1. Introduction

Our goal in this paper is to describe some of the interesting topology that arises in the dynamics of entire functions such as the complex exponential family $E(z) = \lambda e^z$. We will see that the important invariant sets for this family possesses a extremely rich topological structure, including such objects as Cantor bouquets, Knaster continua, hair transplants, and explosion points.

For a complex analytic function $E$, the interesting orbits lie in the Julia set, which we denote by $J(E)$. This is the set on which the map is chaotic. For the exponential family, the Julia set of $E_\lambda$ has three characterizations:

1. $J(E_\lambda)$ is the set of points at which the family of iterates of $E_\lambda$, $\{E_\lambda^n\}$ is not a normal family in the sense of Montel. This is the characterization that is most useful to prove theorems.
2. $J(E_\lambda)$ is the closure of the set of repelling periodic points of $E_\lambda$. This is the dynamical definition of the Julia set.
3. $J(E_\lambda)$ is the closure of the set of points whose orbits tend to $1$. This is the characterization that is most useful to compute the Julia set.

We remark that characterization 3 differs from the case of polynomial iterations, where the Julia set is the boundary of the set of escaping orbits. The reason for the difference is $E_\lambda$ has an essential singularity at $\infty$, while polynomials have superattracting fixed points at $\infty$. The equivalence of (1) and (2) was shown by Baker, see [6]. The equivalence of (1) and (3) is shown in [19].

In this paper we will concentrate on the dynamics of $E_\lambda$ where $\lambda$ is real. For $\lambda$ positive, the Julia set for $E_\lambda$ undergoes a remarkable transformation as $\lambda$ passes through $1/e$. We will show below that $E_\lambda$ possesses an attracting fixed point when $0 < \lambda < 1/e$. All points in the left half plane have orbits that tend to this fixed point. Indeed, the full basin of attraction of this fixed point is open and dense in the plane.

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We will show that the complement of the basin, $J(E_\lambda)$, is a Cantor bouquet for $0 < \lambda \leq 1/e$. Roughly speaking, a Cantor bouquet has the property that all points in the set lie on a curve (or “hair”) homeomorphic to a closed half line. Each of these curves in $J(E_\lambda)$ extend to $\infty$ in the right half-plane. All repelling periodic points and points with bounded orbits lie at the endpoints of the curves, while points that do not lie at the endpoints have unbounded orbits. Since repelling periodic points are dense in $J(E_\lambda)$, the endpoints of the Cantor bouquet must be dense in $J(E_\lambda)$. Indeed, we will show that the set of endpoints is a totally disconnected set, but that the set of endpoints together with the point at infinity forms a connected set, a result due to Mayer [24].

In Figs. 1 and 2 we display a computer graphics rendering of the Julia set of $E_\lambda$ for a particular $\lambda$ with $0 < \lambda < 1/e$. This image was computed using characterization (3) of the Julia set: Points are shaded in white and grey if their orbits ever enter the region $\text{Re} \ z > 50$. The complement of the Julia set is displayed in black. It appears that this Julia set contains large open sets, but this in fact is not the case. The Julia set actually consists of uncountably many curves lying in the Cantor bouquet and extending to $\infty$ in the right half plane. These curves are packed together so tightly that the resulting set has Hausdorff dimension 2, thus giving the appearance of an open set. See Fig. 2.

At $\lambda = 1/e$, $E_\lambda$ undergoes a simple saddle-node bifurcation. The attracting fixed point merges with a repelling fixed point at this $\lambda$-value, producing a neutral fixed point. When $\lambda > 1/e$, this neutral fixed point gives way to a pair of repelling fixed points.

This apparently simple bifurcation has profound global ramifications. When $\lambda \leq 1/e$, the Julia set is a nowhere dense subset of the right half plane. However, when $\lambda > 1/e$,
$J(E_\lambda)$ suddenly becomes the whole plane. No new repelling periodic points (except the two fixed points involved in the saddle-node) are born in this bifurcation; all simply move smoothly as $\lambda$ crosses through $1/e$. Yet somehow, as soon as $\lambda$ exceeds $1/e$, the repelling periodic points become dense in $\mathbb{C}$. 
In Fig. 3 we display the Julia set for $E_\lambda$ for a particular $\lambda > 1/e$. Note the striking difference between this image and that in Fig. 1.

At this bifurcation, the attracting fixed point and its entire basin of attraction disappear. Most of the points in the Cantor bouquet remain in the Julia set. However, a new and interesting topological invariant set arises. We will show that this set is an indecomposable continuum on which most orbits cycle toward the orbit of 0 and $\infty$.

When $\lambda$ is complex but close to the real axis ($\text{Re} \lambda > 1/e$), the dynamics of $E_\lambda$ undergoes remarkable changes. We will show that in any neighborhood $U$ of $\lambda > 1/e$ in the parameter plane, there is a positive integer $N$ such that for any $n \geq N$ there is a $\lambda_n \in U$ for which the corresponding exponential has an attracting cycle of period $n$. Hence $E_\lambda$ is not structurally stable for any $\lambda > 1/e$.

This paper is a summary of a series of lectures delivered by the author at the Summer Conference on Topology and Its Applications held at the C.W. Post Campus of Long Island University in August, 1999.

2. Exponential dynamics

As in the often-studied quadratic family $Q_c(z) = z^2 + c$, it is the orbit of 0 that plays a crucial role in determining the dynamics of $E_\lambda$. For the exponential family, 0 is an asymptotic value (an omitted value) rather than a critical point. Nevertheless, any stable domain in the complement of the Julia set of $E_\lambda$ must be associated with the orbit of 0 in the following sense:

**Theorem 2.1.** Suppose $E_\lambda$ has an attracting or rationally neutral (parabolic) periodic point. Then $E_\lambda^n(0)$ must tend to the attracting or neutral cycle. If, on the other hand, $E_\lambda^n(0) \to \infty$, then $J(E_\lambda) = \mathbb{C}$.

The proof of the first statement in this theorem is a classical fact that goes back to Fatou. The second follows from the Sullivan No Wandering Domains Theorem [29], as extended to the case of the exponential by Goldberg and Keen [21] and Eremenko and Lyubich [20]. Rather than rely on this big machinery, we will give a bare-hands approach due to Misiurewicz [26] to show that $J(E_\lambda) = \mathbb{C}$ when $\lambda > 1/e$ in Section 5.

Consider for the moment the restriction of $E_\lambda$ to the real line. The exponential family undergoes a saddle node bifurcation at $\lambda = 1/e$ since, when $\lambda = 1/e$, the graph of $E_{1/e}$ is tangent to the diagonal at 1. See Fig. 4. We have $E_{1/e}(1) = 1$ and $E'_{1/e}(1) = 1$. When $\lambda < 1/e$, the graph of $E_\lambda$ lies above the diagonal and all orbits (including 0) tend to $\infty$.

When $\lambda < 1/e$, the graph of $E_\lambda$ crosses the diagonal twice, at an attracting fixed point $a_\lambda$ and a repealing fixed point $r_\lambda$. For later use note that $0 < a_\lambda < 1 < r_\lambda$. Note also that the orbit of 0 tends to $a_\lambda$, as it must by Fatou’s theorem. See Fig. 4.
3. Cantor bouquets

In this section, we begin the study of the dynamics of $E_\lambda$ by considering the case where $\lambda \leq 1/e$. We show here that $J(E_\lambda)$ is a Cantor bouquet.

Here is a rough idea of the construction of a Cantor bouquet. We will “tighten up” these ideas in following sections.

Let $E(z) = (1/e)e^z$. We have $E(1) = 1$ and $E'(1) = 1$. If $x_0 \in \mathbb{R}$ and $x_0 < 1$, then $E^n(x_0)$ tends to the fixed point at 1. If $x_0 > 1$, then $E^n(x_0) \to \infty$ as $n \to \infty$. This can be shown using the web diagram as shown in Fig. 5.

The vertical line $\text{Re} z = 1$ is mapped to the circle of radius 1 centered at the origin. In fact, $E$ is a contraction in the half plane $H$ to the left of this line, since

$$|E'(z)| = \frac{1}{e} \exp(\text{Re} z) < 1$$
if $z \in H$. Consequently, all points in $H$ have orbits that tend to 1. Hence this half plane lies in the stable set, i.e., in the complement of the Julia set. We will try to paint the picture of the Julia set of $E$ by painting instead its complement.

Since the half plane $H$ is forward invariant under $E$, we can obtain the entire stable set by considering all preimages of this half plane. Now the first preimage of $H$ certainly contains the horizontal lines $\text{Im} z = (2k + 1)\pi$, $\text{Re} z \geq 1$, for each integer $k$, since $E$ maps these lines to the negative real axis which lies in $H$. Hence there are open neighborhoods of each of these lines that lie in the stable set. The first preimage of $H$ is shown in Fig. 6. The complement of $E^{-1}(H)$ consists of infinitely many “fingers”. The fingers are $2k\pi i$ translates of each other, and each is mapped onto the complementary half plane $\text{Re} z \geq 1$.

We denote the fingers in the complement of $E^{-1}(H)$ by $C_j$ with $j \in \mathbb{Z}$, where $C_j$ contains the half line $\text{Im} z = 2j\pi$, $\text{Re} z \geq 1$, which is mapped into the positive real axis. That is, the $C_j$ are indexed by the integers in order of increasing imaginary part. Note that $C_j$ is contained within the strip

$$-\frac{\pi}{2} + 2j\pi \leq \text{Im} z \leq \frac{\pi}{2} + 2j\pi.$$

Now each $C_j$ is mapped in one-to-one fashion onto the entire half plane $\text{Re} z \geq 1$. Consequently each $C_j$ contains a preimage of each other $C_k$. Each of these preimages forms a subfinger which extends to the right in the half plane $H$. See Fig. 7. The complement of these subfingers necessarily lies in the stable set.

Now we continue inductively. Each subfinger is mapped onto one of the original fingers by $E$. Consequently, there are infinitely many sub-subfingers which are mapped to the $C_j$'s by $E^2$. So at each stage we remove the complement of infinitely many subfingers from each remaining finger.

This process is reminiscent of the construction of the Cantor set in the dynamics of polynomials when all critical points tend to $\infty$. In that construction, the complements of
disks are removed at each stage; here we remove the complement of infinitely many fingers. As a result, after performing this operation infinitely many times, we do not end up with points. Rather, as we will see, the intersection of all of these fingers, if nonempty, is a simple curve extending to \( \infty \).

This collection of curves forms the Julia set. \( E \) permutes these curves and each curve consists of a well-defined endpoint together with a “hair” which extends to \( \infty \). It is tempting to think of this structure as a “Cantor set of curves”, i.e., a product of the set of endpoints and the half-line. However, this is not the case as the set of endpoints is not closed.

Note that we can assign symbolic sequences to each point on these curves. To do this, we attach an infinite sequence \( s_0s_1s_2\ldots \) to each hair in the Julia set via the rule: \( s_j \in \mathbb{Z} \) and \( s_j = k \) if the \( j \)th iterate of the hair lies in \( C_k \). The sequence \( s_0s_1s_2\ldots \) is called the itinerary of the curve.

For example, the portion of the real line \( \{ x \mid x > 1 \} \) lies in the Julia set since all points (except 1) tend to \( \infty \) under iteration, not to the fixed point. These points all have itinerary \( 000 \ldots \).

One temptation is to say that there is a hair corresponding to every possible sequence \( s_0s_1s_2\ldots \). This, unfortunately, is not true, as certain sequences simply grow too quickly to correspond to orbits of \( E \).

So this is \( J(E) \): a “hairy” object extending toward \( \infty \) in the right-half plane. We call this object a \textit{Cantor bouquet}. We will see that this bouquet has some rather interesting topological properties as we investigate further.

We remark that the same construction works if \( 0 < \lambda < 1/e \). We still define the half plane \( H \) as the set \( \text{Re} \, z < 1 \). As we saw earlier, the point 1 on the real axis sits between the attracting fixed point \( a_\lambda \) and the repelling fixed point \( r_\lambda \), and so \( E_\lambda(1) < 1 \) and as a consequence \( E_\lambda(H) \) is strictly contained in \( H \). The construction of the fingers now proceeds exactly as above.

3.1. Straight brushes

To describe the structure of a Cantor bouquet in more detail, we need to introduce the notion of a \textit{straight brush}.
To each irrational number \( \zeta \), we assign an infinite string of integers \( n_0n_1n_2\ldots \) as follows. We will break up the real line into open intervals \( I_{n_0n_1n_2\ldots} \) which have the following properties

1. \( I_{n_0n_1n_2\ldots} \supset I_{n_0n_1n_2n_3\ldots} \).
2. The endpoints of \( I_{n_0n_1n_2\ldots} \) are rational.
3. \( \zeta = \bigcap_{k=1}^{\infty} I_{n_0n_1n_2\ldots} \).

Now there are many ways to do this. We choose the following method based on the Farey tree. Inductively, we first define \( I_k = (k, k + 1) \). Given \( I_{n_0n_1n_2\ldots} \), we define \( I_{n_0n_1n_2n_3\ldots} \) as follows. Let

\[
I_{n_0n_1n_2\ldots} = \left( \frac{\alpha}{\beta}, \frac{\gamma}{\delta} \right).
\]

Let \( p_0/q_0 = (\alpha + \gamma)/(\beta + \delta) \), the Farey child of \( \alpha/\beta \) and \( \gamma/\delta \). We write

\[
\frac{\alpha}{\beta} \oplus \frac{\gamma}{\delta} = \frac{p_0}{q_0}
\]

to indicate the Farey child of these fractions. Let \( p_n/q_n \) be the Farey child of \( p_{n-1}/q_{n-1} \) and \( \gamma/\delta \) for \( n > 0 \), and let \( p_{n-1}/q_{n-1} \) be the Farey child for \( \alpha/\beta \) and \( p_n/q_n \) for \( n \geq 0 \). We then set \( I_{n_0n_1n_2n_3\ldots} \) to be the open interval \( (p_j/q_j, p_{j+1}/q_{j+1}) \).

**Example.** \( I_0 = (0, 1) \). The Farey child of \( 0/1 \) and \( 1/1 \) is \( 1/2 \), so \( p_0/q_0 = 1/2 \). Then \( p_1/q_1 = 1/2 \oplus 1/2 = 1/3 \), \( p_2/q_2 = 1/3 \oplus 1/3 = 1/4 \), and \( p_n/q_n = \frac{n+1}{n+2} \) for \( n > 0 \).

For negative \( n \) we have

\[
\frac{p_{-1}}{q_{-1}} = \frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3},
\]
\[
\frac{p_{-2}}{q_{-2}} = \frac{0}{1} \oplus \frac{1}{3} = \frac{1}{4},
\]
\[
\frac{p_{-n}}{q_{-n}} = \frac{1}{n+2}.
\]

Therefore, if \( n \geq 0 \),

\[
I_{0n} = \left( \frac{n+1}{n+2}, \frac{n+2}{n+3} \right)
\]

and if \( n < 0 \),

\[
I_{0n} = \left( \frac{1}{-n+2}, \frac{1}{-n+1} \right).
\]

See Fig. 8. Note that we exhaust all of the rationals via this procedure, so each irrational is contained in a unique \( I_{n_0n_1n_2\ldots} \).

We now define a straight brush, a notion due to Aarts and Oversteegen [2].
Definition 3.1. A straight brush $B$ is a subset of $[0, \infty) \times \mathcal{N}$, where $\mathcal{N}$ is a dense subset of $\mathbb{R} - \mathbb{Q}$. $B$ has the following 3 properties.

1) $B$ is “hairy” in the following sense. If $(y, \alpha) \in B$, then there exists a $y_\alpha \leq y$ such that $(t, \alpha) \in B$ iff $t \geq y_\alpha$. That is, the “hair” $(t, \alpha)$ is contained in $B$ where $t \geq y_\alpha$. $y_\alpha$ is called the endpoint of the hair corresponding to $\alpha$.

2) Given an endpoint $(y_\alpha, \alpha) \in B$ there are sequences $\beta_n \uparrow \alpha$ and $\gamma_n \downarrow \alpha$ in $\mathcal{N}$ such that $(y_{\beta_n}, \beta_n) \rightarrow (y_\alpha, \alpha)$ and $(y_{\gamma_n}, \gamma_n) \rightarrow (y_\alpha, \alpha)$. That is, any endpoint of a hair in $B$ is the limit of endpoints of other hairs from both above and below.

3) $B$ is a closed subset of $\mathbb{R}^2$.

The following facts are easily verified:

1) For any rational number $v$ and any sequence of irrationals $\alpha_n \in \mathcal{N}$ with $\alpha_n \rightarrow v$, it can be shown that the hairs $[y_{\alpha_n}, \alpha_n]$ tend to $(\infty, v)$ in $[0, \infty] \times \mathbb{R}$.

2) Condition (2) above may be changed to: if $(y, \alpha)$ is any point in $B$ (you need not be the endpoint of the $\alpha$-hair), then there are sequences $\beta_n \uparrow \alpha$, $\gamma_n \downarrow \alpha$ so that $(y_{\beta_n}, \beta_n) \rightarrow (y, \alpha)$ and $(y_{\gamma_n}, \gamma_n) \rightarrow (y, \alpha)$ in $B$.

3) Let $(y, \alpha) \in B$ and suppose $y$ is not the endpoint $y_\alpha$. Then $(y, \alpha)$ is inaccessible in $\mathbb{R}^2$ in the sense that there is no continuous curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ such that $\gamma(t) \notin B$ for $0 \leq t < 1$ and $\gamma(1) = (y, \alpha)$.

4) On the other hand, the endpoint $(y_\alpha, \alpha)$ is accessible in $\mathbb{R}^2$.

These facts show that a straight brush is a remarkable object from the topological point of view. We view a straight brush as a subset of the Riemann sphere and set $B^* = B \cup \infty$, i.e., the straight brush with the point at infinity added. Let $E$ denote the set of endpoints of $B$, and let $E^* = E \cup \infty$. Then we have the following result, due to Mayer [24]:

Theorem 3.2. The set $E^*$ is a connected set, but $E$ is totally disconnected.

That is, the set $E^*$ is a connected set, but if we remove just one point from this set, the resulting set is totally disconnected. Topology really is a weird subject!

The reason for this is that, if we draw the straight line in the plane $(\gamma, t)$ where $\gamma$ is a fixed rational, and then we adjoin the point at infinity, we find a disconnection of $E^*$. This, however, is not a disconnection of $E^*$. Moreover, the fact that any non-endpoint in $B$ is inaccessible shows that we cannot disconnect $E^*$ by any other curve.

Remark. Aarts and Oversteegen have shown that any two straight brushes are ambiently homeomorphic, i.e., there is a homeomorphism of $\mathbb{R}^2$ taking one brush onto the other. This leads to a formal definition of a Cantor bouquet.
Definition 3.3. A Cantor bouquet is a subset of $\mathbb{C}$ that is homeomorphic to a straight brush (with $\infty$ mapped to $\infty$).

Our main goal in this section is to sketch a proof of the following result. For more details, see [2].

Theorem 3.4. Suppose $0 < \lambda < 1/e$. Then $J(E_\lambda)$ is a Cantor bouquet.

We now describe the construction of the homeomorphism between the brush and $J(E_\lambda)$. To do this, we first introduce symbolic dynamics. Recall that $E_\lambda$ has a repelling fixed point $r_\lambda > 0$ in $\mathbb{R}$ and that the half plane $\text{Re} \ z < r_\lambda$ lies in the stable set. Similarly the horizontal strips $2k < \text{Im} \ z < 2k$ are contained in the stable set since $E_\lambda$ maps these strips to $\text{Re} \ z < 0$ which is contained in $\text{Re} \ z < r_\lambda$.

Let $v_\lambda = -\ln \lambda$. So $E_\lambda(v_\lambda) = 1$ and so $v_\lambda < r_\lambda$. Fix $\omega_\lambda \in (v_\lambda, r_\lambda)$. Then the half plane $\text{Re} \ z < \omega_\lambda$ also lies in the stable set, and we have $|E'_\lambda(z)| > 1$ in the half plane $\text{Re} \ z \geq \omega_\lambda$.

We denote by $S_k$ the closed halfstrip given by

$$\text{Re} \ z \geq \omega_\lambda \quad \text{and} \quad -\frac{\pi}{2} + 2k\pi \leq \text{Im} \ z \leq \frac{\pi}{2} + 2k\pi.$$ 

Note that these strips contain the Julia set since the complement of the strips lies in the stable set.

Given $z \in J(E_\lambda)$, we define the itinerary of $z$, $S(z)$, as usual by

$$S(z) = s_0 s_1 s_2 \ldots,$$

where $s_j \in \mathbb{Z}$ and $s_j = k$ iff $E^j_\lambda(z) \in S_k$. Note that $S(z)$ is an infinite string of integers that indicates the order in which the orbit of $z$ visits the $S_k$. We will associate to $z$ the irrational number given by the itinerary of $z$ (in the decomposition of the irrationals described above). This will determine the hair in the straight brush to which $z$ is mapped. See Fig. 9. Thus we need only define the $y$-value along this hair. This takes a little work.

Given $z$ on a hair, we will construct a sequence of closed rectangles $R_k(z)$ for each $k \geq 0$. By construction, the $R_k(z)$ will be nested. Each $R_k(z)$ will have sides parallel to the axes and be contained in some strip $S_{s_j}$ where $s_j = k$. We assume that $R_0(z) \supseteq \{z\}$ and for all $w \in R_0(z)$ and for all $j$; otherwise we choose the rectangle $R_0(z)$ to be the real part of the left hand edge of this limiting rectangle.

To begin the construction, for $z \in J(E_\lambda)$, we set $R_0(E^j_\lambda(z))$ to be the square centered around the line $\text{Re} \ z = \text{Re} E^j_\lambda(z)$ with side length $\pi$ and contained in the strip $S_{s_j}$ where $s_j = s_j$. We assume that $\text{Re} w \geq \omega_\lambda$ in $R_0(E^j_\lambda(z))$ for all $w \in R_0(E^j_\lambda(z))$ and for all $j$; otherwise we choose the rectangle $R_0(E^j_\lambda(z)) \cap \{z \geq \omega_\lambda\}$.
Fig. 9. The itinerary of $z$ is $0, 1, -1, \ldots$.

Fig. 10. Construction of $R_0(0)$ and $R_0(1)$.

for the initial box. Observe that $E(z)(R_0(E_\lambda^j(z)) \supset R_0(E_\lambda^{j+1}(z))$. Indeed, the image of $R_0(E_\lambda^j(z))$ is an annulus whose inner radius is $e^{-\pi/2}|E_\lambda^{j+1}(z)|$ and outer radius $e^{\pi/2}|E_\lambda^{j+1}(z)|$. Now $e^{\pi/2} > 4$ and $e^{-\pi/2} < \frac{1}{4}$ so the image annulus is much larger than $R_0(E_\lambda^{j+1})$. See Fig. 10.

It follows that we may find a narrower rectangle $R_1(E_\lambda^j(z))$ strictly contained in $R_0(E_\lambda^j(z))$ having the property that the height of $R_1(E_\lambda^j(z))$ is $\pi$ and the image $E_\lambda(z)(R_1(E_\lambda^j(z)))$ just covers $R_0(E_\lambda^{j+1}(z))$. That is, $R_1(E_\lambda^j(z))$ is the smallest rectangle in $R_0(E_\lambda^j(z))$ whose image annulus is just wide enough so that $R_0(E_\lambda^{j+1}(z))$ fits inside. See Fig. 11. Note that $E_\lambda^j(z) \in R_1(E_\lambda^j(z))$ for each $j$. 
We now continue inductively by setting \( R_k(E_{j}^k(z)) \) to be the subrectangle of \( R_{k-1}(E_{k}^j(z)) \) whose image just covers \( R_{k-1}(E_{k}^{j+1}(z)) \). The \( R_k(E_{k}^j(z)) \) are clearly nested for each fixed \( j \).

Example. Suppose \( z = r_j \). We have that \( R_0(z) \) is the square bounded by \( \text{Re} \, z = r_k \pm \pi/2 \) (or the rectangle bounded by \( \text{Re} \, z = r_k + \pi/2 \) and \( \text{Re} \, z = \omega_k \) if \( r_k \) is close to 1) and \( \text{Im} \, z = \pm \pi/2 \) for each \( j \). One may check that \( \bigcap_{k=0}^{\infty} R_k(z) \) is the strip bounded by \( \text{Re} \, z = r_k \) and \( \text{Re} \, z = \zeta \) where the circle of radius \( \lambda e^{\pi} \) passes through \( \zeta \pm i\pi/2 \). See Fig. 12.

Suppose \( z \) has itinerary \( S(z) = s_0s_1s_2 \ldots \). Let \( I(S(z)) \) denote the irrational number determined by the sequence \( S(z) \) as above. Then set \( \phi(z) = (h(z), I(S(z))) \). We now claim that \( \phi \) is a homeomorphism onto a straight brush. The proof is adapted from [17] and [2].

Let \( B_0 \) denote the image of \( J(E_{z}) \) under \( \phi \). We first show that \( \phi \) is a homeomorphism onto \( B_0 \). We will show later that \( B_0 \) is a straight brush.
We need the following lemma.

**Lemma 3.5 (Expansion Lemma).** Suppose $z_1, z_2 \in J(E_\lambda)$ and $S(z_1) = S(z_2)$. If $z_1 \neq z_2$ then $|E^k_\lambda(z_1) - E^k_\lambda(z_2)| \to \infty$ as $k \to \infty$.

**Proof.** Suppose that $|E^k_\lambda(z_1) - E^k_\lambda(z_2)| \leq M$ for all $k$. There is a well-defined branch $L_{\lambda,j}$ of the inverse of $E_\lambda$ that maps the strip $S_{n_j+1}$ into $S_{n_j}$ and $E^{j+1}_\lambda(z_1)$ to $E^j_\lambda(z_1)$. Clearly, $L_{\lambda,j}$ is a contraction for each $j$ in $\mathbb{R} z \geq r_\lambda$. Thus

$$|L_{\lambda,0} \circ \cdots \circ L_{\lambda,j}(E^{j+1}_\lambda(z_1)) - L_{\lambda,0} \circ \cdots \circ L_{\lambda,j}(E^{j+1}_\lambda(z_2))|$$

tends to $0$ since $|E^{j+1}_\lambda(z_1) - E^{j+1}_\lambda(z_2)|$ is bounded.

We first claim that $\phi$ is one-to-one. Suppose not. Then there exists $z_1, z_2 \in J(E_\lambda)$ with $z_1 \neq z_2$ and $\phi(z_1) = \phi(z_2) = (t, I(S(z_1)))$. Then the left-hand edges of $R_\infty(E^j_\lambda(z_1))$ and $R_\infty(E^j_\lambda(z_2))$ are the same for all $j$. But then $|E^j_\lambda(z_1) - E^j_\lambda(z_2)| \leq 2\pi$ for all $j$. This contradicts the Expansion Lemma.

Next we prove continuity of $\phi$. If $z_0$ and $z_1$ are close, then the first $n$ terms of their itineraries must agree and so $I(S(z_0))$ must be close to $I(S(z_1))$. So suppose there is a sequence $z_i \to z_0$ in $J(E_\lambda)$, but $|h(z_i) - h(z_0)| > \delta$. Then the difference of the inner radii of the annuli covered by $E_\lambda(R_\infty(z_i))$ and $E_\lambda(R_\infty(z_0))$ is at least $c\delta$ for some constant $c > 1$. Some elementary geometry shows that the distance between $h(E_\lambda(z_i))$ and $h(E_\lambda(z_0))$ is also larger than $c\delta > \delta$. Continuing, we see that the left hand edges of $R_\lambda(z_i)$ must eventually differ from $h(E^k_\lambda(z_0))$ by at least $\pi$. This contradicts the fact that $E^k_\lambda(z_1) \to E^k_\lambda(z_0)$.

The continuity of $\phi^{-1}$ follows similarly. Hence $\phi$ is a homeomorphism onto its image. It remains to show that $\phi(J(E_\lambda))$ is a straight brush.

We first show that the image of $J(E_\lambda)$ under $\phi$ projects onto a dense set of irrationals. Let $\xi \in \mathbb{R} - \mathbb{Q}$. Say $\xi = l_{m_0}l_{m_1}l_{m_2} \cdots$. Let $\overline{m}_k = (m_{n_0}m_{n_1}m_{n_2})$. Then $\overline{m}_k = l_{\overline{m}_k}$ converges to $\xi$ as $k \to \infty$ with itinerary $\overline{m}_k$. We will produce a point in $J(E_\lambda)$ with this itinerary.

Suppose $m = \max_{0 \leq j \leq n_j} |n_j|$. Choose $x$ so that $E_\lambda(x) > x + |n|\pi + \pi/2$. Let $S_0(x)$ denote the rectangle in $S_\xi$ with height $\pi$ and satisfying $r_\lambda \leq \text{Re } z \leq x$. Then $E_\lambda(S_0(x))$ covers $S_{n_j}(x)$ for any pair of indices $n_i$ and $n_j$ since the right border of $S_{n_j}(x)$ is mapped to a semi-circle which lies further from the origin than $x \pm im$. Consider

$$L_{\lambda,n_{n_0}} \circ \cdots \circ L_{\lambda,n_{n_{j-1}}} \circ L_{\lambda,n_j}|S_{n_j}(x).$$

This inverse contracts $S_{n_j}(x)$ strictly inside itself. Hence by the Schwarz Lemma, this map has a unique fixed point $z_{\overline{m}_k}$ and $z_{\overline{m}_k}$ has itinerary $\overline{m}_k$. Note that $z_{\overline{m}_k}$ has period $k + 1$ for $E_\lambda$ and is repelling.

We single this fact out as a corollary.

**Corollary 3.6.** Let $s = s_{m_0} \cdots s_{m_\ell}$ be a repeating sequence. Then there is a unique repelling periodic point in $J(E_\lambda)$ that has itinerary $s$.

The uniqueness part of this result follows from the Expansion Lemma.
The fact that \( \phi(J(E_j)) \) is hairy follows from the fact that if \( \phi(z) = (t, s) \) then for any \( t' > t \), \( \phi^{-1}(t', s) \) is a point in \( J(E_j) \). Indeed, let \( R_0 \) denote the square in \( S_{s_0} \) with side length \( \pi \) and left hand boundary on the line \( \text{Re} z = t' \). \( E_j(R_0) \) is an annulus that maps across \( S_{s_1} \) since we know that a rectangle with side on \( \text{Re} z = t < t' \) has this property. Let \( R_1 \) be the square of sidelong \( \pi \) that lies in \( S_{s_1} \) and meets the image of \( \text{Re} z = t' \) at just one point. Then \( E_j(R_0) \supseteq R_1 \). Now continue in this fashion, defining the square \( R_{k+1} \) to be the square in \( S_{s_{k+1}} \) with side length \( \pi \) just meeting the inner boundary of \( E_j(R_k) \). Then using the expansion lemma there is a unique point \( z_0 \) whose orbit travels through the \( R_i \) in order. Note that \( h(z_0) = t' \) by construction.

**Corollary 3.7.** If \( s = (s_0 \ldots s_k) \), then the endpoint of the hair corresponding to \( I(s) \) is a repelling periodic point.

**Proof.** We have already constructed such a point \( z_0 \). If \( z \) has itinerary \( s \) and \( h(z) < h(z_0) \), the above proof shows that the orbit of \( z \) must be bounded. This contradicts the Expansion Lemma.

We emphasize that, even though most hairs spiral in to their respective endpoints, the map \( \phi \) is still one-to-one.

**Remarks.**

1. If \( s = (s_0 s_1 s_2 \ldots) \) is a bounded sequence, then the endpoint of the hair also has a bounded orbit. If \( s \) is unbounded then the endpoint has an unbounded orbit. This orbit tends to \( \infty \) but does so at a slower rate than points on the corresponding hair.

2. It can be shown that the construction above works for any exponential for which there exists an attracting or neutral periodic point. See [19]. However, in the general case, some of the hairs in the Cantor bouquet may be attached to the same point in the crown. We discuss this briefly in Section 6. See [8].

3. McMullen [25] has shown that the Hausdorff dimension of the Cantor bouquet constructed above is 2 but its Lebesgue measure is zero. This accounts for why Figs. 1 and 2 seem to have open regions in the Julia set.

### 3.2. Uniformization of the attracting basin

The basin of attraction \( \Omega_\lambda \) of \( E_\lambda \) is an open, dense, and simply connected subset of the Riemann sphere. Hence the Riemann Mapping Theorem guarantees the existence of a uniformization \( \phi_\lambda : D \rightarrow \Omega_\lambda \). Given such a uniformization, it is natural to ask if the uniformizing map extends to the boundary of \( D \).

In order to extend \( \phi_\lambda \) to the boundary, we need that the image under \( \phi_\lambda \) of a straight ray \( re^{\theta} \), where \( \theta \) is constant, to converge to a single point as \( r \rightarrow 1 \). It is known that if the boundary of the uniformizing region is locally connected, then in fact \( \phi_\lambda \) does extend continuously to \( D \). On the other hand, if the boundary of the region is not locally connected, then not all rays need converge (though a full measure set of them must converge). In our
case, the boundary of $\Omega_\lambda$ is nowhere locally connected (except at $\infty$). However, it is a fact that all rays do converge. Moreover, they land at precisely the endpoints of the Cantor bouquet and $\infty$. This means that we can induce a map on the set of endpoints, but that map is necessarily nowhere continuous [27].

In the case of a straight brush, it is clear that all rays do land at the crown of the bouquet. A direct proof for $E_\lambda$ is given in [14].

In fact, it can be shown that, if we normalize the Riemann map $\phi_\lambda$ so that 0 is mapped to 0, then the induced map $\phi_\lambda^{-1} \circ E_\lambda \circ \phi_\lambda$ on the unit disk is given by

$$T_\mu(z) = \exp\left(i \left( \frac{\mu + \pi}{1 + z} \right) \right).$$

Here $\mu$ is a parameter that lies in the upper half plane and depends upon $\lambda$.

4. Indecomposable continua

We now consider the case $\lambda > 1/e$. Since the orbit of 0 tends to $\infty$, the Julia set is now the entire plane (we will prove this in the next section). For these $\lambda$ values, the attracting basin for the attracting fixed point $a_\lambda$ disappears. What replaces it is a complicated invariant set that is an indecomposable continuum. We describe the construction of this set in this section.

Consider the horizontal strip

$$S = \{ z \mid 0 \leq \text{Im} z \leq \pi \}$$

(or its symmetric image under $z \to \overline{z}$). The exponential map $E_\lambda$ takes the boundary of $S$ to the real axis and the interior of $S$ to the upper half plane. Thus, $E_\lambda$ maps certain points outside of $S$ while other points remain in $S$ after one application of $E_\lambda$. Our goal is to investigate the set of points whose entire orbit lie in $S$. Call this set $\Lambda$. The set $\Lambda$ is clearly invariant under $E_\lambda$. There is a natural way to compactify this set in the plane to obtain a new set $\Gamma$. Moreover, the exponential map extends to $\Gamma$ in a natural way. Our main results in this section include:

**Theorem 4.1.** $\Gamma$ is an indecomposable continuum.

Moreover, we will see that $\Lambda$ is constructed in similar fashion to a family of indecomposable continua known as Knaster continua. See [13] for additional details.

As we will show in Section 4.2, the topology of $\Lambda$ is quite intricate. Despite this, we will show that the dynamics of $E_\lambda$ on $\Lambda$ is quite tame. Specifically, we will prove:

**Theorem 4.2.** The restriction of $E_\lambda$ to $\Lambda - \{ \text{orbit of 0} \}$ is a homeomorphism. This map has a unique repelling fixed point $w_\lambda \in \Lambda$, and the $\alpha$-limit set of all points in $\Lambda$ is $w_\lambda$. On the other hand, if $z \in \Lambda$, $z \neq w_\lambda$, then the $\omega$-limit set of $z$ is either

1. The point at $\infty$, or
2. The orbit of 0 under $E_\lambda$ together with the point at $\infty$. 


Thus we see that $E_\lambda$ possesses an interesting mixture of topology and dynamics in the case where the Julia set is the whole plane. In the plane the dynamics of $E_\lambda$ are quite chaotic, but the overall topology is tame. On our invariant set $A$, however, it is the topology that is rich, but the dynamics are tame.

4.1. Topological preliminaries

In this section we review some of the basic topological ideas associated with indecomposable continua. See [22] for a more extensive introduction to these concepts.

Recall that a continuum is a compact, connected space. A continuum is decomposable if it is the union of two proper subcontinua. Otherwise, it is indecomposable. A well-known example of an indecomposable continuum is the Knaster continuum, $K$. One way to construct this set is to begin with the Cantor middle-thirds set. Then draw the semicircles lying in the upper half plane with center at $\frac{1}{2}; 0$ that connect each pair of points in the Cantor set that are equidistant from $1/2$. Next draw all semicircles in the lower half plane which have for each $n > 1$ centers at $\frac{5}{2}; \frac{3n}{2}, 0)$ and pass through each point in the Cantor set lying in the interval $2/3^n \leq x \leq 1/3^{n-1}$.

The resulting set is partially depicted in Fig. 13.

Theorem 4.3. Suppose $X$ is a one-dimensional nonseparating plane continuum which is the closure of a ray that limits upon itself. Then $X$ is indecomposable.
We refer to [11] for a proof.

Another view of the Knaster continuum which is intimately related to our own construction is as follows. Begin with the unit square $S_0$ in the plane. Next remove a “canal” $C_1$ from $S_0$ whose boundary lies within a distance $1/3$ of each boundary point of $S_0$ as depicted in Fig. 2. Call this set $S_1$. Next remove a new canal $C_2$ from $S_1$. This time the boundary of $C_2$ should be within $1/9$ of the boundary of $S_1$ as depicted in Fig. 14. It is possible to continue this construction inductively in such a way that the resulting set is homeomorphic to the Knaster continuum.

4.2. Construction of $\Lambda$

Recall that the strip $S$ is given by $\{z \mid 0 \leq \text{Im}(z) \leq \pi \}$. Note that $E_\lambda$ maps $S$ in one-to-one fashion onto $\{z \mid \text{Im } z \geq 0 \} - \{0\}$. Hence $E_\lambda^{-1}$ is defined on $S - \{0\}$ and, in fact, $E_\lambda^{-n}$ is defined for all $n$ on $S - \{\text{orbit of } 0\}$. We will always assume that $E_\lambda^{-n}$ means $E_\lambda^{-n}$ restricted to this subset of $S$.

Define

$$\Lambda = \{z \mid E_\lambda^n(z) \in S \text{ for all } n \geq 0 \}.$$ 

If $z \in \Lambda$ it follows immediately that $E_\lambda^n(z) \in S$ for all $n \in \mathbb{Z}$ provided $z$ does not lie on the orbit of 0. Our goal is to understand the structure of $\Lambda$.

Toward that end we define $L_n$ to be the set of points in $S$ that leave $S$ at precisely the $n$th iteration of $E_\lambda$. That is,

$$L_n = \{z \in S \mid E_\lambda^i(z) \in S \text{ for } 0 \leq i < n \text{ but } E_\lambda^n(z) \notin S\}.$$ 

Let $B_n$ be the boundary of $L_n$.

Recall that $E_\lambda$ maps a vertical segment in $S$ to a semi-circle in the upper half plane centered at 0 with endpoints in $\mathbb{R}$. Either this semi-circle is completely contained in $S$ or else an open arc lies outside $S$. As a consequence, $L_1$ is an open simply connected region which extends to $\infty$ toward the right in $S$ as shown in Fig. 15. There is a natural parametrization $\gamma_1 : \mathbb{R} \to B_1$ defined by

$$E_\lambda(\gamma_1(t)) = t + i\pi.$$ 

As a consequence,

$$\lim_{t \to \pm \infty} \text{Re } \gamma_1(t) = \infty.$$
If \( c > 0 \) is large, the segment \( \text{Re} \, z = c \) in \( S \) meets \( S - L_1 \) in two vertical segments \( v_+ \) and \( v_- \) with \( \text{Im} \, v_- > \text{Im} \, v_+ \). \( E_3 \) maps \( v_- \) to an arc of a circle in \( S \cap \{ z \mid \text{Re} \, z < 0 \} \) while \( E_3 \) maps \( v_+ \) to an arc of a circle in \( S \cap \{ z \mid \text{Re} \, z > 0 \} \). As a consequence, if \( c \) is large, \( v_+ \) meets \( L_2 \) in an open interval. Since \( L_2 = E_3^{-1}(L_1) \), it follows that \( L_2 \) is an open simply connected subset of \( S \) that extends to \( \infty \) in the right half plane below \( L_1 \).

Continuing inductively, we see that \( L_n \) is an open, simply connected subset of \( S \) that extends to \( \infty \) toward the right in \( S \). We may also parametrize the boundary \( B_n \) of \( L_n \) by \( \gamma_n : \mathbb{R} \to B_n \) where

\[
E_3^n(\gamma_n(t)) = t + i\pi
\]
as before. Again

\[
\lim_{t \to \pm \infty} \text{Re} \, \gamma_n(t) = \infty.
\]

Since each \( L_n \) is open, it follows that \( \Lambda \) is a closed subset of \( S \).

**Proposition 4.4.** Let \( J_n = \bigcup_{i=n}^{\infty} B_i \). Then \( J_n \) is dense in \( \Lambda \) for each \( n > 0 \).

**Proof.** Let \( z \in \Lambda \) and suppose \( z \not\in B_i \) for any \( i \). Let \( U \) be an open connected neighborhood of \( z \). Fix \( n > 0 \). Since \( E_3^i(z) \in S \) for all \( i \), we may choose a connected neighborhood \( V \subset U \) of \( z \) such that \( E_3^i(V) \subset S \) for \( i = 0, \ldots, n \).

Now the family of functions \( \{ E_3^i \} \) is not normal on \( V \), since \( z \) belongs to the Julia set of \( E_3 \). Consequently, \( \bigcup_{i=0}^{\infty} E_3^i(V) \) covers \( \mathbb{C} - [0] \). In particular, there is \( m > n \) such that \( E_3^m(V) \) meets the exterior of \( S \). Since \( E_3^m(z) \in S \), it follows that \( E_3^m(V) \) meets the boundary of \( S \). Applying \( E_3^{-m} \), we see that \( B_m \) meets \( V \).

In fact, it follows that for any \( z \in \Lambda \) and any neighborhood \( U \) of \( z \), all but finitely many of the \( B_m \) meet \( V \). This follows from the fact that \( E_3 \) has fixed points outside of \( S \) (in fact one such point in each horizontal strip of width \( 2\pi \)—see [17]), so we may assume that \( E_3^m(V) \) contains this fixed point for all sufficiently large \( m \). In particular, we have shown:

**Proposition 4.5.** Let \( z \in \Lambda \) and suppose that \( V \) is any connected neighborhood of \( z \). Then \( E_3^m(V) \) meets the boundary of \( S \) for all sufficiently large \( m \).

**Proposition 4.6.** \( \Lambda \) is a connected subset of \( S \).
Proof. Let $G$ be the union of the boundaries of the $L_i$ for all $i$. Since $A$ is the closure of $G$, it suffices to show that $G$ is connected. Suppose that this is not true. Then we can write $G$ as the union of two disjoint sets $A$ and $B$. One of $A$ or $B$ must contain infinitely many of the boundaries of the $L_i$. Say $A$ does. But then, if $b \in B$, the previous proposition guarantees that infinitely many of these boundaries meet any neighborhood of $b$. Hence $b$ belongs to the closure of $A$. This contradiction establishes the result.

We can now prove:

Theorem 4.7. There is a natural compactification $\Gamma$ of $\Lambda$ that makes $\Gamma$ into an indecomposable continuum.

Proof. We first compactify $\Lambda$ by adjoining the backward orbit of 0. To do this we identify the “points” $(-\infty, 0)$ and $(-\infty, \pi)$ in $S$: this gives $E^{-1}_0(0)$. We then identify the points $(\infty, \pi)$ and $\lim_{t \to -\infty} \gamma(t)$. This gives $E^{-2}_0(0)$. For each $n > 1$ we identify $\lim_{t \to \infty} \gamma_n(t)$ and $\lim_{t \to -\infty} \gamma_{n+1}(t)$ to yield $E^{-n-1}_0(0)$. This augmented space $\Gamma$ may easily be embedded in the plane. See Fig. 16. Moreover, if we extend the $B_i$ and the lines $y = 0$ and $y = \pi$ in the natural way to include these new points, then this yields a curve which accumulates everywhere on itself but does not separate the plane. See the proposition above. By a theorem of Curry [11], it follows that $\Gamma$ is indecomposable. \qed

As a consequence of this theorem, $\Lambda$ must contain uncountably many composants (see [22, p. 213]). In fact, in [17] it is shown that $\Lambda$ contains uncountably many curves.

4.3. Dynamics on $\Lambda$

In this section we describe completely the dynamics of $E_\lambda$ on $\Lambda$.

Proposition 4.8. There exists a unique fixed point $w_\lambda$ in $S$ if $\lambda > 1/e$. Moreover, $w_\lambda$ is repelling and, if $z \in S$ orbit of 0, $E^{-n}_\lambda(z) \to w_\lambda$ as $n \to \infty$. 
Proof. First consider the equation
\[ \lambda e^{y \cot y} \sin y = y. \]
Since \( y \cot y \to 1 \) as \( y \to 0 \) and \( \lambda e > 1 \), we have \( \lambda e^{y \cot y} \sin y > y \) for \( y \) small and positive. Since the left-hand side of this equation vanishes when \( y = \pi \), it follows that this equation has at least one solution \( y_\lambda \) in the interval \( 0 < y < \pi \).

Let \( x_\lambda = y_\lambda \cot y_\lambda \). Then one may easily check that \( w_\lambda = x_\lambda + iy_\lambda \) is a fixed point for \( E_\lambda \) in the interior of \( S \). Since the interior of \( S \) is conformally equivalent to a disk and \( E_\lambda^{-1} \) is holomorphic, it follows from the Schwarz Lemma that \( w_\lambda \) is an attracting fixed point for the restriction of \( E_\lambda^{-1} \) to \( S \) and that \( E_\lambda^{-n}(z) \to w_\lambda \) for all \( z \in S \).

Remarks.
(1) Thus the \( \omega \)-limit set of any point in \( A \) is \( w_\lambda \).

(2) The bound \( \lambda > 1/e \) is necessary for this result, since we know that \( E_\lambda \) has two fixed points on the real axis for any positive \( \lambda < 1/e \). These fixed points coalesce at \( 1 \) as \( \lambda \to 1/e \) and then separate into a pair of conjugate fixed points, one of which lies in \( S \).

We now describe the \( \omega \)-limit set of any point in \( A \). Clearly, if \( z \in B_n \) then \( E^{n+1}_\lambda(z) \in \mathbb{R} \) and so the \( \omega \)-limit set of \( z \) is infinity. Thus we need only consider points in \( A \) that do not lie in \( B_n \). We will show:

**Theorem 4.9.** Suppose \( z \in A \) and \( z \neq w_\lambda, \ z \notin B_n \) for any \( n \). Then the \( \omega \)-limit set of \( z \) is the orbit of \( 0 \) under \( E_\lambda \) together with the point at infinity.

To prove this we first need a lemma.

**Lemma 4.10.** Suppose \( z \in A, \ z \neq w_\lambda \). Then \( E^n_\lambda(z) \) approaches the boundary of \( S \) as \( n \to \infty \).

**Proof.** Let \( h \) be the uniformization of the interior of \( S \) taking \( S \) to the open unit disk and \( w_\lambda \) to \( 0 \). Recall that \( E_\lambda^{-1} \) is well defined on \( S \) and takes \( S \) inside itself. Then \( g = h \circ E_\lambda^{-1} \circ h^{-1} \) is an analytic map of the open disk strictly inside itself with a fixed point at \( 0 \). This fixed point is therefore attracting by the Schwarz Lemma. Moreover, if \( |z| > 0 \) we have \( |g(z)| < |z| \). As a consequence, if \( \{z_n\} \) is an orbit in \( A \), we have \( |h(z_{n+1})| > |h(z_n)| \), and so \( |h(z_n)| \to 1 \) as \( n \to \infty \). \( \square \)

The remainder of the proof is essentially contained in [17] (see pp. 45–49). In that paper it is shown that there is a “quadrilateral” \( Q \) containing a neighborhood of \( 0 \) in \( \mathbb{R} \) as depicted in Fig. 17. The set \( Q \) has the following properties:

(1) If \( z \in A - \bigcup_n B_n \) and \( z \neq w_\lambda \), then the forward orbit of \( z \) meets \( Q \) infinitely often.

(2) \( Q \) contains infinitely many closed “rectangles” \( R_k, R_{k+1}, R_{k+2}, \ldots \) for some \( k > 1 \) having the property that if \( z \in R_i \), then \( E\lambda^i(z) \in Q \) but \( E\lambda^j(z) \notin Q \) for \( 0 < i < j \).

(3) If \( z \in Q \) but \( z \notin \bigcup_{j=k}^{\infty} R_j \), then \( z \in L_n \) for some \( n \).
(4) $E^1_j(R_j)$ is a “horseshoe” shaped region lying below $R_j$ in $Q$ as depicted in Fig. 17.

(5) $\lim_{j \to \infty} E^1_j(R_j) = \{0\}$.

As a consequence of these facts, any point in $A$ has orbit that meets the $\bigcup R_j$ infinitely often. We may thus define a return map

$$\Phi : A \cap \left( \bigcup_{j} R_j \right) \to A \cap \bigcup_{j} R_j$$

by

$$\Phi(z) = E^1_j(z)$$

if $z \in R_j$. By item (4), $\Phi(z)$ lies in some $R_k$ with $k > j$. By item (5), it follows that

$$\Phi^n(z) \to 0$$

for any $z \in A \cap Q$. Consequently, the $\omega$-limit set of $z$ contains the orbit of 0 and infinity.

For the opposite containment, suppose that the forward orbit of $z$ accumulates on a point $q$. By the lemma, $q$ lies in the boundary of $S$. Now the orbit of $z$ must also accumulate on the preimages of $q$. If $q$ does not lie on the orbit of 0, then these preimages form an infinite set, and some points in this set lie on the boundaries of the $L_n$. But these points lie in the interior of $S$, and this contradicts the lemma. Thus the orbit of $z$ can only accumulate in the finite plane on points on the orbit of 0. Since the “preimage” of 0 is infinity, the orbit also accumulates at infinity. □

It is known [7] that there are uncountably many curves in the $\lambda$-plane having the property that, if $\lambda$ lies on one of these curves, then $E^\omega_\lambda(0) \to \infty$. Consequently, for each such $\lambda$-value, the Julia set of $E_\lambda$ is again the complex plane. For these $\lambda$-values, a variant of the above construction also yields invariant indecomposable continua in the Julia set. Whether these continua are homeomorphic to any of those constructed above is an open question. We plan to discuss these constructions in a later paper.

Douady and Goldberg [18] have shown that if $\lambda, \mu > 1/e$, then $E_\lambda$ and $E_\mu$ are not topologically conjugate. Each such map possesses invariant indecomposable continua $A_\lambda$ and $A_\mu$ in $S$, and the dynamics on each are similar, as shown above. In fact, one can show that each pair of these invariant sets is non-homeomorphic.

As a final remark, Lyubich has shown that each $A_\lambda$ is a set of measure 0 in $S$. Indeed, it follows from his work [23] that the set of points in $\mathbb{C}$ whose orbits have arguments that are
equidistributed on the unit circle have full measure. In \( A_\lambda \), the arguments of all orbits tend to 0 and/or \( \pi \), and so \( A_\lambda \) has measure 0 in \( S \).

5. After the explosion

As we have mentioned, when \( \lambda > 1/e \), the Julia set of \( E_\lambda \) is the entire plane. In 1981, Misiurewicz showed that \( J(E_1) = \mathbb{C} \), answering a sixty-year-old question of Fatou. We present his proof of this fact below, generalizing it to the case \( \lambda > 1/e \).

The following proposition highlights one of the differences between \( E_\lambda(z) \) and polynomials: points which tend to 1 under iteration of \( E_\lambda \) need not be in the stable set.

**Proposition 5.1.** When \( \lambda > 1/e \), the real line is contained in \( J(E_\lambda) \) and hence all preimages of the real line lie in \( J(E_\lambda) \).

**Proof.** Let \( S \) denote the strip \( |\text{Im}(z)| \leq \pi \). Suppose \( E_\lambda^j(z) \in \mathbb{R} \). Hence \( E_\lambda^j(z) \to \infty \). Let \( U \) be a neighborhood of \( z \). Then \( E_\lambda^j(U) \) meets the real line for all \( i \geq j \). For sufficiently large \( n \), we have that \( \text{Im}(E_\lambda^n(z)) > 2 \). Let \( B_\delta \) be a ball of radius \( \delta \) about \( E_\lambda^n(z) \) that is strictly contained in \( E_\lambda^n(U) \). Then \( E_\lambda^k(B_\delta) \) contains a ball of radius \( 2^k \delta \) about \( E_\lambda^{n+k}(z) \), provided the successive images of this ball lies in \( S \) (so that \( E_\lambda \) is one-to-one). This cannot happen for all \( k \), however, since eventually these balls must grow to meet the lines \( \text{Im} z = \pm \pi \). Therefore the next iteration of \( E_\lambda \) maps points in this ball to the far left half plane. This happens for all sufficiently large \( k \). Thus there are points arbitrarily close to \( E_\lambda^n(z) \) whose images eventually lie in the unit disk about 0. Thus the family of iterates \( \{E_\lambda^n\} \) does not converge uniformly to \( 1 \) on \( U \). Hence the family of iterates of \( E_\lambda \) is not a normal family on \( U \) and so \( z \in J(E_\lambda) \). \( \square \)

Thus to show that \( J(E_\lambda) = \mathbb{C} \), it suffices to show that inverse images of the real line are dense in \( \mathbb{C} \). For this, we need several lemmas.

**Lemma 5.2.** \( |\text{Im}(E_\lambda^n(z))| \leq |(E_\lambda^n)'(z)| \).

**Proof.** If \( z = x + iy \), we have

\[
|\text{Im}(E_\lambda(z))| = \lambda e^x |\sin y| \leq \lambda e^x |y| = |E_\lambda'(z)| |\text{Im}(z)|
\]

so that

\[
\frac{|\text{Im}(E_\lambda(z))|}{|\text{Im}(z)|} \leq |E_\lambda'(z)|
\]

if \( z \notin \mathbb{R} \). More generally, if \( E_\lambda^n(z) \notin \mathbb{R} \), we may apply this inequality repeatedly to find

\[
\frac{|\text{Im}(E_\lambda^n(z))|}{|\text{Im}(E_\lambda(z))|} = \prod_{i=1}^{n-1} \frac{|\text{Im}(E_\lambda(E_\lambda^i(z)))|}{|\text{Im}(E_\lambda^i(z))|} \leq \prod_{i=1}^{n-1} \left| E_\lambda'(E_\lambda^i(z)) \right|.
\]
Since $|\text{Im}(E_n(z))| \leq |E_n(z)| = |E'_n(z)|$ we may write

$$|\text{Im}(E^n_n(z))| \leq \prod_{i=0}^{n-1} |E_n(E'_n(z))| = |(E^n_n)'(z)|.$$

Now let $W$ denote the strip $\{z \mid |\text{Im}z| \leq \pi/3\}$ which is contained in $S$. Since orbits in the right half plane tend to diverge from the real axis, it follows that an open set of points in $W \subset S$ have orbits that leave $S$ and hence $W$ under iteration. The next lemma shows, however, that the orbits of most of these points must eventually return.

**Lemma 5.3.** Let $U$ be an open connected set. Then only finitely many of the $E^n(U)$ can be disjoint from $W$.

**Proof.** Let us assume that infinitely many of the images of $U$ are disjoint from $W$. If there is an $n$ for which $E^n(U)$ is not a homeomorphism taking $U$ onto its image, then there exist $z_1, z_2 \in U$, $z_1 \neq z_2$, for which $E^n_n(z_1) = E^n_n(z_2)$. Consequently, there is a $j$ for which $E_j(z_1) = E_j(z_2) + 2k\pi i$ for some $k \in \mathbb{Z} - \{0\}$. But then $E^n_j(U)$ must meet a horizontal line of the form $y = 2m\pi$ for $m \in \mathbb{Z}$ and so $E^{j+1}_n(U)$ meets $\mathbb{R}$. Hence $E^{j+\alpha}_n(U)$ meets $\mathbb{R}$ for all $\alpha > 0$ and only finitely many of the images of $U$ can be disjoint from $W$. We thus conclude that each $E^n_n$ must be a homeomorphism on $U$.

Now suppose there is a sequence $n_j$ such that for each $j$, $E^{n_j}_n(U) \cap W = \emptyset$. By the previous lemma, $$|(E^{n_j}_n)'(z)| \geq (\pi/3)^j$$ for each $j$ and all $z \in U$. It follows that, if $U$ contains a disk of radius $\delta > 0$, then $E^{\alpha}_{n_j}(U)$ contains a disk of radius $\delta(\pi/3)^j$. Hence for $j$ large enough, $E^{\alpha}_{n_j}(U)$ must meet a line of the form $y = 2\pi$ and again we are done.

We can now prove

**Theorem 5.4.** $J(E_n) = \mathbb{C}$.

**Proof.** By Proposition 5.1, it suffices to show that any open set in $\mathbb{C}$ contains some preimage of $\mathbb{R}$. To that end, let $U$ be open and connected and suppose $E^n_n(U) \cap \mathbb{R} = \emptyset$ for each $n$. By Montel’s Theorem, $\{E^n_n\}$ is a normal family on $U$.

By the previous lemma, we have that at most finitely many iterates of $U$ are disjoint from $W$. Since none of the iterates of $U$ meet the boundary of $S$, it follows that all but finitely many of the iterates of $U$ lie in $S$. By replacing $U$ by $E^n_n(U)$, we may assume that all of the iterates of $U$ lie in $S$.

Now we invoke the results of the previous section. The $\omega$-limit set of any point in $U$ must be the orbit of $0$ and $\infty$. Hence the orbit of $U$ must enter any small neighborhood of $0$ infinitely often. But we saw above that, after entering this neighborhood, subsequent iterates of $U$ move along the real axis until suddenly jumping above the exit set $L_1$. But this image lies outside the strip $W$. Since this happens infinitely often, we have infinitely many images of $U$ that do not meet $W$. This contradiction establishes the theorem.
6. Hair transplants

In this section we will consider \( \lambda \)-values for which \( \lambda \) is negative. We will see that there is a unique bifurcation that occurs at \( \lambda = -e \). On the real line this is a simple period doubling bifurcation. But in the plane this simple bifurcation has global ramifications, though they are not as spectacular as occurred in the saddle-node bifurcation.

When \( \lambda = -e \), \( E_\lambda \) has a fixed point at \(-1\) and \( E'_\lambda(-1) = -1 \). For \( \lambda < -e \), \( E_\lambda \) has a single repelling fixed point \( r_2 \) on the real axis; when \(-e < \lambda < 0\) this fixed point is attracting.

**Proposition 6.1.** Suppose \( \lambda < -e \). Then \( E_\lambda \) has a unique attracting 2-cycle on the negative real axis. Moreover, the orbit of any point on the real axis (except \( r_2 \)) tends to this 2-cycle.

**Proof.** Consider the graph of
\[
y(x) = E^2_\lambda(x) = \lambda e^{\lambda e^x}.
\]
We have \( y'(x) > 0 \) and
\[
\lim_{x \to -\infty} y(x) = 0, \quad \lim_{x \to \infty} y(x) = \lambda.
\]
Moreover
\[
y''(x) = y(x)(\lambda e^x)(\lambda e^x + 1).
\]
Hence \( y''(x) < 0 \) if \( \lambda e^x + 1 < 0 \) whereas \( y''(x) > 0 \) if \( \lambda e^x + 1 > 0 \). Thus there is a unique inflection point for \( y \).

Now \( E_\lambda(-1) = \lambda e^{-1} < -1 \) since \( \lambda < -e \). Hence \( r_2 < -1 \) since the graph of \( E_\lambda \) decreases. Therefore
\[
y''(r_2) < 0
\]
and so \( y'' < 0 \) for \( x > r_2 \). It follows that the graph of \( y \) crosses the diagonal exactly once in each of the intervals \(( -\infty, r_2 )\) and \(( r_2, \infty )\). This yields the 2-cycle, which must be attracting since \( y' \) lies between 0 and 1 at each point. The graph of \( y \) shows that the intervals \(( r_2, \infty )\) and \(( -\infty, r_2 )\) are mapped inside themselves by \( E^2_\lambda \), and so all orbits except \( r_2 \) tend to the 2-cycle.

Thus we have a typical period-doubling bifurcation when \( \lambda = -e \). In the complex plane, this bifurcation is accompanied by a “hair transplant”. By this we mean: When \( \lambda > -e \) the Julia set is a Cantor bouquet. As \( \lambda \) approaches \(-e \), a repelling 2-cycle approaches the attracting fixed point, dragging with it the attached hairs. See Fig. 18. When \( \lambda = -e \), the fixed point becomes neutral and it now has a pair of hairs attached. When \( \lambda < -e \), the fixed point becomes repelling, but retains the two hairs—they have been transplanted from the 3-cycle to the fixed point.

For the remainder of this section we will consider only the case \( \lambda = -e \). We will see how two hairs can be attached to a single point. We write \( E(z) = E_{-e}(z) \) for simplicity.

Let \( H \) be the half plane \( \text{Re} \ z \leq -1 \). Then \( E_\lambda(H) \) is the disk \( 0 < |z| \leq 1 \).
Proposition 6.2. $E^2(H) \subset H$.

Proof. Consider the vertical line $\text{Re} z = -1 + it$ for $-\pi \leq t \leq \pi$. We claim that $\text{Re}(E^2(1 + it)) < -1$ if $t \neq 0$. Indeed, we have

$$y(t) = \text{Re} E^2(1 + it) = -e^{-\cos t} \cos(t).$$

Then

$$y'(t) = -e^{-\cos t} \left( \sin(t - \sin t) \right).$$

If $0 < t < \pi$, $\sin(t - \sin t) > 0$ and so $y'(t) < 0$. If $-\pi < t < 0$, $\sin(t - \sin t) < 0$ and so $y'(t) > 0$. Therefore $y(t)$ decreases from $y(0) = -1$ to $y(\pi) = -e^2$ as $t$ increases from 0 to $\pi$, and, similarly, $y(t)$ increases from $-e^2$ to $-1$ as $t$ runs from $-\pi$ to 0.

Since $E^2(H)$ is contained inside the simple closed curve bounded by $E^2(1 + it)$, we have the result. \qed

Since $E^2(H) \subset H$ it follows that all orbits in $H$ and $E(H)$ tend to the neutral fixed point at $-1$. These two regions are called attracting petals.

Consider $E^{-1}(H)$. This set consists of infinitely many “fingers” that surround the horizontal line $\text{Im} z = 2k\pi$, $\text{Re} z \geq -1$. We denote each finger by $F_k$, where $-1 + 2k\pi i$ is the preimage of $-1$ lying in $F_k$. Since $E(F_k) = H$, it follows that all points in $F_k$ have orbits tending to $-1$. Note that $F_0$ contains the disk $E(H)$ and in fact the segment $[-1, \infty)$ in the reals. See Fig. 19.

Now consider $E^{-1}(F_0)$. Certainly $E^{-1}(F_0) \supset H$ since $H$ is mapped to $E(H) \subset F_0$. $E^{-1}(F_0)$ also contains the straight lines $\text{Im} z = (2k + 1)\pi$, $\text{Re} z \geq -1$, since these lines are mapped to the half line $(-\infty, -e)$ which is properly contained in $H$. Finally, $E^{-1}(F_0)$ is simply connected and contains all preimages of any $z \in F_0$ (except 0). $E^{-1}(F_0)$ resembles a “glove” since it separates each of the fingers as shown in Fig. 20.
Let $G$ denote the glove $E^{-1}(F_0)$. Then all points in the interior of $G \cup (\bigcup_n F_n)$ lie in the stable set for $E$ as all orbits tend to $-1$. Note that $-1$ and its preimages lie on the boundary of $G \cup (\bigcup_n F_n)$. These points do not lie in the stable set as rationally indifferent fixed points always lie in $J(E)$. Now consider the component of the complement of $G$ that contains $F_0$. $F_0$ divides this region into two open sets, $S_\pm$, $S_+$ lying above $F_0$, $S_-$ below. Using arguments as in Section 3 one can check that, there is a unique hair $y_\pm$ in $J(E)$ that lies in $S_\pm \cap \{\Re z > y\}$ for $y$ sufficiently large. Indeed, $E(y_+) = y_-$ and $E(y_-) = y_+$. The existence of these hairs in the far right half plane may be verified by considering rectangles $R_\pm(t)$ centered at $t \pm i\pi/2$ and having vertical height $\pi$. Note that $E$ maps $R_+ (t)$ to the half plane $\Im z \leq 0$ while $R_-$ is mapped to the upper half plane. Constructing strings of such rectangles give points whose orbits hop back and forth between $S_+$ and $S_-$. 

Fig. 19. The fingers $F_k$.

Fig. 20. The glove $E^{-1}(F_0)$. 


One checks easily that the bounded regions

\[ S_+ \cap \{ \Re z \leq n \} \]

are mapped completely over each other, as \( E(S_+) \supset S_- \) and \( E(S_-) \supset S_+ \). Hence we can pull the hairs \( \gamma_\lambda \) back by a branch of the inverse of \( E \) in each of \( S_+ \) and \( S_- \). The only place that these pulled back hairs can limit is the fixed point at \( -1 \). Hence we see that, when \( \lambda = -e \), the neutral fixed point has two hairs attached.

What happens is this: as long as \( E_\lambda \) has an attracting fixed point, all repelling periodic points have unique hairs attached. But as \( \lambda \) tends to \(-e\) along \( \mathbb{R}^+ \), a repelling 2-cycle merges with the attracting fixed point to produce the neutral fixed point at \(-1\). This bifurcation is a "hair transplant".

Note that there are infinitely many preimages of \(-1\) under \( E^{-1} \), \( E^{-2} \), \ldots. Hence there are infinitely many other points in \( J(E) \) that have 2 hairs attached.

### 6.1. Other examples

For complex \( \lambda \)-values, one often encounters Julia sets with multiple hairs attached.

For example, in Fig. 21, we display the Julia set when \( \lambda = 5 + i\pi \). It is relatively easy to check that this exponential admits an attracting 3-cycle. Note that there seem to be three hairs attached at various points in the plane.

In Fig. 22, we display the Julia set when \( \lambda = 10 + 3\pi i \). This map also has an attracting cycle of period 3. Note that different hairs now seem to be attached. In contrast, the Julia set for \( \lambda = 3.14i \) (Fig. 23) shows that the structure of the attached hairs can be extremely complicated.

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Fig. 21. The Julia set for \( \lambda = 5 + \pi i \).
One can use a “kneading invariant” to characterize which hairs are attached in case $E_\lambda$ admits an attracting cycle. See [7]. The full story here, however, is not yet complete.

References