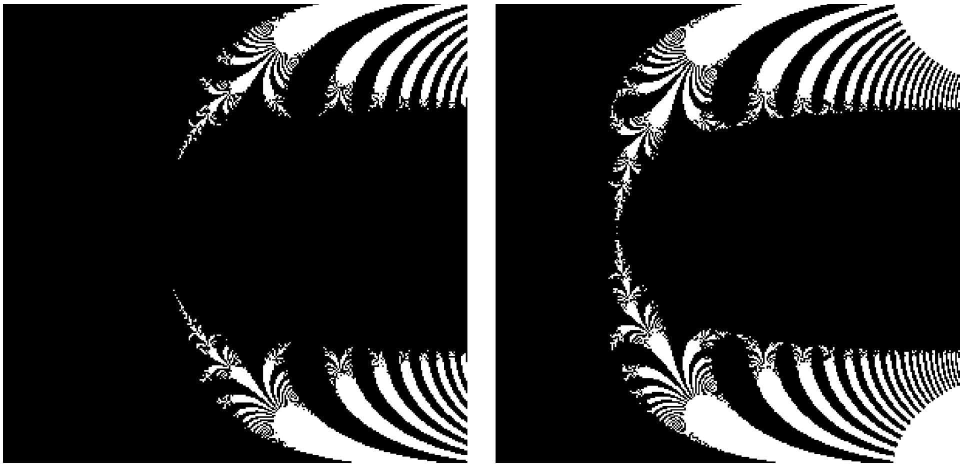

Fig. 18. The Julia sets for $\lambda = -2.5$ and $\lambda = -3.5$.

Proposition 6.2. $E^2(H) \subset H$.

Proof. Consider the vertical line $\operatorname{Re} z = -1 + \mathrm{i}t$ for $-\pi \leqslant t \leqslant \pi$. We claim that $\operatorname{Re}(E^2(1 + \mathrm{i}t)) < -1$ if $t \neq 0$. Indeed, we have

$$y(t) = \operatorname{Re} E^2(1 + \mathrm{i}t) = -\mathrm{e}\mathrm{e}^{-\cos t} \cos(\sin t).$$

Then

$$y'(t) = -\mathrm{e}\mathrm{e}^{-\cos t}\big(\sin(t - \sin t)\big).$$

If $0 < t < \pi$, $\sin(t - \sin t) > 0$ and so $y'(t) < 0$. If $-\pi < t < 0$, $\sin(t - \sin t) < 0$ and so $y'(t) > 0$. Therefore $y(t)$ decreases from $y(0) = -1$ to $y(\pi) = -\mathrm{e}^2$ as t increases from 0 to π, and, similarly, $y(t)$ increases from $-\mathrm{e}^2$ to -1 as t runs from $-\pi$ to 0.

Since $E^2(H)$ is contained inside the simple closed curve bounded by $E^2(1 + \mathrm{i}t)$, we have the result. $\square$

Since $E^2(H) \subset H$ it follows that all orbits in H and $E(H)$ tend to the neutral fixed point at -1. These two regions are called attracting petals.

Consider $E^{-1}(H)$. This set consists of infinitely many "fingers" that surround the horizontal line $\operatorname{Im} z = 2k\pi$, $\operatorname{Re} z \geqslant -1$. We denote each finger by F_k, where $-1 + 2k\pi\mathrm{i}$ is the preimage of -1 lying in F_k. Since $E(F_k) = H$, it follows that all points in F_k have orbits tending to -1. Note that F_0 contains the disk $E(H)$ and in fact the segment $[-1, \infty)$ in the reals. See Fig. 19.

Now consider $E^{-1}(F_0)$. Certainly $E^{-1}(F_0) \supset H$ since H is mapped to $E(H) \subset F_0$. $E^{-1}(F_0)$ also contains the straight lines $\operatorname{Im} z = (2k + 1)\pi$, $\operatorname{Re} z \geqslant -1$, since these lines are mapped to the half line $(-\infty, -\mathrm{e})$ which is properly contained in H. Finally, $E^{-1}(F_0)$ is simply connected and contains all preimages of any $z \in F_0$ (except 0). $E^{-1}(F_0)$ resembles a "glove" since it separates each of the fingers as shown in Fig. 20.

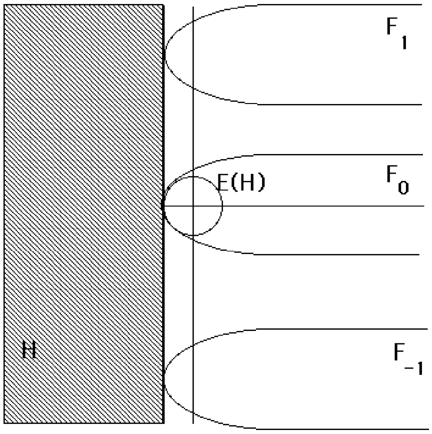

Fig. 19. The fingers F_k.

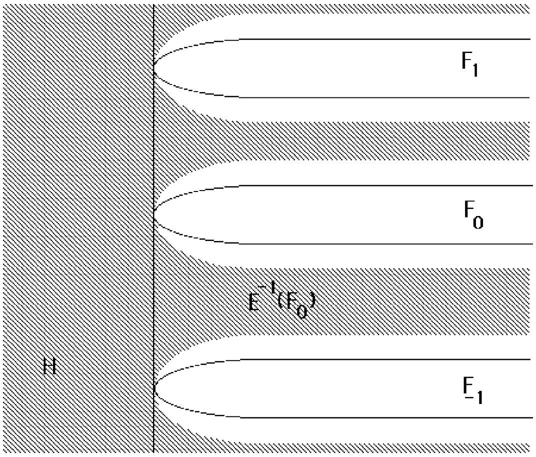

Fig. 20. The glove $E^{-1}(F_0)$.

Let G denote the glove $E^{-1}(F_0)$. Then all points in the interior of $G \cup (\bigcup_n F_n)$ lie in the stable set for E as all orbits tend to -1. Note that -1 and its preimages lie on the boundary of $G \cup (\bigcup_n F_n)$. These points do not lie in the stable set as rationally indifferent fixed points always lie in $J(E)$.

Now consider the component of the complement of G that contains F_0. F_0 divides this region into two open sets, $S_\pm$, S_+ lying above F_0, S_- below. Using arguments as in Section 3 one can check that, there is a unique hair $\gamma_\pm$ in $J(E)$ that lies in $S_\pm \cap \{\operatorname{Re} z > \gamma\}$ for γ sufficiently large. Indeed, $E(\gamma_+) = \gamma_-$ and $E(\gamma_-) = \gamma_+$. The existence of these hairs in the far right half plane may be verified by considering rectangles $R_\pm(t)$ centered at $t \pm \mathrm{i}\pi/2$ and having vertical height π. Note that E maps $R_+(t)$ to the half plane $\operatorname{Im} z \leqslant 0$ while R_- is mapped to the upper half plane. Constructing strings of such rectangles give points whose orbits hop back and forth between S_+ and S_-.

One checks easily that the bounded regions

$$S_{\pm} \cap \{\operatorname{Re} z \leqslant \eta\}$$

are mapped completely over each other, as $E(S_+) \supset S_-$ and $E(S_-) \supset S_+$. Hence we can pull the hairs $\gamma_{\pm}$ back by a branch of the inverse of E in each of S_+ and S_-. The only place that these pulled back hairs can limit is the fixed point at -1. Hence we see that, when $\lambda = -\mathrm{e}$, the neutral fixed point has two hairs attached.

What happens is this: as long as E_λ has an attracting fixed point, all repelling periodic points have unique hairs attached. But as λ tends to $-\mathrm{e}$ along $\mathbb{R}^-$, a repelling 2-cycle merges with the attracting fixed point to produce the neutral fixed point at -1. This bifurcation is a "hair transplant".

Note that there are infinitely many preimages of -1 under $E^{-1}, E^{-2}, \ldots$. Hence there are infinitely many other points in $J(E)$ that have 2 hairs attached.

6.1. Other examples

For complex λ-values, one often encounters Julia sets with multiple hairs attached.

For example, in Fig. 21, we display the Julia set when $\lambda = 5 + \mathrm{i}\pi$. It is relatively easy to check that this exponential admits an attracting 3-cycle. Note that there seem to be three hairs attached at various points in the plane.

In Fig. 22, we display the Julia set when $\lambda = 10 + 3\pi\mathrm{i}$. This map also has an attracting cycle of period 3. Note that different hairs now seem to be attached. In contrast, the Julia set for $\lambda = 3.14\mathrm{i}$ (Fig. 23) shows that the structure of the attached hairs can be extremely complicated.

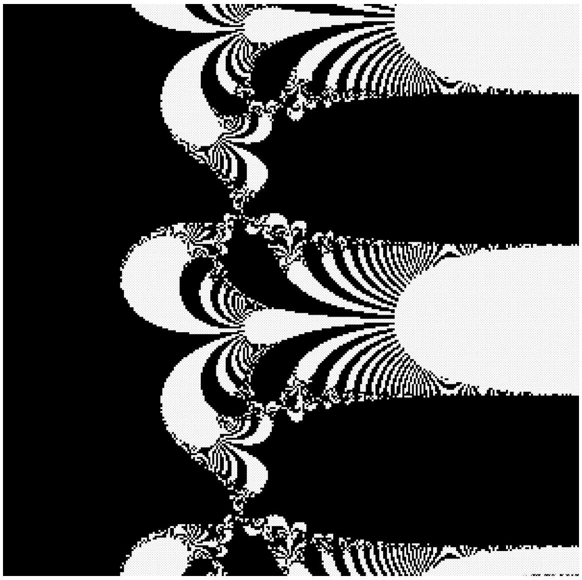

Fig. 21. The Julia set for $\lambda = 5 + \pi\mathrm{i}$.

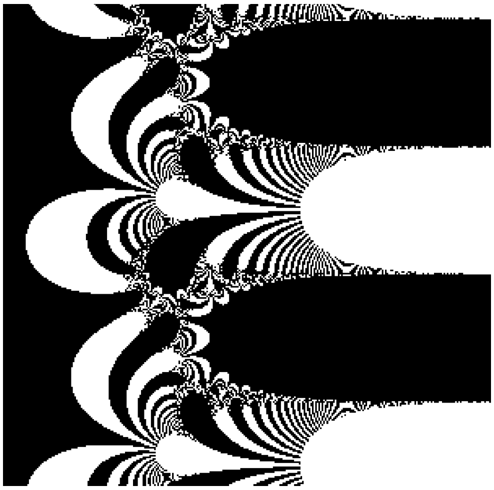

Fig. 22. The Julia set for $\lambda = 10 + 3\pi i$.

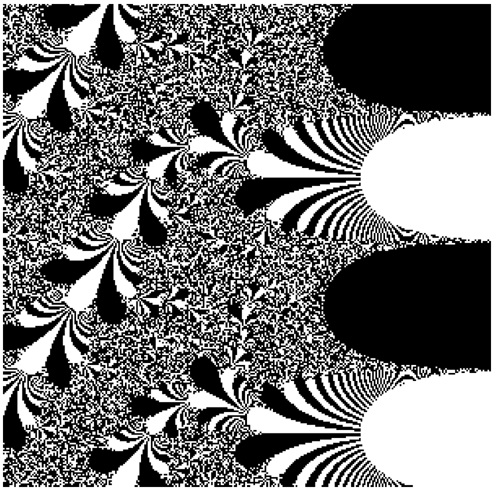

Fig. 23. The Julia set for $\lambda = 3.14i$.

One can use a "kneading invariant" to characterize which hairs are attached in case E_λ admits an attracting cycle. See [7]. The full story here, however, is not yet complete.

References

[1] P. Atela, Bifurcations of dynamic rays in complex polynomials of degree two, Ergodic Theory Dynamical Systems 12 (1991) 401–423.

[2] J. Aarts, L. Oversteegen, The geometry of Julia sets, Trans. Amer. Math. Soc. 338 (1993) 897–918.

[3] B. Branner, The Mandelbrot set, in: Chaos and Fractals: The Mathematics Behind the Computer Graphics, Amer. Math. Soc., Providence, RI, 1989, pp. 75–106.
[4] M. Barge, Horseshoe maps and inverse limits, Pacific J. Math. 121 (1986) 29–39.
[5] I.N. Baker, Wandering domains in the iteration of entire functions, Proc. London Math. Soc. 49 (1984) 563–576.
[6] I.N. Baker, Repulsive fixpoints of entire functions, Math. Z. 104 (1968) 252–256.
[7] R. Bhattacharjee, R.L. Devaney, Tying hairs for structurally stable exponentials, Preprint.
[8] C. Bodelon et al., Hairs for the complex exponential family, Internat. J. Bifurcation and Chaos, to appear.
[9] P. Blanchard, Complex analytic dynamics on the Riemann sphere, Bull. Amer. Math. Soc. 2 (1984) 85–141.
[10] I.N. Baker, P. Rippon, Iteration of exponential functions, Ann. Acad. Sci. Fenn., Ser. 1A Math. 9 (1984) 49–77.
[11] S. Curry, One-dimensional nonseparating plane continua with disjoint ε-dense subcontinua, Topology Appl. 39 (1991) 145–151.
[12] R.L. Devaney, The structural instability of Exp(z), Proc. Amer. Math. Soc. 94 (1985) 545–548.
[13] R.L. Devaney, Knaster-like continua and complex dynamics, Ergodic Theory Dynamical Systems 13 (1993) 627–634.
[14] R.L. Devaney, L. Goldberg, Uniformization of attracting basins for exponential maps, Duke Math. J. 55 (1987) 253–266.
[15] A. Douady, J. Hubbard, Étude dynamique des polynôme complexes, Publications Mathematiques d'Orsay.
[16] A. Douady, J. Hubbard, Itération des polynômes quadratiques complexes, C. R. Acad. Sci. Paris Ser. I 29 (1982) 123–126.
[17] R.L. Devaney, M. Krych, Dynamics of Exp(z), Ergodic Theory Dynamical Systems 4 (1984) 35–52.
[18] A. Douady, L. Goldberg, The nonconjugacy of certain exponential functions, in: Holomorphic Functions and Moduli, MSRI Publ., Springer, Berlin, 1988, pp. 1–8.
[19] R.L. Devaney, F. Tangerman, Dynamics of entire functions near the essential singularity, Ergodic Theory Dynamical Systems 6 (1986) 489–503.
[20] A. Eremenko, M.Yu. Lyubich, Iterates of entire functions, Dokl. Akad. Nauk SSSR 279 (1984) 25–27, English transl. in:, Soviet Math. Dokl. 30 (1984) 592–594.
[21] L.R. Goldberg, L. Keen, A finiteness theorem for a dynamical class of entire functions, Ergodic Theory Dynamical Systems 6 (1986) 183–192.
[22] K. Kuratowski, Topology, Vol. 2, Academic Press, New York, 1968.
[23] M. Lyubich, Measurable dynamics of the exponential, Soviet Math. Dokl. 35 (1987) 223–226.
[24] J. Mayer, An explosion point for the set of endpoints of the Julia set of $\lambda\exp(z)$, Ergodic Theory Dynamical Systems 10 (1990) 177–184.
[25] C. McMullen, Area and Hausdorff dimension of Julia sets of entire functions, Trans. Amer. Math. Soc. 300 (1987) 329–342.
[26] M. Misiurewicz, On iterates of e^z, Ergodic Theory Dynamical Systems 1 (1981) 103–106.
[27] G. Piranian, The boundary of a simply connected domain, Bull. Amer. Math. Soc. 64 (1958) 45–55.
[28] S. Smale, Diffeomorphisms with many periodic points, in: Differential and Combinatorial Topology, Princeton University Press, 1964, pp. 63–80.
[29] D. Sullivan, Quasiconformal maps and dynamical systems. I, Solutions of the Fatou–Julia problem on wandering domains, Ann. Math. 122 (1985) 401–418.