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Oscillation Criteria for Second-Order Quasi-Linear Neutral Difference Equations

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Abstract—In this paper, we obtain some oscillation criteria for the second-order quasi-linear neutral delay difference equation

$$\Delta\left[a_{n-1}\left|\Delta\left(x_{n-1}+p_{n-1}x_{n-1-\sigma}\right)\right|^{\alpha-1}\Delta\left(x_{n-1}+p_{n-1}x_{n-1-\sigma}\right)\right]+q_n f\left(x_{n-\tau}\right)=0,$$

where $\alpha>0$, $\tau\geq 0$, and $\sigma\geq 0$ are constants, $\{a_n\}$, $\{p_n\}$, $\{q_n\}$ are nonnegative sequences and $f\in C(R,R)$.

Keywords—Quasi-linear neutral delay difference equation, Oscillation.

1. INTRODUCTION

In this paper, we are concerned with the oscillatory behavior of solutions of second-order quasi-linear neutral delay difference equation of the form

$$\Delta\left[a_{n-1}\left|\Delta\left(x_{n-1}+p_{n-1}x_{n-1-\sigma}\right)\right|^{\alpha-1}\Delta\left(x_{n-1}+p_{n-1}x_{n-1-\sigma}\right)\right]+q_n f\left(x_{n-\tau}\right)=0, \tag{1}$$

where $n=1,2,3,\ldots,\alpha$ is a positive constants, τ and σ are nonnegative integers. Throughout this paper, we assume the following.

(a) $0\leq p_n\leq 1$ for $n=0,1,2,\ldots$.
(b) $\{q_n\}_{n=1}^{\infty}$ is a nonnegative sequence with infinitely many positive terms.
(c) $a_n>0$, $n=0,1,2,\ldots$, and $\sum^{\infty}(1/a_n^{1/\alpha})=\infty$.
(d) There exists a differential function $\phi:R\to R$ such that

$$\frac{f(x)}{|\phi(x)|^{\alpha-1}\phi(x)}\geq\gamma>0,\quad \phi'(x)\geq\varepsilon>0,\quad x\phi(x)>0,\qquad \text{for } x\neq 0.$$

By a solution of (1), we mean a nontrivial sequence $\{x_n\}$ which is defined for $n\geq-\max\{\sigma,\tau\}$ and satisfies equation (1) for $n=1,2,3,\ldots$. A solution $\{x_n\}$ of equation (1) is said to be oscillatory if for every $N>0$, there exists an $n\geq N$ such that $x_n x_{n+1}\leq 0$, otherwise, it is nonoscillatory. Equation (1) is oscillatory if all its solutions are oscillatory.

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Many authors have investigated the special case of equation (1). For example, Li and Yeh [2], Agarwal, Manuel and Thandapani [2,3] have considered the following neutral delay difference equation:

$$\Delta\left[a_{n-1}\Delta\left(x_{n-1}+p_{n-1}x_{n-1-\sigma}\right)\right]+q_n f(x_{n-\tau})=0, \qquad n=1,2,3,\dots. \tag{2}$$

Thandapani, Manuel and Agarwal [4] have studied the quasi-linear difference equation

$$\Delta\left[a_{n-1}\left|\Delta x_{n-1}\right|^{\alpha-1}\Delta x_{n-1}\right]+q_n f\left(x_n\right)=0, \qquad n=1,2,3,\dots. \tag{3}$$

Szafranski and Szmanda [5] and Lalli and Grace [6] have investigated the delay difference equation

$$\Delta\left[a_{n-1}\Delta x_{n-1}\right]+q_n f\left(x_{n-k}\right)=0, \qquad n=1,2,3,\dots. \tag{4}$$

The purpose here is to develop oscillation theory for such a general case of (1). Theorems 1–3 obtained here include and extend all the results in [1]. Theorem 4 is new, up to now, even for equations (2)–(4).

2. MAIN RESULTS

In order to prove our theorems, we use the following lemmas. The first is due to Hardy, Littlewood and Polya [7].

LEMMA 1. *If X and Y are nonnegative, then*

$$X^q+(q-1)Y^q-qXY^{q-1}\ge 0, \qquad q>1,$$

where equality holds if and if $X=Y$.

LEMMA 2. *If $x,y\in R$, $x\ge 0$, then*

$$y(x+2y)-\frac{1}{(\alpha+1)^{\alpha+1}}|x+2y|^{\alpha+1}\le \max\left\{2y^2,\alpha y^{\alpha+1/\alpha}\right\}.$$

PROOF. Let $G(x,y)=y(x+2y)-1/((\alpha+1)^{\alpha+1})|x+2y|^{\alpha+1}$.

(i) If $y\le 0$, then $G(x,y)\le 2y^2$.

(ii) If $y\ge 0$, then

$$G_x(x,y)=y-\frac{1}{(\alpha+1)^{\alpha}}(x+2y)^{\alpha},$$
$$G_{xx}(x,y)=-\frac{\alpha}{(\alpha+1)^{\alpha}}(x+2y)^{\alpha-1}\le 0.$$

Hence, let $G_x(x_0,y)=0$, then $G(x,y)\le G(x_0,y)=\alpha y^{\alpha+1/\alpha}$. The proof is complete.

THEOREM 1. *Let $\{H_{m,n}\mid m\ge n\ge 0\}$ be a double sequence satisfying the following two conditions.*

(i)
$$H_{m,m}=0, \qquad \text{for } m\ge 0,$$
$$H_{m,n}>0, \qquad \text{for } m>n\ge 0.$$

(ii)
$$\Delta_2 H_{m,n}=H_{m,n+1}-H_{m,n}\le 0, \qquad \text{for } m>n\ge 0.$$

Suppose that $\{h_{m,n} \mid m > n \geq 0\}$ is a double sequence with

$$\Delta H_2 m, n = -h_{m,n}\sqrt{H_{m,n}}, \qquad \text{for all } m > n \geq 0.$$

If there exists a positive sequence $\{\xi_n \mid \xi_n > 0,\ n = 1, 2, 3, \dots\}$ such that

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} H_{m,n}\xi_n \left[\psi_n + a_{n-\tau}\beta_n\eta_{m,n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} |\eta_{m,n}|^{\alpha+1}\right] = \infty, \tag{5}$$

where

$$\psi_n = \gamma q_{n+1}\varepsilon^\alpha (1 - p_{n+1-\tau})^\alpha - \Delta(a_{n-1-\tau}\beta_{n-1}), \tag{6}$$

$$\eta_{m,n} = \frac{h_{m,n}\sqrt{H_{m,n}}}{H_{m,n}} \frac{\xi_{n+1}}{\xi_n} + 2\beta_n, \qquad \beta_n = -\frac{\Delta\xi_n}{2\xi_n}. \tag{7}$$

Then equation (1) is oscillatory.

PROOF. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1) and let $z_n = x_n + p_n x_{n-\sigma}$ for $n = 0, 1, 2, \dots$. Then by Conditions (a)–(d) and equation (1), it is not difficult to prove that $z_n \Delta z_n$ is eventually positive. Without loss of generality, we assume that $z_{n-\tau} > 0$, $\Delta z_n > 0$, and $x_{n-\sigma-\tau} > 0$, for $n \geq N$, for some positive integer N. Now observe that from (1), we have

$$\Delta\left[a_{n-1} |\Delta z_{n-1}|^{\alpha-1} \Delta z_{n-1}\right] + q_n f(x_{n-\tau}) = 0. \tag{8}$$

Using Condition (d) in (8), we get

$$\Delta\left[a_{n-1}(\Delta z_{n-1})^\alpha\right] + \gamma q_n \phi^\alpha (z_{n-\tau} - p_{n-\tau}x_{n-\tau-\sigma}) \leq 0,$$

which, in view of the fact that $z_n \geq x_n > 0$ and z_n is increasing, from Condition (d) yields

$$\Delta\left[a_{n-1}(\Delta z_{n-1})^\alpha\right] + \gamma q_n \varepsilon^\alpha (1 - p_{n-\tau})^\alpha z_{n-\tau}^\alpha \leq 0,$$

for $n \geq N$. Define

$$w_n = \xi_n \left[\frac{a_n (\Delta z_n)^\alpha}{z_{n-\tau}^\alpha} + a_{n-1-\tau}\beta_{n-1}\right], \qquad n \geq N,$$

then

$$\begin{aligned}\Delta w_n \leq{}& \frac{\Delta\xi_n}{\xi_{n+1}} w_{n+1} + \xi_n \left[-\gamma q_{n+1}\varepsilon^\alpha (1 - p_{n+1-\tau})^\alpha \right.\\ &\left. - \frac{\alpha}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right)^{\alpha+1/\alpha} + \Delta(a_{n-1-\tau}\beta_{n-1})\right]\\ ={}& \frac{\Delta\xi_n}{\xi_{n+1}} w_{n+1} + \xi_n \left[-\psi_n - \frac{\alpha}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right)^{\alpha+1/\alpha}\right],\end{aligned} \tag{9}$$

for $n \geq N$, where ψ_n is defined in (6). Therefore,

$$\sum_{n=k}^{m-1} H_{m,n}\Delta w_n \leq \sum_{n=k}^{m-1} \frac{\Delta\xi_n}{\xi_{n+1}} w_{n+1} H_{m,n} - \sum_{n=k}^{m-1} H_{m,n}\xi_n \left[\psi_n + \frac{\alpha}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right)^{\alpha+1/\alpha}\right],$$

i.e.,

$$\begin{aligned}&\sum_{n=k}^{m-1} H_{m,n}\xi_n (\psi_n + \eta_{m,n}a_{n-\tau}\beta_n)\\ &\leq H_{m,k}w_k + \sum_{n=k}^{m-1} H_{m,n}\xi_n \left[|\eta_{m,n}| \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right) - \frac{\alpha}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right)^{\alpha+1/\alpha}\right],\end{aligned} \tag{10}$$

where $\eta_{m,n}$ is defined in (7). Taking

$$X = \frac{\alpha^{\alpha/\alpha+1}}{a_{n-\tau}^{1/\alpha+1}} \left[\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n \right],$$

$$Y = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha}} \frac{a_{n-\tau}^{\alpha/\alpha+1}}{\alpha^{\alpha^2/\alpha+1}} \left|\eta_{m,n}\right|^{\alpha},$$

according to Lemma 1, we obtain

$$|\eta_{m,n}| \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n \right) - \frac{\alpha}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n \right)^{\alpha+1/\alpha} \leq \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1}.$$

Hence, from (10), we get

$$\sum_{n=k}^{m-1} H_{m,n}\xi_n \left[\psi_n + \eta_{m,n} a_{n-\tau}\beta_n - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1} \right] \leq H_{m,k} w_k. \tag{11}$$

Then, using Lemma 2, we get

$$\sum_{n=0}^{m-1} H_{m,n}\xi_n \left[\psi_n + \eta_{m,n} a_{n-\tau}\beta_n - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1} \right]$$
$$\leq H_{m,0} \left|w_N\right| + \sum_{n=0}^{N-1} H_{m,0}\xi_n \left[\psi_n + a_{n-\tau} \max\left\{ 2\beta_n^2, \alpha\beta_n^{\alpha+1/\alpha} \right\} \right].$$

Hence,

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} H_{m,n}\xi_n \left(\psi_n + a_{n-\tau}\beta_n\eta_{m,n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1} \right) < \infty,$$

which contradicts (5). This contradiction completes the proof.

THEOREM 2. *Let $\{H_{m,n}\}$ and $\{h_{m,n}\}$ be as in Theorem 1, and let*

$$0 < \inf_{n\geq 0} \left\{ \lim_{m\to\infty} \inf \frac{H_{m,n}}{H_{m,0}} \right\} \leq \infty, \tag{12}$$

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} H_{m,n}\xi_n a_{n-\tau} \left|\eta_{m,n}\right|^{\alpha+1} < \infty. \tag{13}$$

Suppose that there exists a sequence $\{c_k\}_{k=0}^{\infty}$ satisfying

$$\lim_{n\to\infty} \sup \frac{1}{H_{m,k}} \sum_{n=k}^{m-1} H_{m,k}\xi_n \left[\psi_n + a_{n-\tau}\beta_n\eta_{m,n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1} \right] \geq c_k \tag{14}$$

and

$$\sum_{n=0}^{\infty} \frac{\xi_n}{a_{n-\tau}^{1/\alpha}} \left(\left[\frac{c_{n+1}}{\xi_{n+1}} - a_{n+1-\tau}\beta_{n+1} \right]^{+} \right)^{\alpha+1/\alpha} = \infty, \tag{15}$$

where

$$\left[\frac{c_{n+1}}{\xi_{n+1}} - a_{n+1-\tau}\beta_{n+1} \right]^{+} = \max\left\{ \frac{c_{n+1}}{\xi_{n+1}} - a_{n+1-\tau}\beta_{n+1}, 0 \right\}, \tag{16}$$

ψ_n is defined in (6), and $\eta_{m,n}$, β_n are defined in (7), then equation (E) is oscillatory.

PROOF. Suppose that $\{x_n\}$ is an eventually positive solution of equation (1). As in the proof of Theorem 1, (10) holds for all $m \geq k \geq N$. Hence,

$$\lim_{n\to\infty} \sup \frac{1}{H_{m,k}} \sum_{n=k}^{m-1} H_{m,k}\xi_n \left[\psi_n + a_{n-\tau}\beta_n\eta_{m,n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} |\eta_{m,n}|^{\alpha+1}\right] \leq w_k,$$

for $k \geq N$. It follows from (14) that

$$c_k \leq w_k, \qquad \text{for every } k \geq N, \tag{17}$$

and

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\xi_n (\psi_n + a_{n-\tau}\beta_n\eta_{m,n}) \geq c_N. \tag{18}$$

Define

$$u_m = \frac{1}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\xi_n |\eta_{m,n}| \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right),$$
$$v_m = \frac{\alpha}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\xi_n \frac{1}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right)^{\alpha+1/\alpha},$$

for $m = N+1, N+2, \ldots$. Then (10) and (18) imply that

$$\begin{aligned} &\lim_{m\to\infty} \inf [v_m - u_m] \\ &\leq w_N - \lim_{m\to\infty} \sup \frac{1}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\xi_n (\psi_n + a_{n-\tau}\beta_n\eta_{m,n}) \leq w_N - c_N < \infty. \end{aligned} \tag{19}$$

We shall next prove that

$$\sum_{n=N}^{\infty} \frac{\xi_n}{a_{n-\tau}^{1/\alpha}} \left[\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right]^{\alpha+1/\alpha} < \infty. \tag{20}$$

If we suppose that (20) fails, there exists an $N_1 > N$ such that

$$\sum_{n=N}^{m} \frac{\xi_n}{a_{n-\tau}^{1/\alpha}} \left[\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right]^{\alpha+1/\alpha} \geq \frac{M_2}{\alpha M_1}, \qquad \text{for all } m \geq N_1, \tag{21}$$

where M_2 is an arbitrary positive number and M_1 is a positive constant such that

$$\inf_{n\geq 0} \left\{\lim_{m\to\infty} \inf \frac{H_{m,n}}{H_{m,0}}\right\} > M_1 > 0. \tag{22}$$

Therefore,

$$\begin{aligned} v_m &= \frac{\alpha}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\xi_n \frac{1}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right)^{\alpha+1/\alpha} \\ &\geq \frac{\alpha}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\Delta \left[\sum_{k=N}^{n-1} \frac{\xi_k}{a_{k-\tau}^{1/\alpha}} \left(\frac{w_{k+1}}{\xi_{k+1}} - a_{k-\tau}\beta_k\right)^{\alpha+1/\alpha}\right] \\ &\geq -\frac{\alpha}{H_{m,N}} \sum_{n=N}^{m-1} \left[\sum_{k=N}^{n} \frac{\xi_k}{a_{k-\tau}^{1/\alpha}} \left(\frac{w_{k+1}}{\xi_{k+1}} - a_{k-\tau}\beta_k\right)^{\alpha+1/\alpha}\right] \Delta_2 H_{m,n} \\ &\geq \frac{M_2}{M_1 H_{m,N}} \sum_{n=N_1}^{m-1} (-\Delta_2 H_{m,n}) \\ &\geq \frac{M_2 H_{m,N_1}}{M_1 H_{m,0}}, \end{aligned}$$

for all $m \geq N_1$. By (22), there is an $N_2 \geq N_1$ such that $H_{m,N_1}/H_{m,0} \geq M_1$, for all $m \geq N_2$, and accordingly $v_m \geq M_2$, for all $m \geq N_2$. Since M_2 is arbitrary,

$$\lim_{m\to\infty} v_m = \infty. \tag{23}$$

Further, consider a sequence $\{m_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} m_k = \infty$ and

$$\lim_{k\to\infty} [v_{m_k} - u_{m_k}] = \lim_{m\to\infty} \inf [v_m - u_m].$$

Then, from (19), there exists a constant M such that

$$v_{m_k} - u_{m_k} \leq M, \qquad \text{for } k = 0, 1, 2, \dots. \tag{24}$$

Since (23) ensures that

$$\lim_{k\to\infty} v_{m_k} = \infty, \tag{25}$$

this and (24) implies

$$\lim_{k\to\infty} u_{m_k} = \infty. \tag{26}$$

By taking into account (25), from (24), we derive for k sufficiently large,

$$\frac{u_{m_k}}{v_{m_k}} - 1 \geq -\frac{M}{v_{m_k}} > -\frac{1}{2}.$$

Therefore,

$$\frac{u_{m_k}}{v_{m_k}} > \frac{1}{2}, \qquad \text{for all large } k,$$

which together with (26) implies

$$\lim_{k\to\infty} \frac{u_{m_k}^{\alpha+1}}{v_{m_k}^{\alpha}} = \infty. \tag{27}$$

On the other hand, by Holder's inequality, we have

$$\begin{aligned} u_{m_k} &= \frac{1}{H_{m_k,N}} \sum_{n=N}^{m_k-1} H_{m_k,n}\xi_n \left|\eta_{m_k,n}\right| \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right) \\ &\leq \left(\frac{\alpha}{H_{m_k,N}} \sum_{n=N}^{m_k-1} \frac{H_{m_k,n}\xi_n}{a_{n-\tau}^{\frac{1}{\alpha}}} \left[\frac{w_{n+1}}{\xi_{n+1}} - a_{n-\tau}\beta_n\right]^{\alpha+1/\alpha}\right)^{\alpha/\alpha+1} \\ &\quad \times \left(\frac{1}{\alpha^\alpha H_{m_k,N}} \sum_{n=N}^{m_k-1} H_{m_k,n}\xi_n a_{n-\tau} \left|\eta_{m_k,n}\right|^{\alpha+1}\right)^{1/\alpha+1}, \end{aligned}$$

and accordingly,

$$\frac{u_{m_k}^{\alpha+1}}{v_{m_k}^{\alpha}} \leq \frac{1}{\alpha^\alpha H_{m_k,N}} \sum_{n=N}^{m_k-1} H_{m_k,n}\xi_n a_{n-\tau} \left|\eta_{m,n}\right|^{\alpha+1}.$$

So, because of (28), we have

$$\lim_{k\to\infty} \frac{1}{\alpha^\alpha H_{m_k,N}} \sum_{n=N}^{m_k-1} H_{m_k,n}\xi_n a_{n-\tau} \left|\eta_{m,n}\right|^{\alpha+1} = \infty,$$

i.e.,

$$\lim_{m\to\infty} \frac{1}{H_{m,0}} \sum_{n=N}^{m-1} H_{m,n}\xi_n a_{n-\tau} \left|\eta_{m,n}\right|^{\alpha+1} = \infty,$$

which contradicts condition (13). Then (20) holds. Hence, by (17),

$$\sum_{n=0}^{\infty} \frac{\xi_n}{a_{n-\tau}^{1/\alpha}} \left(\left[\frac{c_{n+1}}{\xi_{n+1}} - a_{n+1-\tau}\beta_{n+1} \right]^+ \right)^{\alpha+1/\alpha} \leq \sum_{n=0}^{\infty} \frac{\xi_n}{a_{n-\tau}^{1/\alpha}} \left(\frac{w_{n+1}}{\xi_{n+1}} - a_{n+1-\tau}\beta_{n+1} \right)^{\alpha+1/\alpha} < \infty,$$

which contradicts (15). This completes the proof.

THEOREM 3. *Let $\{H_{m,n}\}$ and $\{h_{m,n}\}$ be as in Theorem 1, and let (12) hold. Suppose that*

$$\lim_{m\to\infty} \inf \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} H_{m,n}\xi_n \left(\psi_n + a_{n-\tau}\beta_n\eta_{m,n}\right) < \infty, \tag{28}$$

and there exists a sequence $\{c_k\}_{k=0}^{\infty}$ satisfying (16) and for $k > 0$,

$$\lim_{m\to\infty} \inf \frac{1}{H_{m,k}} \sum_{n=k}^{m-1} H_{m,k}\xi_n \left[\psi_n + a_{n-\tau}\beta_n\eta_{m,n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1}\right] \geq c_k, \tag{29}$$

where $\psi_n, \beta_n\eta_{m,n}$ are as in Theorem 1, then equation (E) is oscillatory.

PROOF. Suppose that $\{x_n\}$ is an eventually positive solution of equation (1). As in the proof of Theorem 2, (17) holds for all $m > k \geq N$. Using (28), we conclude that

$$\lim_{m\to\infty} \sup \left[v_m - u_m\right] \leq w_N - \lim_{m\to\infty} \inf \frac{1}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\xi_n \left(\psi_n + \eta_{m,n}a_{n-\tau}\beta_n\right) < \infty.$$

It follows from condition (29) that

$$\begin{aligned} c_0 &\leq \lim_{m\to\infty} \inf \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} H_{m,n}\xi_n \left[\psi_n + a_{n-\tau}\beta_n\eta_{m,n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1}\right] \\ &\leq \lim_{m\to\infty} \inf \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} H_{m,n}\xi_n \left[\psi_n + a_{n-\tau}\beta_n\eta_{m,n}\right] \\ &\quad - \lim_{m\to\infty} \inf \frac{1}{H_{m,N}} \sum_{n=N}^{m-1} H_{m,n}\xi_n \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left|\eta_{m,n}\right|^{\alpha+1}, \end{aligned}$$

so that (28) implies

$$\lim_{m\to\infty} \inf \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} H_{m,n}\xi_n a_{n-\tau} \left|\eta_{m,n}\right|^{\alpha+1} < \infty.$$

Consider a sequence $\{m_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} m_k = \infty$ satisfying

$$\lim_{k\to\infty} \left[v_{m_k} - u_{m_k}\right] = \lim_{m\to\infty} \sup \left[v_m - u_m\right].$$

Then, using the procedure of the proof of Theorem 2, we conclude that (20) is satisfied. The remainder of the proof proceeds as in the proof of Theorem 2.

THEOREM 4. *Assume that there exist two sequences $\{\xi_n \mid \xi_n > 0,\ n = 1,2,3,\dots\}$ and $\{F_n \mid n = 1,2,3,\dots\}$ such that*

$$F_n + \Delta\xi_n \geq 0, \qquad \text{for } n = 1,2,3,\dots, \tag{30}$$

$$\begin{gathered} \lim_{m\to\infty} \sup \sum_{n=0}^{m-1} \left[\prod_{i=0}^{n-1} \frac{\xi_{i+1} + F_i}{\xi_i}\right] \\ \left[\psi_n - \frac{\Delta\xi_n}{\xi_n} a_{n-\tau}\beta_n - \frac{\Delta\left(a_{n-1-\tau}\beta_{n-1}\right)}{\xi_n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left(\frac{F_n + \Delta\xi_n}{\xi_n}\right)^{\alpha+1}\right] = \infty, \end{gathered} \tag{31}$$

where

$$\psi_n = \gamma q_{n+1}\varepsilon^\alpha (1 - p_{n+1-\tau})^\alpha - \Delta (a_{n-1-\tau}\beta_{n-1}), \qquad \beta_n = -\frac{\Delta\xi_n}{2\xi_n}. \tag{32}$$

Then equation (1) is oscillatory.

PROOF. Assume that $\{x_n\}$ is a nonoscillatory of equation (1). As in the proof of Theorem 1, (9) holds for $n \geq N$. Let $y_n = w_n/\xi_n - a_{n-1-\tau}\beta_{n-1}$, then $y_n > 0$. From (9), we get

$$\Delta w_n \leq -\xi_n\psi_n + a_{n-\tau}\beta_n\Delta\xi_n - F_n y_{n+1} + (F_n + \Delta\xi_n)\, y_{n+1} - \frac{\alpha\xi_n}{a_{n-\tau}^{1.\alpha}} y_{n+1}^{\alpha+1/\alpha},$$

i.e.,

$$\begin{aligned}&\xi_{n+1}y_{n+1} - \xi_n y_n + \Delta (a_{n-1-\tau}\beta_{n-1}) \\ &\leq -\xi_n\psi_n + a_{n-\tau}\beta_n\Delta\xi_n - F_n y_{n+1} + (F_n + \Delta\xi_n)\, y_{n+1} - \frac{\alpha\xi_n}{a_{n-\tau}^{1/\alpha}} y_{n+1}^{\alpha+1/\alpha}.\end{aligned} \tag{33}$$

Using Lemma 1, we obtain for $n \geq N$,

$$(F_n + \Delta\xi_n)\, y_{n+1} - \frac{\alpha\xi_n}{a_{n-\tau}^{1/\alpha}} y_{n+1}^{\alpha+1/\alpha} \leq \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left(\frac{F_n + \Delta\xi_n}{\xi_n}\right)^{\alpha+1}. \tag{34}$$

Hence, from (33), we obtain

$$\frac{\xi_{n+1} + F_n}{\xi_n} y_{n+1} - y_n \leq -\psi_n - \frac{\Delta\xi_n}{\xi_n} a_{n-\tau}\beta_n - \frac{\Delta (a_{n-1-\tau}\beta_{n-1})}{\xi_n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left(\frac{F_n + \Delta\xi_n}{\xi_n}\right)^{\alpha+1},$$

i.e.,

$$\begin{aligned}&\Delta \left[\left(\prod_{i=0}^{n-1} \frac{\xi_{i+1} + F_i}{\xi_i}\right) y_n\right] \\ &\leq \left(\prod_{i=0}^{n-1} \frac{\xi_{i+1} + F_i}{\xi_i}\right) \left[-\psi_n - \frac{\Delta\xi_n}{\xi_n} a_{n-\tau}\beta_n - \frac{\Delta (a_{n-1-\tau}\beta_{n-1})}{\xi_n} - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left(\frac{F_n + \Delta\xi_n}{\xi_n}\right)^{\alpha+1}\right], \\ &\times \sum_{n=N}^{m-1} \left[\prod_{i=0}^{n-1} \frac{\xi_{i+1} + F_i}{\xi_i}\right] \left[\psi_n - \frac{\Delta\xi_n}{\xi_n} a_{n-\tau}\beta_n - \frac{\Delta(a_{n-1-\tau}\beta_{n-1})}{\xi_n} \right. \\ &\left. - \frac{a_{n-\tau}}{(\alpha+1)^{\alpha+1}} \left(\frac{F_n + \Delta\xi_n}{\xi_n}\right)^{\alpha+1}\right] \\ &\leq \left[\prod_{i=0}^{N} \frac{F_i + \xi_{i+1}}{\xi_i}\right] y_N - \left[\prod_{0}^{m} \frac{F_i + \xi_{i+1}}{\xi_i}\right] y_m \leq \left[\prod_{i=0}^{N} \frac{F_i + \xi_{i+1}}{\xi_i}\right] y_N,\end{aligned}$$

which contradicts (30). This completes the proof.

REMARK 1. In the case when $\phi(x) \equiv x$, $\alpha = 1$, our main results, Theorems 1–3, reduce to all the results of [1].

REMARK 2. In the case when $p_n \equiv 0$, $\alpha = 1$, $\gamma = 1$, Theorem 1 improves Theorem 2 of [5].

REMARK 3. In the case when $p_n \equiv 0$, $\tau = 0$, $f(x) = |x|^{\alpha-1}x$, Theorems 1–3 improve the discrete analogue of the results in [8], which studies the continuous case of equation (3).

REMARK 4. To the authors' knowledge, Theorem 4 is new, even for equations (2)–(4).

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