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A geometric characterization of automatic semigroups

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Abstract

In the study of automatic groups, the geometrical characterization of automaticity (in terms of the "fellow traveller property") plays a fundamental role. When we move to the study of automatic semigroups, we no longer have this simple formulation. The purpose of this paper is to give a general geometric characterization of automaticity in semigroups.

Keywords: Automatic semigroup; Regular language; Cayley graph; Fellow traveller property; Continuation graph

1. Introduction

The notion of an automatic group (see [1,7]) has played an important role in the study of finitely generated groups. It has been pointed out (see [15,17] for example) that the definition of automaticity generalizes naturally from groups to semigroups and a systematic exploration of the basic properties of automatic semigroups was undertaken in [3]. Since then the theory has developed considerably and much is now known about automaticity in semigroups. However, there appears to be one fundamental difference between the theory of automatic groups and the situation when we generalize to semigroups.

In the study of automatic groups, the geometrical characterization of automaticity (in terms of the "fellow traveller property") plays a fundamental role. When we move to the study of automatic semigroups, we no longer have this simple and elegant formulation in general (although we do have it in some special cases; see [4] for example). This has been a significant obstacle to the development of the theory of automatic semigroups. The issue here is that not all properties generalize to the semigroup case and, where properties do generalize, the necessity for new methods of proof has arisen.

The purpose of this paper is to generalize the fellow traveller property from groups to semigroups. The main result is the following (see Theorem 4.8):

Theorem. *Let S be a semigroup, A be a finite generating set for S and L be a regular language over A that maps onto S. Then S has a finite number of continuation graphs with respect to L if and only if (A, L) is an automatic structure for S.*

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The concept of a "finite number of continuation graphs" is explained below in Definitions 4.4 and 4.7.

Given that the fellow traveller property is thought of as being a geometric property of automatic groups (describing, as it does, the structure of the Cayley graph of an automatic group), we think of this extension as being a geometric characterization of automatic semigroups.

We should mention that an interesting alternative approach has been taken by Silva and Steinberg in [19] (see also [18]). They introduce the notion of "prefix-automatic" monoids; this assumes a potentially stronger definition of the notion of automaticity. It is an open question as to whether being automatic and being prefix-automatic are equivalent for arbitrary semigroups, although this is known to be the case if one only considers right-cancellative semigroups (see [10,19]). If one assumes, in addition, that the monoid M is of "finite geometric type" (i.e. if, for every $p \in M$, there exists $k \in \mathbb{N}$ such that the equation $xp = q$ has at most k solutions for every $q \in M$), then they obtain an elegant geometric characterization in this particular case. It is not immediately obvious how this description relates to the characterization given in this paper. The notion of finite geometric type is equivalent to that of "bounded indegree" (see Definition 2.11).

2. Preliminaries

In this section, we set up the notation and definitions and also recall some basic results we shall need in this paper. For general background on semigroups see [14,16] and for formal languages see [9,13].

For any set S, let $\wp(S)$ denote the set of all subsets of S. For any finite set A, let A^+ denote the set of all non-empty words over A, and let A^* denote the set of all words over A (including the empty word ε). For any word α in A^*, we let $|\alpha|$ denote the length of α (where $|\varepsilon| = 0$). If $a \in A$, we let $|\alpha|_a$ denote the number of occurrences of a in the word α. For any $k \in \mathbb{N}$, we let $A^{\leqslant k}$ denote the set of all words α in A^* with $|\alpha| \leqslant k$.

If S is a semigroup and $A \subseteq S$ is a set of generators of S, then there is a natural homomorphism $\theta : A^+ \to S$ where each word α in A^+ is mapped to the corresponding element of S; we write $S = \langle A \rangle$ if S is generated by A. We will normally be concerned with finite sets A, so that the semigroup S is finitely generated. Where there is no danger of confusion, we will sometimes suppress the reference to θ, simply writing α for the element of the semigroup represented by α. In this context, if α and β are elements of A^+, we will write $\alpha \equiv \beta$ if α and β are identical as words, and $\alpha = \beta$ if α and β represent the same element of S (i.e., if $\alpha\theta = \beta\theta$). If we wish to stress which semigroup we are working in, we will write $\alpha =_S \beta$ if α and β represent the same element of the semigroup S. We may also then write $\alpha = s$, where $\alpha \in A^+$ and $s \in S$, which says that $\alpha\theta = s$ in S. Given this, if $v \in A$ and $\alpha \in A^*$, then αv makes sense as a (uniquely defined) element of S; in particular, if $\alpha \equiv \varepsilon$ we have that $\alpha v = v$. For $\alpha \equiv a_1 a_2 \ldots a_n \in A^*$ and $t > 0$ we define

$$\alpha(t) \equiv \begin{cases} a_1 a_2 \ldots a_t & \text{if } t \leqslant n, \\ a_1 a_2 \ldots a_n & \text{if } t > n. \end{cases}$$

We will use the notation S^1 for the semigroup created by adjoining an isolated identity element to S.

We will use the concept of a "string rewriting system" and we recall the basic definitions here. Given a finite alphabet Σ, a *string rewriting system* $\mathcal{R}$ over Σ is a set of rules of the form $\alpha \to \beta$, where $\alpha, \beta \in \Sigma^+$ (it is more normal to allow rules $\alpha \to \beta$ with $\alpha \in \Sigma^+$ and $\beta \in \Sigma^*$, but, as we are interested in semigroups as opposed to monoids, this is more natural for our purposes); we shall only be interested in finite rewriting systems in this paper.

We define the *reduction relation* $\Longrightarrow_{\mathcal{R}}^*$ to be the reflexive transitive closure of $\Longrightarrow_{\mathcal{R}}$, where $\gamma\alpha\eta \Longrightarrow_{\mathcal{R}} \gamma\beta\eta$ if $\gamma, \eta \in \Sigma^*$ and $(\alpha \to \beta) \in \mathcal{R}$. The equivalence relation generated by $\Longrightarrow_{\mathcal{R}}$ will be written as $\Longleftrightarrow_{\mathcal{R}}^*$, and is called the *Thue congruence* of $\mathcal{R}$. We say that a rewriting system $\mathcal{R}$ over Σ *presents* the semigroup

$$\langle \Sigma : \{\alpha = \beta : (\alpha \to \beta) \in \mathcal{R}\} \rangle,$$

which is isomorphic to the semigroup $\Sigma^+ / \Longleftrightarrow_{\mathcal{R}}^*$.

A rewriting system $\mathcal{R}$ is said to be *confluent* if, whenever $\alpha, \beta_1, \beta_2 \in \Sigma^*$ with $\alpha \Longrightarrow_{\mathcal{R}}^* \beta_1$ and $\alpha \Longrightarrow_{\mathcal{R}}^* \beta_2$, then there exists $\gamma \in \Sigma^*$ such that $\beta_1 \Longrightarrow_{\mathcal{R}}^* \gamma$ and $\beta_2 \Longrightarrow_{\mathcal{R}}^* \gamma$. A rewriting system $\mathcal{R}$ is said to be *Noetherian* if there is no infinite chain

$$\alpha_1 \Longrightarrow_{\mathcal{R}} \alpha_2 \Longrightarrow_{\mathcal{R}} \alpha_3 \Longrightarrow_{\mathcal{R}} \cdots .$$

A rewriting system which is both confluent and Noetherian is said to be *complete*. If we have a complete rewriting system $\mathcal{R}$, then, for each $\alpha \in \Sigma^*$, there is a unique irreducible $\beta \in \Sigma^*$ with $\alpha \Longrightarrow_{\mathcal{R}}^* \beta$; we call β the *normal form* of α. For a general account of string rewriting systems, see [2].

When studying automatic groups and semigroups, we consider finite automata and the languages they accept, i.e. the regular languages. One often considers deterministic automata $M = (Q, A, \tau, s, E)$, where Q is the set of states of the machine, A is the input alphabet, $\tau : Q \times A \to Q$ is the transition partial function, s is the start state and E is the set of accept states; this implies that, for every $\alpha \in A^*$, there exists at most one state $q \in Q$ such that $\tau(s, \alpha) = q$. If the automaton M is non-deterministic, then $\tau(s, \alpha)$ may consist of a set of states as opposed to a single state. In addition we can assume, without loss of generality, that M has no dead states, which means that, for every state $q \in Q$, there exists an element $\alpha \in A^*$ with $\tau(q, \alpha) \in E$ (or that $\tau(q, \alpha) \cap E \neq \emptyset$ in the non-deterministic case).

As in the case of automatic groups, we will want to consider automata accepting pairs (α, β) of words with $\alpha, \beta \in A^+$. If $\alpha \equiv a_1 a_2 \ldots a_n$ and $\beta \equiv b_1 b_2 \ldots b_m$, this is accomplished by having an automaton with input alphabet $A \times A$ and reading pairs (a_1, b_1), (a_2, b_2), and so on. To deal with the case where $n \neq m$, we introduce a *padding symbol* \$. More formally, as with automatic groups, we define a mapping $\delta_A : A^* \times A^* \to A(2, \$)^*$, where $\$ \notin A$ and $A(2, \$) = (A \cup \{\$\}) \times (A \cup \{\$\})$, by

$$(\alpha, \beta)\delta_A = \begin{cases} (a_1, b_1) \ldots (a_n, b_n) & \text{if } n = m, \\ (a_1, b_1) \ldots (a_n, b_n)(\$, b_{n+1}) \ldots (\$, b_m) & \text{if } n < m, \\ (a_1, b_1) \ldots (a_m, b_m)(a_{m+1}, \$) \ldots (a_n, \$) & \text{if } n > m. \end{cases}$$

If S is a semigroup generated by a finite set A, $L \subseteq A^+$ and $a \in A \cup \{\varepsilon\}$, then we define

$$L_a^\$ = \{(\alpha, \beta)\delta_A : \alpha, \beta \in L, \alpha a = \beta\}.$$

We sometimes omit the reference to the padding symbol \$ and simply write L_a when the context is clear. Given all this, we have

Definition 2.1. If S is a semigroup, A is a finite set, L is a regular subset of A^+ and $\phi : A^+ \to S$ is a homomorphism with $L\phi = S$, we say that (A, L) is an *automatic structure* for S if $L_a^\$$ is regular in $A(2, \$)^*$ for each $a \in A \cup \{\varepsilon\}$. If, in addition, L maps bijectively to S we say that (A, L) is an *automatic structure with uniqueness*.

If a semigroup S has an automatic structure (A, L) for some A and L, then we say that S is *automatic*.

Remark 2.2. It is known (see [3]) that, if a semigroup S has an automatic structure (A, L), then there is a subset K of L such that (A, K) is an automatic structure for S with uniqueness.

One should note in passing that there is some natural choice in the conventions as to which side one performs the multiplication and where one takes the paddings (see [12]), but this will not concern us here.

We will need the following basic facts about regular languages:

Proposition 2.3. *If $K \subseteq A^*$ and $L \subseteq A^*$ are regular, then $K \cup L$, $K \cap L$, $K - L$, KL and K^* are regular.*

Proposition 2.4. *If A and B are finite sets, $\phi : A^* \to B^*$ is a homomorphism, and L is a regular subset of A^*, then $L\phi$ is a regular subset of B^*.*

Proposition 2.5. *If $K, L \subseteq A^*$ are regular, then $(K \times L)\delta_A$ is regular.*

Proposition 2.6. *If L is a regular language over the set A and if $U_1, \ldots, U_n$ are regular languages over $A(2, \$)$, then the set*

$$\{(\alpha, \beta)\delta_A : \alpha, \beta \in L \text{ and there exist } \omega_1, \ldots, \omega_{n-1} \in L \text{ such that } (\alpha, \omega_1)\delta_A \in U_1,\ (\omega_1, \omega_2)\delta_A \in U_2,\ ,(\omega_{n-2}, \omega_{n-1})\delta_A \in U_{n-1},\ (\omega_{n-1}, \beta)\delta_A \in U_n\}$$

is regular.

Following [8], where the concept was defined for groups, we have the notion of a "rational cross section" of a semigroup (see also [5]):

Definition 2.7. Let S be a finite generated semigroup, let A be a finite generating set for S. Let L be a regular subset of A^+. If L maps bijectively to S, then we call L a *rational cross section* of S.

We recall the following result from [3]:

Proposition 2.8. *Let S be a semigroup and S^1 be the monoid formed by adding an identity element to S; then S is automatic if and only if S^1 is automatic.*

The following result from [6] is very useful:

Theorem 2.9. *If M is a monoid which is automatic and if A is any finite (semigroup) generating set for M, then there is a regular language L over A^+ such that (A, L) is an automatic structure for M.*

Note that Theorem 2.9 is not true for arbitrary semigroups (see [3] for example).

When considering the geometric properties of semigroups, we will look at the following distance functions:

Definition 2.10. Let S be a semigroup and A be a finite generating set of S. The function $d : S \times S \to \mathbb{N}$ defined by

$$d(v, w) = \min\{|\alpha| : \alpha \in A^* \text{ and } (v\alpha = w \text{ or } v = w\alpha)\}$$

is called the *directed distance function* of S with respect to A. We take $d(v, w)$ to be undefined if there is no $\alpha \in A^*$ with $v\alpha = w$ or $w\alpha = v$.

Given this, the function $d' : S \times S \to \mathbb{N}$ defined by

$$d'(v, w) = \min\{n \in \mathbb{N} : n = d(v, s_1) + d(s_1, s_2) + \cdots + d(s_r, w) \qquad \text{for some } s_1, s_2, \ldots, s_r \in S,\ r \geqslant 0\}$$

is called the *undirected distance function* of S with respect to A. We take $d'(v, w)$ to be undefined if no such s_i exist. Note that this cannot happen in a monoid as we could take $r = 1$ and $s_1 = 1$ there to get

$$d'(v, w) \leqslant \min\{|\alpha| + |\beta| : \alpha, \beta \in A^+, \alpha =_S v, \beta =_S w\}.$$

However, it is possible to have d' undefined in a semigroup that is not a monoid.

These are distance functions in the Cayley graph of S. Recall that, if S is a semigroup generated by a finite set A, then the (right) *Cayley graph* Γ of S with respect to A is the directed graph with vertex set S and an edge labelled a from s to sa for every vertex $s \in S$ and every $a \in A$.

We will also use the following concept from [11]:

Definition 2.11. A semigroup S generated by a finite set A is said to have *bounded indegree* if there exists a constant k such that, for all $s \in S$, the set $\{t \in S : s = ta \text{ for some } a \in A\}$ has size at most k.

This property does not depend on which finite generating set we choose for S, in that, if S has bounded indegree with respect to one finite generating set, then it has bounded indegree with respect to any finite generating set (see [11] for example). In terms of the Cayley graph Γ of S with respect to A, we are saying that the number of directed edges coming into each vertex of Γ is bounded by k. Note that, if S has bounded indegree, $v \in S$, $k \geqslant 0$, and d and d' are the distance functions introduced in Definition 2.10, then the sets

$$\{w \in S : d(v, w) \leqslant k\} \text{ and } \{w \in S : d'(v, w) \leqslant k\}$$

are both finite.

3. Fellow traveller property

As we pointed out in the Introduction, the "fellow traveller property" is of critical importance in the study of automatic groups (see [1,7] for example). We can also talk about such a concept in the context of semigroups. One possible version of this is the following:

Definition 3.1. Let S be a semigroup and A be a finite generating set for S. Let L be a regular subset of A^+ and $k \in \mathbb{N}$ be a constant. Let d be the directed distance function of S with respect to A. We say S has the *directed fellow traveller property* with respect to L with constant k if $d(\alpha(t), \beta(t)) \leqslant k$ for all $t > 0$ and for all $\alpha, \beta \in L$ with either $\alpha = \beta$ or $\alpha a = \beta$ for some $a \in A$.

We note the obvious fact that, if L maps bijectively onto S in Definition 3.1, then the assumption there that $d(\alpha(t), \beta(t)) \leqslant k$ for all $t > 0$ and all $\alpha, \beta \in L$ with $\alpha =_S \beta$ is redundant.

In general, even this strong version of the fellow traveller property in semigroups is not sufficient to ensure automaticity. However this version of the concept is a sufficient condition for a semigroup to be automatic in the following circumstances:

Proposition 3.2. *Let S be a semigroup satisfying the following condition*:

(H) *if* $u, v, w \in S$ *with* $uv = uw$ *then* $sv = sw$ *for all* $s \in S$.

Let A be a finite generating set for S and L be a regular subset of A^+ that maps onto S. Suppose that S has the directed fellow traveller property with respect to L; then (A, L) is an automatic structure for S.

Proof. Let $B = A \cup \{\$\}$ and k be the constant of the directed fellow traveller property of S with respect to L. For $a \in B$ we let

$$\overline{a} = \begin{cases} a & \text{for } a \in A, \\ \varepsilon & \text{for } a = \$. \end{cases}$$

We will show that, for all $a \in A \cup \{\varepsilon\}$, the set

$$\begin{aligned} K_a = \{&(a_1, b_1)(a_2, b_2) \ldots (a_n, b_n) : d(\overline{a_1 \ldots a_i}, \overline{b_1 \ldots b_i}) \leqslant k \text{ for all } i \geqslant 0, \\ &a_1, b_1 \in A,\ a_j, b_j \in B \text{ for all } j > 1,\ \overline{a_1 a_2 \ldots a_n} a = \overline{b_1 b_2 \ldots b_n}\} \end{aligned}$$

is regular.

First we define, for any fixed $u \in S$, $b, c \in A$, $\gamma \in A^{\leqslant k}$ and $g \in \{\mathsf{up}, \mathsf{down}\}$, the following finite sets (see Fig. 1):

$$\begin{aligned} T_{b,c,\gamma,g} = \{(\beta, h) :\ &\beta \in A^{\leqslant k}, h \in \{\mathsf{up}, \mathsf{down}\}, \\ &u\gamma\overline{b} = u\overline{c}\beta \text{ for } g = \mathsf{up} \text{ and } h = \mathsf{up} \\ &u\gamma\overline{b}\beta = u\overline{c} \text{ for } g = \mathsf{up} \text{ and } h = \mathsf{down} \\ &u\overline{b} = u\gamma\overline{c}\beta \text{ for } g = \mathsf{down} \text{ and } h = \mathsf{up} \\ &u\overline{b}\beta = u\gamma\overline{c} \text{ for } g = \mathsf{down} \text{ and } h = \mathsf{down}\}. \end{aligned}$$

By our hypothesis (H), the set $T_{b,c,\gamma,g}$ is independent of the choice of u.

We now construct a non-deterministic finite automaton

$$P_a = (Q, B \times B, \tau, s, E)$$

accepting K_a (where $a \in K \cup \{e\}$) as follows.

First let $Q = (A^{\leqslant k} \times \{\mathsf{up}, \mathsf{down}\}) \cup \{s\}$ with $s \notin A^{\leqslant k} \times \{\mathsf{up}, \mathsf{down}\}$. For all $\alpha \in A^{\leqslant k}$, $b, c \in B$ and $g \in \{\mathsf{up}, \mathsf{down}\}$ we have the following transitions:

$$\begin{aligned} \tau(s, (b, c)) &= (\beta, \mathsf{down}) \quad \text{where} b, c \in A \text{ and } \beta \in A^{\leqslant k} \text{ with } b\beta = c, \\ \tau(s, (b, c)) &= (\beta, \mathsf{up}) \quad \text{where} b, c \in A \text{ and } \beta \in A^{\leqslant k} \text{ with } b = c\beta, \\ \tau((\alpha, g), (b, c)) &= (\beta, h) \quad \text{where} (\beta, h) \in T_{b,c,\alpha,g}. \end{aligned}$$

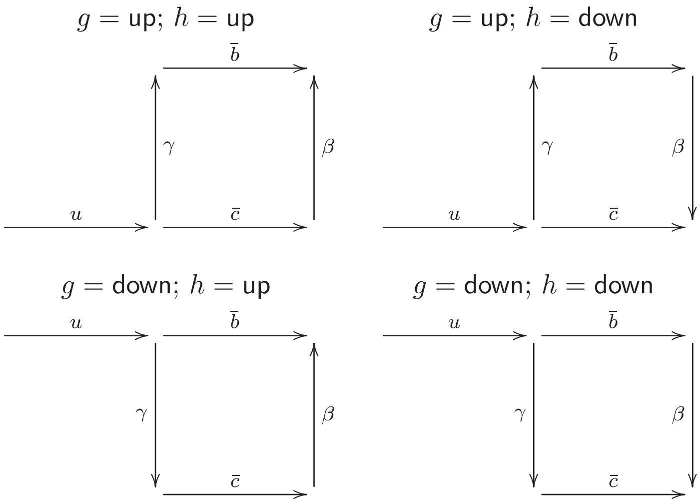

Fig. 1. The definition of $T_{b,c,\gamma,g}$.

Note that, in general, we have that h is uniquely defined as either up or down for any particular β, but it is possible to have both pairs (β, up) and (β, down) in $T_{b,c,\gamma,g}$ (for some b, c, γ and g); in particular, this will happen in the case $\beta = \varepsilon$.

Let $E = \{a\} \times \{\mathsf{down}\}$. In the case where $a = \varepsilon$, we could just as well have chosen $\{\varepsilon\} \times \{\mathsf{up}\}$, but $\{\varepsilon\} \times \{\mathsf{down}\}$ certainly suffices.

It is easy to see, by induction over n, that $(\gamma, g) \in \tau(s, (a_1, b_1) \dots (a_n, b_n))$ if and only if

$$a_1, b_1 \in A,\; d(\overline{a_1 a_2 \dots a_i}, \overline{b_1 b_2 \dots b_i}) \leqslant k \quad \text{for all } 0 < i \leqslant n$$

and either $\overline{a_1 \dots a_n}\gamma = \overline{b_1 \dots b_n}$ and $g = \mathsf{down}$ or else $\overline{a_1 \dots a_n} = \overline{b_1 \dots b_n}\gamma$ and $g = \mathsf{up}$. Hence $L(P_a) = K_a$ and K_a is regular.

By the directed fellow traveller property of S with respect to L we can express L_a as

$$L_a = K_a \cap (L \times L)\delta_A.$$

Since K_a is regular, we have that L_a is regular by Lemma 2.5. $\square$

In general, any left-cancellative semigroup will satisfy hypothesis (H) of Proposition 3.2. In the case of a monoid, left-cancellativity is clearly equivalent to this condition (given (H), if $uv = uw$, then $1v = 1w$, and so $v = w$). On the other hand, the semigroup S defined by the presentation

$$\langle a, b : aa = ab, ba = bb \rangle$$

is not left-cancellative but does satisfy hypothesis (H) (as two words will represent the same element of S if and only if they start with the same letter and have the same length); so the equivalence of hypothesis (H) and left-cancellativity does not hold for arbitrary semigroups.

The next example shows that Proposition 3.2 need not hold for monoids which do not satisfy hypothesis (H) (i.e. which are not left-cancellative). Since the directed fellow traveller property would seem to be the strongest natural generalization of the fellow traveller property from groups to semigroups, it appears that we must seek some further conditions if we are to get a geometric characterization of automaticity in semigroups.

Example 3.3. Let $M = \langle A : \mathfrak{R} \rangle$ with $A = \{e, a, b, x, y, z\}$ and

$$\mathfrak{R} = \{(xa^i z, xb^i) : i \in \mathbb{N}\} \cup \{(xa^i y, xa^i) : i \in \mathbb{N}, i \neq 2^j \text{ for any } j \in \mathbb{N}\}$$
$$\cup\ \{(xa^i y, xb^i) : i = 2^j \text{ for some } j \in \mathbb{N}\} \cup \{(ue, u) : u \in A\} \cup \{(eu, u) : u \in A\}.$$

Note that M is a monoid with identity e but we are taking a semigroup presentation for M here.

Let $B = A - \{e\}$ and let $L = (B^+ - B^*\{x\}\{a\}^*\{y,z\}B^*) \cup \{e\}$. Then L maps bijectively onto M and M has the directed fellow traveller property with respect to L, but M is not automatic.

To prove this, we first consider the string rewriting system $\mathcal{R}$ with rules

$$xa^iz \to xb^i\ (i \in \mathbb{N}),\quad xa^iy \to xa^i\ (i \neq 2^j),\quad xa^iy \to xb^i\ (i = 2^j),\quad ue \to u\ (u \in A),\quad eu \to u\ (u \in A).$$

We may easily check that $\mathcal{R}$ is a complete rewriting system for S and that L is the set of normal forms of $\mathcal{R}$; so L maps bijectively onto S.

Next we show that M has the directed fellow traveller property with respect to L. Let $\alpha, \beta \in L$ with $\alpha u = \beta$ for some $u \in A$; we must have either $\alpha \equiv \beta$ or $\alpha u \equiv \beta$ or ($\alpha \equiv \omega x a^i$ and $\beta \equiv \omega x b^i$). In each case we have that $d(\alpha(t), \beta(t)) \leqslant 1$ for all $t > 0$ (where d is the directed distance function of S with respect to A) as required.

Now we show that M is not automatic. So suppose that M is automatic. By Remark 2.2, Proposition 2.8 and Theorem 2.9 we may assume that there exists an automatic structure (A, K) with uniqueness for M. Let $\phi : A(2,\$)^* \to A^*$ be the homomorphism defined by

$$(u, v)\phi = \begin{cases} a & \text{if } u = a, \\ \varepsilon & \text{for } u \neq a. \end{cases}$$

Since $K_y = \{(\alpha, \beta)\delta_A : \alpha, \beta \in K, \alpha y = \beta\}$ is regular, we have that

$$K_y \cap \left(\{e\}^*\{x\}\{e,a\}^* \times \{e\}^*\{x\}\{e,b\}^*\right)\delta_A$$

is regular by Propositions 2.3 and 2.5. Note that each of the sets

$$\{e\}^*\{x\}\{e,a\}^* \quad \text{and} \quad \{e\}^*\{x\}\{e,b\}^*$$

is closed under applications of rules from $\mathfrak{R}$. Applying Proposition 2.4 we would have that

$$\left(K_y \cap (\{e\}^*\{x\}\{e,a\}^* \times \{e\}^*\{x\}\{e,b\}^*)\delta_A\right)\phi = \{a^i : i = 2^j \text{ for some } j \in \mathbb{N}\}$$

is regular, a contradiction. Therefore M is not automatic.

An interesting question is whether we can have an example as in Example 3.3 that is finitely presented; it does seem likely that this will be the case although it would appear that the construction might be a little involved.

Given that S is automatic if and only if S^1 is automatic (Proposition 2.8), we might expect that we have a similar result with respect to the directed fellow traveller property. As the next result shows, this does hold if we add the assumption that our semigroup S has bounded indegree. We should mention that the proof of Proposition 2.8 gives that, if L is an automatic structure for S, then $L \cup \{e\}$ is an automatic structure for S^1 (where e represents the adjoined identity).

Proposition 3.4. *Let S be a semigroup with bounded indegree and A be a finite generating set for S. Let L be a subset of A^+ and let e represent the identity element of S^1. Then S has the directed fellow traveller property with respect to L if and only if S^1 has the directed fellow traveller property with respect to $L \cup \{e\}$.*

Proof. For simplicity, we let T denote S^1 throughout this proof. Note that, as S has bounded indegree, T also has bounded indegree.

"$\Leftarrow$" Assume that T has the directed fellow traveller property with respect to $L \cup \{e\}$ with constant k. Let $\alpha, \beta \in L$ with $\alpha a = \beta$ for some $a \in A \cup \{\varepsilon\}$. So, for every $t > 0$, there exists $\gamma_t \in (A \cup \{e\})^*$ with

$$\alpha(t)\gamma_t =_T \beta(t) \quad \text{or} \quad \alpha(t) =_T \beta(t)\gamma_t$$

and such that $|\gamma_t| \leqslant k$; note that $\alpha(t) \neq e \neq \beta(t)$. Deleting any instances of e in γ_t gives a word $\overline{\gamma}_t \in A^*$ with

$$\alpha(t)\overline{\gamma}_t =_S \beta(t) \quad \text{or} \quad \alpha(t) =_S \beta(t)\overline{\gamma}_t;$$

note that $|\overline{\gamma}_t| \leqslant |\gamma_t| \leqslant k$. So S has the directed fellow traveller property with respect to L with constant k.

"$\Rightarrow$" Assume that S has the directed fellow traveller property with respect to L with constant k. Let d_S be the directed distance function of S with respect to A and d_T be the directed distance function of T with respect to $A \cup \{e\}$.

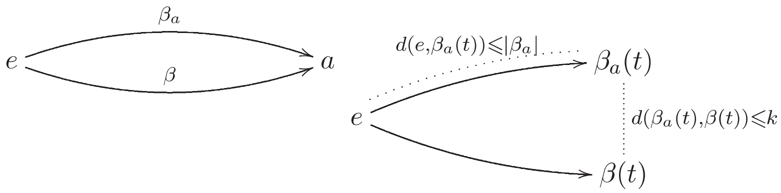

Fig. 2. $d'_T(e, \beta(t)) \leqslant k + |\beta_a|$.

Since every directed path in T not starting or ending at e can be transformed into a directed path in S by deleting any edges labelled by e, it follows that

$$d_S(v, w) = d_T(v, w) \quad \text{for all } v, w \in S.$$

Therefore, for all $t > 0$, $a \in A \cup \{e\}$ and $\alpha, \beta \in L$ with either $\alpha a = \beta$ or $\alpha = \beta$, we have that

$$d_T(\alpha(t), \beta(t)) \leqslant k.$$

We also trivially have that $d_T(e, e) = 0 \leqslant k$. We will show that, for each $a \in A$, there exists a constant k_a such that, for all $t > 0$ and all $\beta \in L$ with $\beta = a$, we have

$$d_T(e, \beta(t)) \leqslant k_a.$$

Let $a \in A$ be fixed; then let $\beta_a \in L$ with $\beta_a = a$ be fixed. So

$$d_T(\beta(t), \beta_a(t)) \leqslant k$$

for all $t > 0$ and all $\beta \in L$ with $\beta = a$. Let d'_T be the undirected distance function of T with respect to $A \cup \{e\}$; then

$$d'_T(e, \beta(t)) \leqslant k + |\beta_a(t)| \leqslant k + |\beta_a|$$

for all $\beta \in L$ with $\beta = a$ (see Fig. 2).

By the bounded indegree of T, the set

$$B_a = \{v \in T : d'_T(e, v) \leqslant k + |\beta_a|\}$$

is finite; so there exists $k_a \in \mathbb{N}$ with $d_T(e, b) \leqslant k_a$ for all $b \in B_a$. (Note that there exists at least one directed path from e to every element of S.) If $\beta \in L$ and $\beta = a$, then $\beta(t)$ lies in B_a for all t, and so $d_T(e, \beta(t)) \leqslant k_a$ for all t.

Let $k' = \max(\{k_a : a \in A\} \cup \{k\})$; if $a \in A \cup \{e\}$ then $d_T(\alpha(t), \beta(t)) \leqslant k'$ for all $t > 0$ and all $\alpha, \beta \in L \cup \{e\}$ with $\alpha = \beta a$ as required. □

If we assume that L maps bijectively onto S (i.e. that L is a rational cross section), then Proposition 3.4 holds for arbitrary semigroups (since, for each $a \in A$, we have a unique word $\beta \in L$ with $\beta = a$, and the argument we had to produce in the "⇒" part of the proof is now superfluous). It seems to be a reasonable question as to whether Proposition 3.4 holds for arbitrary semigroups if we only assume that L maps surjectively (as opposed to bijectively) onto S; as far as we are aware, this is still open.

Suppose that S is a cancellative semigroup with generating set A and then that L is a regular subset of A^+ that maps onto S. Given that S having the directed fellow traveller property with respect to L implies automaticity by Proposition 3.2, one might ask whether we could weaken this to only assuming the undirected fellow traveller property. We know, from [3], that this will not be true for non-cancellative semigroups, the obvious example being formed by adding a zero to a non-automatic semigroup. However, one might think that, if S is a cancellative semigroup, then the hypothesis of S having the undirected fellow traveller property with respect to L might imply automaticity; however, as the following example shows, this is not the case.

Example 3.5. Let $S = \langle A : \Re \rangle$ with

$$A = \{a, b, c, p, q, r, t, u, v, w, x, y, z\}$$
$$\Re = \{(ua^i v, wb^i x) : i \in \mathbb{N}\} \cup \{(ua^i y, wb^i z) : i = 2^j, j \in \mathbb{N}\}$$
$$\cup \{(ua^i y, qc^i r) : i \in \mathbb{N}, i \neq 2^j, j \in \mathbb{N}\} \cup \{(ua^i p, qc^i t) : i \in \mathbb{N}\}.$$

Let L be the regular language

$$A^+ - (A^*(\{u\}\{a\}^*\{v\} \cup \{u\}\{a\}^*\{y\} \cup \{u\}\{a\}^*\{p\})A^*);$$

then we claim that L is a rational cross section for S, S is cancellative and S has the undirected fellow traveller property with respect to L, but S is not automatic. We will treat these four claims in turn.

Claim 1. *L is a rational cross section for S.*

Let $\mathcal{R}$ be the set of rewrite rules

$$ua^i v \to wb^i x \ (i \geqslant 0), \quad ua^i y \to wb^i z \ (i = 2^j), \ ua^i y \to qc^i r \ (i \neq 2^j), \quad ua^i p \to qc^i t \ (i \geqslant 0).$$

We see that $\mathcal{R}$ is a complete rewriting system for S and that L is the set of normal forms of $\mathcal{R}$; so L maps bijectively onto S as required.

Claim 2. *The semigroup S is cancellative.*

Note that, if $\alpha_1 \to \beta_1$ and $\alpha_2 \to \beta_2$ are any two distinct rewrite rules in $\mathcal{R}$, then no prefix of α_1 is a suffix of α_2; moreover, if $\alpha \to \beta$ is a rule in $\mathcal{R}$ and $\zeta \equiv \gamma\beta\delta$ is any word in A^+, then no rule in $\mathcal{R}$ can be applied to ζ which rewrites any part of the subword β. So any word η in A^+ can be written uniquely in the form $\eta \equiv \theta_1\alpha_1\theta_2\alpha_2 \ldots \theta_r\alpha_r\theta_{r+1}$, where we have rewrite rules $\alpha_1 \to \beta_1, \ldots, \alpha_r \to \beta_r$ in $\mathcal{R}$, where the α_i are the only subwords of η that can be rewritten, and where the rewritten word $\theta_1\beta_1\theta_2\beta_2 \ldots \theta_r\beta_r\theta_{r+1}$ is in normal form.

Now suppose that $\omega_1, \omega_2 \in L$ and $d \in A$ with $\omega_1 d = \omega_2 d$. If $\omega_1 d \equiv \omega_2 d$ then $\omega_1 = \omega_2$ and we are done; so assume that $\omega_1 d \not\equiv \omega_2 d$. Since L maps bijectively onto S, we cannot have both $\omega_1 d \in L$ and $\omega_2 d \in L$, and so we can apply a rule of $\mathcal{R}$ to a suffix of either $\omega_1 d$ or $\omega_2 d$. Since the rule will change the last letter of the word, we must be able to apply a rule of $\mathcal{R}$ to suffixes of both $\omega_1 d$ or $\omega_2 d$. Now

$$\omega_1 d \equiv \xi_1\alpha_1 \Longrightarrow_R \xi_1\beta_1 \in L \quad \text{and} \quad \omega_2 d \equiv \xi_2\alpha_2 \Longrightarrow_R \xi_2\beta_2 \in L.$$

Since $\xi_1\beta_1 = \xi_2\beta_2$ and L maps bijectively onto S, we have that $\xi_1\beta_1 \equiv \xi_2\beta_2$, where the β_i are right-hand sides of rewrite rules from $\mathcal{R}$, so that $\xi_1 \equiv \xi_2$ and $\beta_1 \equiv \beta_2$, and then $\alpha_1 \equiv \alpha_2$, so that $\omega_1 d \equiv \omega_2 d$, a contradiction. So S is right-cancellative.

The proof that S is left-cancellative is similar, and we have proved Claim 2.

Claim 3. *The semigroup S has the undirected fellow traveller property with respect to L.*

Let d' be the undirected distance function of S with respect to A. Since L maps bijectively onto S, we only need to prove that there is a constant k such that

$$\alpha, \beta \in L, \quad f \in A, \quad \alpha f = \beta \Rightarrow d'(\alpha(t), \beta(t)) \leqslant k \quad \text{for all } t \geqslant 0.$$

In the case where $\alpha f \equiv \beta$, we have that the distance $d'(\alpha(t), \beta(t))$ is at most 1 for all $t \geqslant 0$. We now consider the case where $\alpha f = \beta$ but $\alpha f \not\equiv \beta$. Since $\alpha \in L$, no substring of α can be rewritten via a rule in $\mathcal{R}$, and so $\alpha f \equiv \zeta\gamma \Longrightarrow_{\mathcal{R}} \zeta\delta \equiv \beta$ where $(\gamma \to \delta) \in \mathcal{R}$. If $t \leqslant |\zeta|$, then $\alpha(t) \equiv \beta(t)$ and so $d'(\alpha(t), \beta(t)) = 0$; so we may assume that $t = |\zeta| + r$ with $r > 0$.

Now $\alpha(t) \equiv \zeta\gamma(r)$ and $\beta(t) \equiv \zeta\delta(r)$. If $r = |\gamma| = |\delta|$, the result is clear; otherwise we have $\gamma \equiv \gamma_1 f'$, $\delta \equiv \delta_1 f''$, and then there exist g and g' such that

$$\gamma(r)g \equiv \gamma_1(r)g = \delta_1(r)g' \equiv \delta(r)g',$$

so that $d'(\alpha(t), \beta(t)) = d(\zeta\gamma(r), \zeta\delta(r)) \leqslant 2$ as required.

Claim 4. *The semigroup S is not automatic.*

Suppose that S is automatic, so that $T = S^1$ is automatic by Proposition 2.8. Let e denote the identity element of T; then there must be an automatic structure (B, K) for T by Theorem 2.9, where $B = A \cup \{e\}$. By Remark 2.2 we can further assume that (B, K) is an automatic structure with uniqueness for T.

Now the sets

$$\{(\alpha, \beta)\delta_B : \alpha, \beta \in K, \alpha y = \beta\} \quad \text{and} \quad \{(\beta, \gamma)\delta_B : \beta, \gamma \in K, \beta = \gamma z\}$$

are regular, and so the set $U = \{(\alpha, \gamma)\delta_B : \alpha, \gamma \in K, \alpha y = \gamma z\}$ is regular by Proposition 2.6.

Let $\phi : A^+ \to K$ be the map defined by $\alpha\phi = \zeta$ where $\alpha =_T \zeta$ and $\zeta \in K$. There is no relation we can apply to a word ua^i except those that allow us to add instances of e. So the set

$$C = \{(ua^i)\phi : i \geqslant 0\} = \{e\}^*\{u\}\{e, a\}^* \cap K$$

is regular and $|(ua^i)\phi|_a = i$ for all $i \geqslant 0$. Then

$$(C \times K)\delta_B \cap U = \{(\alpha, \beta)\delta_B : \alpha, \beta \in K, \alpha =_T ua^i, \beta =_T wb^i, i = 2^j \text{ for some } j \in \mathbb{N}\}$$

is regular. Applying the homomorphism $\psi : A(2, \$)^* \to A^*$ defined by

$$(d, f)\psi = \begin{cases} a & \text{if } d \equiv a, \\ \varepsilon & \text{if } d \not\equiv a \end{cases}$$

we would have that $\{a^i : i = 2^j, j \in \mathbb{N}\}$ is regular by Proposition 2.4, a contradiction.

4. Continuation graphs

In this section, we introduce the idea of having "a finite number of continuation graphs", and we use this to give a geometric characterization of automaticity in semigroups. We first introduce some notation:

Notation 4.1. Let A be a finite set with $\$ \notin A$ and let $\alpha \in A^*\{\$\}^*$; then we let $\overline{\alpha}$ be the unique element of A^* such that $\alpha \equiv \overline{\alpha}\$^i$ with $i \in \mathbb{N}$.

Notation 4.2. Let S be a semigroup and A be finite generating set of S. Let L be a regular subset of A^+ and $M = (Q, A \cup \{\$\}, \tau, s, E)$ be a deterministic finite state automaton accepting $L\{\$\}^*$. Let $v, w \in S$. Then we define:

$$Q_{v,w,M} = \{(\tau(s, \alpha), \tau(s, \beta)) \in Q \times Q : \quad (\alpha, \beta) \in (A^+\{\$\}^* \times A^+) \cup (A^+ \times A^+\{\$\}^*), \quad |\alpha| = |\beta|, \ \overline{\alpha} = v, \ \overline{\beta} = w\}.$$

The following result is clear but we note it for what follows:

Lemma 4.3. *Let A be a finite set and L be a regular language over A. Let* $M = (Q, A \cup \{\$\}, \tau, s, E)$ *be a deterministic finite state automaton accepting* $L\{\$\}^*$ *such that all states in M are reachable. Let* $q \in Q$ *and* $\alpha \in A^*\{\$\}^*$; *then* $\tau(q, \alpha) \in E$ *if and only if* $\tau(q, \overline{\alpha}) \in E$.

Proof. We have $\alpha \equiv \overline{\alpha}\$^i$ for some $i \in \mathbb{N}$. Since all the states in M are reachable (and M is deterministic) there exists a word $\beta \in (A \cup \{\$\})^*$ with $\tau(s, \beta) = q$. We have that

$$\beta\overline{\alpha}\$^i \in L\{\$\}^* \iff \beta\overline{\alpha} \in L.$$

So $\tau(q, \alpha) = \tau(s, \beta\overline{\alpha}\$^i) \in E$ if and only if $\tau(q, \overline{\alpha}) = \tau(s, \beta\overline{\alpha}) \in E$. $\square$

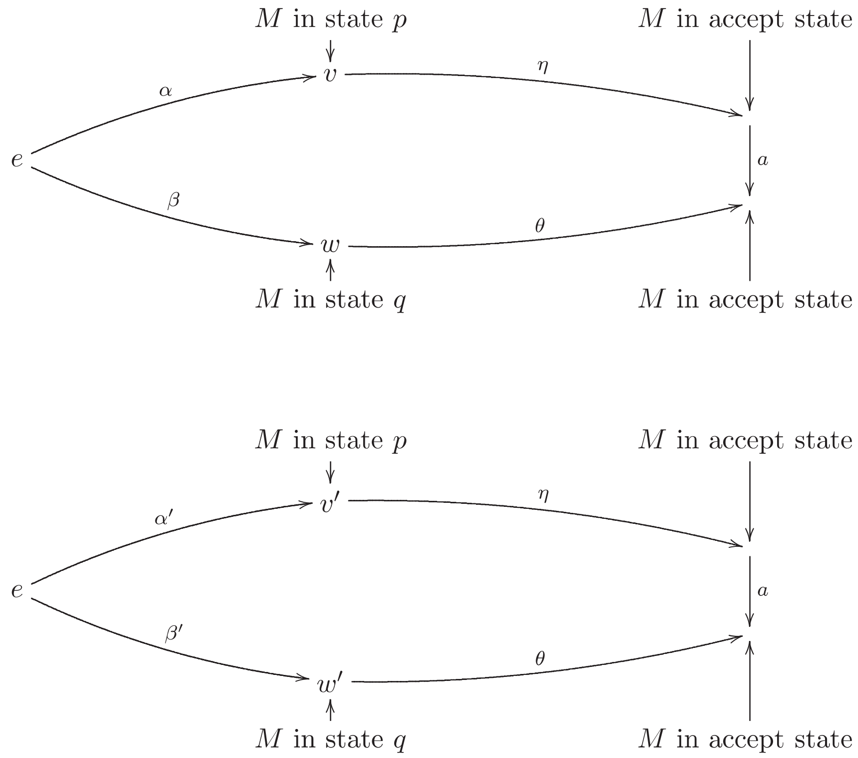

Fig. 3. Continuation graphs.

We now come to the fundamental concept we will use in establishing a geometric condition for automaticity in semigroups:

Definition 4.4. Let S be a semigroup and A be a finite generating set for S. Let L be a regular subset of A^+ such that L maps onto S. Let

$$M = (Q, A \cup \{\$\}, \tau, s, E)$$

be a deterministic finite state automaton accepting $L\{\$\}^*$ with no dead states. For $v, v', w, w' \in S$, we say that (v, w) and (v', w') *have the same continuation graph* with respect to M, and we write $(v, w) \simeq (v', w')$, if

1. $Q_{v,w,M} = Q_{v',w',M}$;
2. for all $a \in A \cup \{\varepsilon\}$, all $(p, q) \in Q_{v,w,M}$ and all $\eta, \theta \in A^*$ with $\tau(p, \eta), \tau(q, \theta) \in E$, we have

$$v\eta a = w\theta \iff v'\eta a = w'\theta.$$

Note that $\simeq$ is an equivalence relation on $S \times S$.

If the set $(S \times S)/\simeq$ is finite then we say that S has a *finite number of continuation graphs* with respect to M.

The reader may find Fig. 3 helpful when considering Definition 4.4.

Theorem 4.5. *Let S be a semigroup and A be a finite generating set for S. Suppose that L is a regular subset of A^+ that maps onto S. Let $A' = A \cup \{\$\}$ and $M = (Q_M, A', \tau_M, s_M, E_M)$ be a deterministic finite state automaton with no dead states accepting $L\{\$\}^*$. Then S has a finite number of continuation graphs with respect to M if and only if (A, L) is an automatic structure for S.*

Proof. Throughout this proof, we let $Q_{v,w,M}$ (for $v, w \in S$) be defined as in Notation 4.2.

"$\Leftarrow$" We let $A = \{a_1, \ldots, a_n\}$ and assume that (A, L) is an automatic structure for S. For $1 \leqslant i \leqslant n$ we let $M_{a_i} = (Q_{a_i}, A(2,\$), \tau_{a_i}, s_{a_i}, E_{a_i})$ be a deterministic finite state automaton accepting the language $L^{\$}_{a_i}$. Similarly, we let $M_\varepsilon = (Q_\varepsilon, A(2,\$), \tau_\varepsilon, s_\varepsilon, E_\varepsilon)$ be a deterministic finite state automaton accepting $L^{\$}_\varepsilon$. Now, for $v, w \in S$, let

$$P_{\varepsilon,v,w} = \{\tau_\varepsilon(s_\varepsilon, (\alpha,\beta)\delta_A) : \alpha, \beta \in A^+, \overline{\alpha} = v, \overline{\beta} = w\},$$
$$P_{a_i,v,w} = \{\tau_{a_i}(s_{a_i}, (\alpha,\beta)\delta_A) : \alpha, \beta \in A^+, \overline{\alpha} = v, \overline{\beta} = w\} \quad (1 \leqslant i \leqslant n),$$
$$R_{v,w} = (P_{a_1,v,w}, P_{a_2,v,w}, \ldots, P_{a_n,v,w}, P_{\varepsilon,v,w}, Q_{v,w,M}).$$

Since $P_{a_i,v,w} \subseteq Q_{a_i}$ for each i, the set $\{P_{a_i,v,w} : v, w \in S\}$ is finite for $1 \leqslant i \leqslant n$ and, similarly, the sets $\{P_{\varepsilon,v,w} : v, w \in S\}$ and $\{Q_{v,w,M} : v, w \in S\}$ are finite; therefore the set $\{R_{v,w} : v, w \in S\}$ is finite.

We will show that, if $R_{v,w} = R_{v',w'}$, then the pairs (v, w) and (v', w') have the same continuation graph. We first note that, if $R_{v,w} = R_{v',w'}$, we clearly have that $Q_{v,w,M} = Q_{v',w',M}$.

Now let $(p, q) \in Q_{v,w,M}$ so that there exists

$$(\alpha, \beta) \in (A^+\{\$\}^* \times A^+) \cup (A^+ \times A^+\{\$\}^*)$$

such that $\tau_M(s_M, \alpha) = p$, $\tau_M(s_M, \beta) = q$, $|\alpha| = |\beta|$, $\overline{\alpha} = v$ and $\overline{\beta} = w$. Moreover, let $\eta, \theta \in A^*$ with $\tau_M(p, \eta) = f \in E_M$, $\tau_M(q, \theta) = g \in E_M$. So $\alpha\eta, \beta\theta \in L\$^*$ and $\overline{\alpha}\eta, \overline{\beta}\theta \in L$. In particular, if $|\overline{\alpha}| < |\overline{\beta}|$, then η will be the empty word and, if $|\overline{\alpha}| > |\overline{\beta}|$, then θ will be the empty word. We deduce that

$$(\overline{\alpha}\eta, \overline{\beta}\theta)\delta_A \equiv (\alpha\eta, \beta\theta)\delta_A \equiv (\alpha, \beta)\delta_A(\eta, \theta)\delta_A \equiv (\overline{\alpha}, \overline{\beta})\delta_A(\eta, \theta)\delta_A.$$

Lastly, suppose that $a \in A \cup \{\varepsilon\}$ and that $v\eta a = w\theta$ (so that $\overline{\alpha}\eta a = \overline{\beta}\theta$); we want to show that $v'\eta a = w'\theta$.

Let $r = \tau_a(s_a, (\overline{\alpha}, \overline{\beta})\delta_A)$, so that $\tau_a(r, (\eta, \theta)\delta_A) \in E_a$ (as $\overline{\alpha}\eta a = \overline{\beta}\theta$) and $r \in P_{a,v,w}$. Since $P_{a,v,w} = P_{a,v',w'}$, we have that $r \in P_{a,v',w'}$ and so there exist $\alpha', \beta' \in A^*$ with $r = \tau_a(s_a, (\alpha', \beta')\delta_A)$, $\overline{\alpha'} = v'$ and $\overline{\beta'} = w'$. Now

$$\tau_a(s_a, (\alpha'\eta, \beta'\theta)\delta_A) = \tau_a(r, (\eta, \theta)\delta_A) \in E_a,$$

so that $(\alpha'\eta, \beta'\theta)\delta_A$ is accepted by M_a. Hence

$$v'\eta a = \overline{\alpha'}\eta a = \overline{\beta'}\theta = w\theta$$

as required. So $(v, w) \simeq (v', w')$.

Given this, we see that the number of equivalence classes of $\simeq$ is bounded by $|\{R_{v,w} : v, w \in S\}|$, and so S has only a finite number of continuation graphs with respect to M.

"$\Rightarrow$" We now assume that S has a finite number of continuation graphs with respect to M.

Let a be an element of $A \cup \{\varepsilon\}$; we will construct the finite state automaton $M_a = (Q_a, A(2,\$) - \{(\$,\$)\}, \tau_a, s_a, E_a)$ which accepts $L^{\$}_a$.

Let $D \subseteq S \times S$ be a set of representatives of the equivalence classes of $\simeq$. Let $P = D \cup \{(\varepsilon, \varepsilon)\}$, and then let $Q_a = (P \times Q_M \times Q_M)$, $s_a = (\varepsilon, \varepsilon, s_M, s_M)$. We define τ_a as follows:

$$\tau_a((v, w, p, q), (c, d)) = (v', w', \tau_M(p, c), \tau_M(q, d)),$$

where $(c, d) \in A(2,\$) - \{(\$,\$)\}$, $(v', w') \in D$ and $(v', w') \simeq (v\overline{c}, w\overline{d})$. Finally, we let

$$E_a = \{(v, w) \in D : va = w\} \times E_M \times E_M.$$

In order to show that $L(M_a) = L^{\$}_a$ we will establish the following two claims.

Claim I. *Let $r = (v, w, p, q)$ be a reachable state in M_a and suppose that $(v, w) \neq (\varepsilon, \varepsilon)$; then $(p, q) \in Q_{v,w,M}$.*

Since r is reachable there exist $c_1, \ldots, c_k, d_1, \ldots, d_k \in A \cup \{\$\}$ with

$$\tau_a(s_a, ((c_1, d_1)(c_2, d_2) \ldots (c_k, d_k)) = r.$$

For $1 \leqslant i \leqslant k$ let $r_i = (v_i, w_i, p_i, q_i) \in Q_a$ with

$$r_i = \tau_a(s_a, ((c_1, d_1) \ldots (c_i, d_i)).$$

We will prove, by induction, that $(p_i, q_i) \in Q_{v_i, w_i, M}$ for all $1 \leqslant i \leqslant k$.

Initial case: $i = 1$. Here $r_1 = ((c_1, d_1), \tau_M(s_M, c_1), \tau_M(s_M, d_1))$ and, clearly,

$$(\tau_M(s_M, c_1), \tau_M(s_M, d_1)) \in Q_{c_1, d_1, M}.$$

Since $(v_1, w_1) \simeq (c_1, d_1)$, we have that $Q_{v_1, w_1, M} = Q_{c_1, d_1, M}$ and therefore $(p_1, q_1) \in Q_{v_1, w_1, M}$. Note that $L \subseteq A^+$; therefore $c_1, d_1 \in A$.

Induction step: Let $1 \leqslant i < k$ and $(p_i, q_i) \in Q_{v_i, w_i, M}$; so we have that $(p_{i+1}, q_{i+1}) \in Q_{v_i \overline{c}_{i+1}, w_i \overline{d}_{i+1}, M}$. Since $(v_i \overline{c}_{i+1}, w_i \overline{d}_{i+1}) \simeq (v_{i+1}, w_{i+1})$ we have that

$$Q_{v_i \overline{c}_{i+1}, w_i \overline{d}_{i+1}, M} = Q_{v_{i+1}, w_{i+1}, M}.$$

Therefore $(p_{i+1}, q_{i+1}) \in Q_{v_{i+1}, w_{i+1}, M}$. This proves Claim I.

Claim II. *Let $r = (v, w, p, q)$ be a reachable state in M_a. For all $\gamma_1, \gamma_2 \in A^*$ we have that*

$$\tau_a(r, (\gamma_1, \gamma_2)\delta_A) \in E_a \iff \begin{cases} v\gamma_1 a = w\gamma_2 \quad \text{and} \\ \tau_M(p, \gamma_1), \tau_M(q, \gamma_2) \in E_M. \end{cases}$$

We will prove Claim II by induction over the length of $(\gamma_1, \gamma_2)\delta_A$.

Initial case: $|(\gamma_1, \gamma_2)\delta_A| = 0$. Then $\gamma_1 \equiv \gamma_2 \equiv \varepsilon$. By the definition of E_a, we have $r \in E_a$ if and only $va = w$ and $p, q \in E_M$.

Induction step: Let $|(\gamma_1, \gamma_2)\delta_A| > 0$; so we have $(\gamma_1, \gamma_2)\delta_A \equiv (x, y)(\gamma_1', \gamma_2')\delta_A$ for some $x, y \in A'$ and $\gamma_1', \gamma_2' \in A^*$.

First case: we assume that $v\gamma_1 a = w\gamma_2$ and $\tau_M(p, \gamma_1), \tau_M(q, \gamma_2) \in E_M$. Let $r' = (v', w', p', q') = \tau_a(r, (x, y))$; then, by definition of τ_a, we have that $(v\overline{x}, w\overline{y}) \simeq (v', w')$; so we have

$$\tau_M(p', \gamma_1') = \tau_M(\tau_M(p, x), \gamma_1') \in E_M$$

and, similarly, $\tau_M(q', \gamma_2') \in E_M$. Since either $(v, w) = (\varepsilon, \varepsilon)$ or, by Claim I, $(p, q) \in Q_{v,w,M}$ and $(v\overline{x}, w\overline{y}) \simeq (v', w')$, we have that $v'\gamma_1' a = w'\gamma_2'$. Using the inductive hypothesis, we have $\tau_a(r', (\gamma_1', \gamma_2')\delta_a) \in E_a$. Therefore $\tau_a(r, (\gamma_1, \gamma_2)\delta_a) \in E_a$.

Second case: we assume that $\tau_a(r, (\gamma_1, \gamma_2)\delta_A) \in E_a$. Then let

$$r' = (v', w', p', q') = \tau_a(r, (x, y)).$$

Since $\tau_a(r', (\gamma_1', \gamma_2')\delta_A) = \tau_a(r, (\gamma_1, \gamma_2)\delta_a)$, and this is therefore an element of E_a, we have, by the inductive hypothesis, that $v'\gamma_1' a = w'\gamma_2'$ and that

$$\tau_M(p', \gamma_1'), \tau_M(q', \gamma_2') \in E_M.$$

This gives that $\tau_M(p, \gamma_1) = \tau_M(p', \gamma_1')$ and $\tau_M(q, \gamma_2) = \tau_M(q', \gamma_2')$ are elements of E_M. Either $(v, w) = (\varepsilon, \varepsilon)$ or, by Claim I, $(p, q) \in Q_{v,w,M}$. This gives, together with the fact that $(v\overline{x}, w\overline{y}) \simeq (v', w')$, that $v\gamma_1 a = w\gamma_2$.

This proves Claim II.

It is easy to see that $L(M_a) \subseteq (L \times L)\delta_A$; therefore, using Claim II, we have that $\tau_a(s_a, (\alpha, \beta)\delta_A)) \in E_a$ if and only if $\alpha, \beta \in L$ and $\alpha a = \beta$. $\square$

As an immediate consequence of Theorem 4.5, we have

Corollary 4.6. *Let S be a semigroup and A be a finite generating set for S. Let L be a regular language over A that maps onto S, and then let M_1 and M_2 be two deterministic finite state automata accepting $L\{\$\}^*$. Then S has finite number of continuation graphs with respect to M_1 if and only if S has finite number of continuation graphs with respect to M_2.*

Given Corollary 4.6, we can take the definition of having a finite number of continuation graphs to be with respect to a language (as opposed to a specific machine accepting that language as in Definition 4.4):

Definition 4.7. Let S be a semigroup, A be a finite generating set for S and L be a regular language over A. Let M be a deterministic finite state automaton accepting $L\{\$\}^*$. If S has a finite number of continuation graphs with respect to M then we say that S has a *finite number of continuation graphs with respect to* L.

Given this definition, we can restate Theorem 4.5 in the form:

Theorem 4.8. *Let S be a semigroup, A a be finite generating set for S and L be a regular language over A that maps onto S. Then S has a finite number of continuation graphs with respect to L if and only if (A, L) is an automatic structure for S.*

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References

[1] G. Baumslag, S.M. Gersten, M. Shapiro, H. Short, Automatic groups and amalgams, J. Pure Appl. Algebra 76 (1991) 229–316.
[2] R.V. Book, F. Otto, String Rewriting Systems, Springer, Berlin, 1993.
[3] C.M. Campbell, E.F. Robertson, N. Ruškuc, R.M. Thomas, Automatic semigroups, Theoret. Comput. Sci. 365 (2001) 365–391.
[4] C.M. Campbell, E.F. Robertson, N. Ruškuc, R.M. Thomas, Automatic completely-simple semigroups, Acta Math. Hungari. 95 (2002) 201–215.
[5] A. Cutting, A. Solomon, Remark concerning finitely generated semigroup having regular sets of unique normal forms, J. Austral. Math. Soc. 70 (2001) 239–309.
[6] A.J. Duncan, E.F. Robertson, N. Ruškuc, Automatic monoids and change of generators, Math. Proc. Cambridge Philos. Soc. 127 (1999) 403–409.
[7] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S. Levy, M.S. Paterson, W. Thurston, Word Processing in Groups, Jones and Barlett, Boston, MA, 1992.
[8] R.H. Gilman, Automatic groups and string rewriting, in: H. Comon, J.P. Jouannaud (Eds.), Term Rewriting, Lecture Notes in Computer Science, Vol. 909, Springer, Berlin, 1995, pp. 127–134.
[9] M.A. Harrison, Introduction to Formal Language Theory, Addison-Wesley, Reading, MA, 1978.
[10] M. Hoffmann, Automatic semigroups, Ph.D. Thesis, University of Leicester, 2000.
[11] M. Hoffmann, R.M. Thomas, Automaticity and commutative semigroups, Glasgow J. Math. 44 (2002) 167–176.
[12] M. Hoffmann, R.M. Thomas, Notions of automaticity in semigroups, Semigroup Forum 66 (2003) 337–367.
[13] J.E. Hopcroft, R. Motwani, J.D. Ullman, Introduction to Automata Theory, Languages and Computation, Addison-Wesley, Reading, MA, 2001.
[14] J.M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, Oxford, 1995.
[15] J.F.P. Hudson, Regular rewrite systems and automatic structures, in: J. Almeida, G.M.S. Gomes, P.V. Silva (Eds.), Semigroups, Automata and Languages, World Scientific, Singapore, 1998, pp. 145–152.
[16] G. Lallement, Semigroups and Combinatorial Applications, Wiley, New York, 1979.
[17] F. Otto, A. Sattler-Klein, K. Madlener, Automatic monoids versus monoids with finite convergent presentations, in: T. Nipkow (Ed.), Rewriting Techniques and Applications—Proceedings RTA ’98, Lecture Notes in Computer Science, Vol. 1379, Springer, Berlin, 1998, pp. 32–46.
[18] P.V. Silva, B. Steinberg, Extensions of submonoids of automatic monoids, Theoret. Comput. Sci. 289 (2002) 727–754.
[19] P.V. Silva, B. Steinberg, A geometric characterization of automatic monoids, Q. J. Math. 55 (2004) 333–356.