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Vector fields and a family of linear type modules related to free divisors[☆]

Cleto B. Miranda Neto

Departamento de Matemática, CCEN, Universidade Federal da Paraíba, 58051-900, João Pessoa, Paraíba, Brazil

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ABSTRACT

This paper has three main goals. We start describing a method for computing the polynomial vector fields tangent to a given algebraic variety; this is of interest, for instance, in view of (effective) foliation theory. We then pass to furnishing a family of modules of *linear type* (that is, the Rees algebra equals the symmetric algebra), formed with vector fields related to suitable pairs of algebraic varieties, one of them being a *free divisor* in the sense of K. Saito. Finally, we derive freeness criteria for modules retaining a certain tangency feature, so that, in particular, well-known criteria for free divisors are recovered.

1. Introduction

Throughout, we denote by $\mathbb{A}^n = \mathbb{A}^n_k$ the n-dimensional affine space over a ground field k, with affine coordinates $t_1, \ldots, t_n$. The ring $\mathscr{O} = \mathscr{O}_{\mathbb{A}^n}$ of regular functions on $\mathbb{A}^n$ may be realized as the polynomial ring $k[t_1, \ldots, t_n]$, with $t_1, \ldots, t_n$ viewed as variables over k. For geometric reasons, we assume that k is algebraically closed of characteristic zero.

An *ambient vector field* is an algebraic vector field defined globally on $\mathbb{A}^n$, by which we mean an ordered collection ϑ of polynomial functions $h_i\colon \mathbb{A}^n \to k$, $i = 1, \ldots, n$, that to each point $p \in \mathbb{A}^n$ associates the vector ϑ_p with coordinates $h_1(p), \ldots, h_n(p)$ with respect to the basis $\{(\frac{\partial}{\partial t_i})_p\}_{i=1}^n$ of $T_p\mathbb{A}^n \simeq \mathbb{A}^n$.

One may write $\vartheta = \sum_{i=1}^n h_i \frac{\partial}{\partial t_i}$, so that the $\mathscr{O}$-module $\mathscr{D}_{\mathbb{A}^n}$ of all ambient vector fields is simply the (free) module $\mathscr{D}_{\mathbb{A}^n} = \oplus_{i=1}^n \mathscr{O}\frac{\partial}{\partial t_i}$ formed with the k-derivations of $\mathscr{O}$. The singular set of any given $\vartheta \in \mathscr{D}_{\mathbb{A}^n}$ is

$$\operatorname{Sing}\vartheta = \{p \in \mathbb{A}^n \mid \vartheta_p = 0\} = \bigcap_{i=1}^n \{h_i = 0\}.$$

By a *variety* we mean an algebraic variety over k. The singular locus of a variety $\mathscr{W} \subset \mathbb{A}^n$ will be denoted $\operatorname{Sing}\mathscr{W}$. Its (embedded) tangent space $T_p\mathscr{W}$ at a smooth point $p \in \mathscr{W}$ is viewed, as usual, as the kernel of the k-linear map $k^n \to k^m$ whose matrix in the canonical bases is the Jacobian matrix $(\frac{\partial f_i}{\partial t_j}(p))$ (evaluated at p), for some (in fact, any) generating set $\{f_1, \ldots, f_m\} \subset \mathscr{O}$ of the vanishing function ideal of $\mathscr{W}$.

The following simple fact will be used several times in this paper.

Proposition 1.1. *Let $\mathscr{V} \subseteq \mathscr{W} \subset \mathbb{A}^n$ be reduced varieties with vanishing function ideals $I_{\mathscr{V}}, I_{\mathscr{W}} \subset \mathscr{O}$, respectively. For any $\vartheta \in \mathscr{D}_{\mathbb{A}^n}$ such that $\mathscr{V} \nsubseteq \operatorname{Sing}\mathscr{W} \cup \operatorname{Sing}\vartheta$, the following are equivalent:*

(i) *The restriction of ϑ to $\mathscr{V}$ is tangent to $\mathscr{W}$, that is, $\vartheta_p \in T_p\mathscr{W}$ for each point $p \in \mathscr{V} \setminus (\operatorname{Sing}\mathscr{W} \cup \operatorname{Sing}\vartheta)$;*
(ii) *As a k-derivation, ϑ satisfies $\vartheta(I_{\mathscr{W}}) \subseteq I_{\mathscr{V}}$, that is, $\vartheta(f) \in I_{\mathscr{V}}$, for all $f \in I_{\mathscr{W}}$.*

In particular, if $\mathscr{V} \nsubseteq \operatorname{Sing}\vartheta$, then ϑ is tangent to $\mathscr{V}$ if and only if $\vartheta(I_{\mathscr{V}}) \subseteq I_{\mathscr{V}}$.

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E-mail address: cleto@mat.ufpb.br.

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Proof. Write generators $I_{\mathscr{W}} = (f_1, \ldots, f_m)$. The tangent space $T_p\mathscr{W}$ of the embedding $\mathscr{W} \subset \mathbb{A}^n$ at a smooth point $p \in \mathscr{W}$ is defined by the linear equations $\sum_{i=1}^n \frac{\partial f_j}{\partial t_i}(p)\, t_i = 0$, $j = 1, \ldots, m$. Given $\vartheta = \sum_{i=1}^n h_i \frac{\partial}{\partial t_i} \in \mathscr{D}_{\mathbb{A}^n}$, set $\mathscr{U} = \mathscr{V} \setminus (\mathrm{Sing}\, \mathscr{W} \cup \mathrm{Sing}\, \vartheta)$, which is non-empty by hypothesis. For each point $p \in \mathscr{U}$, saying that $\vartheta_p \in T_p\mathscr{W}$ means

$$\sum_{i=1}^n h_i(p) \frac{\partial f_j}{\partial t_i}(p) = 0, \quad j = 1, \ldots, m,$$

that is, regarding ϑ as a k-derivation of $\mathscr{O}$, the functions $\vartheta(f_j)$'s vanish on $\mathscr{U}$, and hence they must vanish on $\mathscr{V}$ since $\mathscr{U} \subset \mathscr{V}$ is a Zariski dense open subset. This means that $\vartheta(f_j) \in I_{\mathscr{V}}$, for every j, which by Leibniz's rule is equivalent to $\vartheta(f) \in I_{\mathscr{V}}$ for all $f \in I_{\mathscr{W}}$. For the particular assertion, take $\mathscr{V} = \mathscr{W}$. □

Needless to say, algebraic vector fields play a classical role in Algebraic Geometry, as well as in Commutative and Non-Commutative Algebra if we realize them as derivations of the base ring. Here, essentially under an algebraic viewpoint, we are interested in suitable ambient vector fields, mainly those yielding a family of *linear type* modules, which in turn will lead us to the important notion of *free divisor* (due to K. Saito). Also, as a byproduct of a corollary of a depth-formula obtained, we recover well-known criteria for a divisor to be free.

In order to be more precise, we pass to a description of the content of the paper.

Section 2 is concerned with Proposition 2.1, which presents a simple method for the computation of all the ambient vector fields tangent to a given reduced variety $\mathscr{V} \subset \mathbb{A}^n$ (the projective setup is allowed, so that $\mathbb{A}^n$ may be replaced by a projective space $\mathbb{P}^n$ over k; see Remark 2.2(ii)). Explicitly, one is focused on the set

$$\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}) = \{\vartheta \in \mathscr{D}_{\mathbb{A}^n} \mid \mathscr{V} \nsubseteq \mathrm{Sing}\,\vartheta,\ \vartheta_p \in T_p\mathscr{V},\ \forall\, p \in \mathscr{V} \setminus (\mathrm{Sing}\,\mathscr{V} \cup \mathrm{Sing}\,\vartheta)\},$$

viewed as an $\mathscr{O}$-module. If $\vartheta \in \mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V})$, one says that ϑ *leaves* $\mathscr{V}$ *invariant*, or that $\mathscr{V}$ is ϑ*-invariant*. The investigation of this module is of interest, for instance, in view of holomorphic foliation theory (cf., e.g., [2,3]).

At first sight, one is led to think that working first on the coordinate ring of $\mathscr{V}$ would be necessary for obtaining a generating set for $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V})$; here, the operations required within our algorithm may be performed simply over $\mathscr{O}$. Loosely speaking, if $\mathscr{V}$ has defining ideal $I_{\mathscr{V}} = (f_1, \ldots, f_m) \subset \mathscr{O}$, one first computes free presentations, over $\mathscr{O}$, of suitable ideals related to $I_{\mathscr{V}}$ and to the gradient ideals of the f_i's, in view of obtaining certain modules derived from the presentation matrices; finally, one computes their intersection (preferably with the aid of an algebraic computer system, *e.g.* [1]) so that the resulting matrix yields the wanted generators. The technique is illustrated in Example 2.3, concerning the elliptic quartic curve $\mathscr{E} \subset \mathbb{P}^3$ (in particular, we shall reobtain a vector field tangent to $\mathscr{E}$ used by M. Soares in order to show that one of his bounds is attained, in the context of his main contribution to Poincaré's problem on foliations; cf. [10]). The drawback of our method is that the steps involved do not carry precise information about the module-theoretic structure of $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V})$; on the other hand, it does not depend on any particular class of ideals.

In Section 3, we produce a family of linear type modules (that is, modules for which the Rees and symmetric algebras coincide; see [9]) formed with distinguished vector fields associated to *free divisors* (cf. [8]). One starts with reduced varieties $\mathscr{V} \subset \mathscr{F} \subset \mathbb{A}^n$ (that once again may be taken projective), where $\mathscr{F} = \{f = 0\}$ is a quasi-homogeneous free divisor with gradient ideal $\mathscr{G}_f$. In analogy with the module $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V})$ written above, let $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F})$ stand for the $\mathscr{O}$-module formed with the $\vartheta \in \mathscr{D}_{\mathbb{A}^n}$ such that $\mathscr{V} \nsubseteq \mathrm{Sing}\,\mathscr{F} \cup \mathrm{Sing}\,\vartheta$ and $\vartheta_p \in T_p\mathscr{F}$, for all $p \in \mathscr{V} \setminus (\mathrm{Sing}\,\mathscr{F} \cup \mathrm{Sing}\,\vartheta)$. In this setting, we prove in Proposition 3.2 that if the ideal $\mathscr{G}_f \cap I_{\mathscr{V}} \subset \mathscr{O}$ is of linear type, then so is $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F})$ as an $\mathscr{O}$-module. The proof makes use of the technique of (non-generic) *Bourbaki ideals*, developed in [9] with focus on Rees algebras of finitely generated modules with rank.

Finally, in Section 4 (where we adopt the projective setup), we derive freeness criteria for the $\mathscr{O}$-module $\mathcal{T}_{\mathbb{P}^n/k}(\mathscr{V}, \mathscr{H})$, where $\mathscr{V} \subseteq \mathscr{H} \subset \mathbb{P}^n$ are reduced projective varieties, with $\mathscr{H}$ a (not necessarily free) hypersurface. This is done in Corollary 4.2. The reason for characterizing freeness is that the non-free case is the situation of interest into the preceding section, since clearly free modules are already of linear type. In essence, such criteria will follow from the depth-formula given in Proposition 4.1. As an immediate byproduct, we recover well-known criteria for $\mathscr{H}$ to be a free divisor (Corollary 4.4).

2. Computing the vector fields tangent to a variety

Given an ambient vector field $\vartheta \in \mathscr{D}_{\mathbb{A}^n}$, a classical problem in foliation theory is to describe the algebraic varieties that ϑ leaves invariant (cf., *e.g.*, [2]). However, as it is well-known, such varieties may not exist in general, even in the case $n = 2$ (see [3] for a nice algorithm to check whether a given derivation on the plane admits an invariant algebraic curve).

Here we shall proceed in the opposite direction: given a reduced algebraic variety $\mathscr{V} \subsetneq \mathbb{A}^n$, we shall describe an algorithm that will allow us to compute the ambient vector fields tangent to $\mathscr{V}$ (that is, leaving $\mathscr{V}$ invariant).

Let $I_{\mathscr{V}} = (f_1, \ldots, f_m) \subset \mathscr{O}$ be the (radical) ideal of the polynomial functions vanishing on $\mathscr{V}$. Given $s \in \{1, \ldots, m\}$, one may look at the gradient ideal of f_s,

$$\mathscr{G}_s = \mathscr{G}_{f_s} = \left(\frac{\partial f_s}{\partial t_1}, \ldots, \frac{\partial f_s}{\partial t_n}\right) \subset \mathscr{O}.$$

Also, consider the ideal $\mathbb{J}_s^{\mathscr{V}} = \mathscr{G}_s + I_{\mathscr{V}} \subset \mathscr{O}$, together with a fixed free presentation

$$\mathscr{O}^{r_s} \xrightarrow{\psi_s^{\mathscr{V}}} \mathscr{O}^{n+m} \longrightarrow \mathbb{J}_s^{\mathscr{V}} \longrightarrow 0,$$

where r_s is a positive integer and $\psi_s^{\mathscr{V}}$ is a $(n+m) \times r_s$ presentation matrix of $\mathbb{J}_s^{\mathscr{V}}$, taken with respect to the canonical bases of the free modules involved. We emphasize that the (ordered, signed) generating set

$$\left\{\frac{\partial f_s}{\partial t_1}, \ldots, \frac{\partial f_s}{\partial t_n}, f_1, \ldots, f_m\right\}$$

of $\mathbb{J}_s^{\mathscr{V}}$ is fixed.

A piece of notation: given integers $u, v \geq 1$ and a $(n+u) \times v$ matrix φ with entries in $\mathscr{O}$, we shall denote by $\mathbb{I}(\varphi)$ the submodule of $\mathscr{O}^n$ generated by the column-vectors of the $n \times v$ submatrix of φ obtained after deletion of its last u rows.

For a reason that will be clarified in the proof of the next result, we shall be interested in each generator f_s of $I_{\mathscr{V}}$ with $s \in \Gamma$, where $\Gamma = \{j \in \{1, \ldots, m\} \mid (\mathscr{G}_{f_j}, f_j) \nsubseteq I_{\mathscr{V}}\}$. This is indeed just a technicality, since very often one has $\Gamma = \{1, \ldots, m\}$.

Now, once one is able to furnish a free presentation for each of the ideals $\mathbb{J}_s^{\mathscr{V}} \subset \mathscr{O}$, $s \in \Gamma$, one may also explicit the modules $\mathbb{I}(\psi_s^{\mathscr{V}}) \subset \mathscr{O}^n$, and subsequently a computation may be performed in order to obtain generators for their intersection. This can be done easily with the aid of an algebraic computer system. For instance, by using the well-known *Macaulay* of Bayer and Stillman (cf. [1]), each matrix $\psi_s^{\mathscr{V}}$ may be created with the `mat` command, and the intersection goes in one step with the `intersect` command. The output is an explicit matrix $\psi_{\mathscr{V}}$ (with n rows) fitting into an equality

$$\mathbb{I}(\psi_{\mathscr{V}}) = \bigcap_{s\in\Gamma} \mathbb{I}(\psi_s^{\mathscr{V}}).$$

Such matrix will play a crucial role into the effective computation of suitable vector fields. In fact, write

$$\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}) = \{\vartheta \in \mathscr{D}_{\mathbb{A}^n} \mid \mathscr{V} \nsubseteq \operatorname{Sing}\vartheta,\ \vartheta_p \in T_p\mathscr{V},\ \forall\, p \in \mathscr{V} \setminus (\operatorname{Sing}\mathscr{V} \cup \operatorname{Sing}\vartheta)\},$$

which may be viewed naturally as an $\mathscr{O}$-submodule of $\mathscr{D}_{\mathbb{A}^n} \simeq \mathscr{O}^n$. Note that the condition $\mathscr{V} \nsubseteq \operatorname{Sing}\vartheta$ is natural, since otherwise the vector field ϑ simply vanishes along $\mathscr{V}$. Thus, one obtains the following proposition:

Proposition 2.1. *Keeping the preceding notation, one may identify*

$$\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}) = \mathbb{I}(\psi_{\mathscr{V}}).$$

In other words, a vector field on $\mathbb{A}^n$ is tangent to $\mathscr{V}$ if and only if it may be expressed as an $\mathscr{O}$-linear combination of the column-vectors of $\psi_{\mathscr{V}}$.

Proof. Write $\mathscr{V} = \cap_{s=1}^m \mathscr{V}_s$, with $\mathscr{V}_s = \{f_s = 0\}$ for reduced $f_s \in \mathscr{O}$. Saying that $s \in \Gamma$ means $\mathscr{V} \nsubseteq \operatorname{Sing}\mathscr{V}_s$, and hence, setting

$$\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{V}_s) = \{\vartheta \in \mathscr{D}_{\mathbb{A}^n} \mid \mathscr{V} \nsubseteq \operatorname{Sing}\vartheta,\ \vartheta_p \in T_p\mathscr{V}_s,\ \forall\, p \in \mathscr{V} \setminus (\operatorname{Sing}\mathscr{V}_s \cup \operatorname{Sing}\vartheta)\},$$

one may apply Proposition 1.1 in order to realize $\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{V}_s)$ as the module formed with the k-derivations $\vartheta : \mathscr{O} \to \mathscr{O}$ such that $\vartheta(f_s) \in I_{\mathscr{V}}$. We claim that

$$\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}) = \bigcap_{s\,\in\,\Gamma} \mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{V}_s).$$

Again by Proposition 1.1, the module $\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V})$ may be seen as the set of k-derivations of $\mathscr{O}$ preserving the ideal $I_{\mathscr{V}}$. Now, denote by $\mathscr{N}$ the intersection on the right-hand side of the proposed equality above, and note that the inclusion $\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}) \subset \mathscr{N}$ is clear. For the other one, take $\vartheta \in \mathscr{N}$. It suffices to show that $\vartheta(f_s) \in I_{\mathscr{V}}$, for $s \in \{1, \ldots, m\}$. This is clear if $s \in \Gamma$, by the above algebraic description of the module $\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{V}_s)$; otherwise, one has $\frac{\partial f_s}{\partial t_i} \in I_{\mathscr{V}}$ for every i, and hence any k-derivation will send f_s into $I_{\mathscr{V}}$. This shows the claim.

Now, as $\mathbb{I}(\psi_{\mathscr{V}}) = \cap_{s\in\Gamma}\mathbb{I}(\psi_s^{\mathscr{V}})$, we only need to show that

$$\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{V}_s) = \mathbb{I}(\psi_s^{\mathscr{V}}).$$

Pick $\vartheta \in \mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{V}_s)$ and write $\vartheta = \sum_{i=1}^n h_i \frac{\partial}{\partial t_i}$, for certain h_i's in $\mathscr{O}$. One gets

$$\sum_{i=1}^{n} h_i \frac{\partial f_s}{\partial t_i} + \sum_{t=1}^{m} g_t f_t = 0,$$

with $g_t \in \mathscr{O}$, $t = 1, \ldots, m$. This means that $v_s = (h_1, \ldots, h_n, g_1, \ldots, g_m) \in \mathscr{O}^{n+m}$ is a relation, or first-order syzygy, of the ideal $\mathbb{J}_s^{\mathscr{V}}$. Since the module of relations of $\mathbb{J}_s^{\mathscr{V}}$ is generated by the column-vectors of the $(n+m) \times r_s$ matrix $\psi_s^{\mathscr{V}} = (h_{ij}^{(s)})$ (whose (i,j)-entry will be denoted h_{ij} for simplicity), there exist polynomial functions $q_1, \ldots, q_{r_s} \in \mathscr{O}$ such that $\upsilon_s = \sum_{j=1}^{r_s} q_j \upsilon_j$, where $\upsilon_j = (h_{1j}, \ldots, h_{nj}, h_{n+1,j}, \ldots, h_{n+m,j})$ is the jth column-vector of $\psi_s^{\mathscr{V}}$. Thus

$$\vartheta = \sum_{i=1}^{n} h_i \frac{\partial}{\partial t_i} = \sum_{i=1}^{n} \left(\sum_{j=1}^{r_s} q_j h_{ij} \right) \frac{\partial}{\partial t_i} = \sum_{j=1}^{r_s} q_j \left(\sum_{i=1}^{n} h_{ij} \frac{\partial}{\partial t_i} \right) = \sum_{j=1}^{r_s} q_j \widetilde{\upsilon}_j,$$

where

$$\widetilde{\upsilon}_j = \sum_{i=1}^{n} h_{ij} \frac{\partial}{\partial t_i}, \quad j = 1, \ldots, r_s.$$

Therefore,

$$\mathfrak{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{V}_s) \subseteq \sum_{j=1}^{r_s} \mathscr{O} \widetilde{\upsilon}_j = \mathbb{I}(\psi_s^{\mathscr{V}}),$$

where, in the equality, each $\widetilde{\upsilon}_j$ is being identified with $(h_{1j}, \ldots, h_{nj}) \in \mathscr{O}^n$. To show the opposite inclusion, it suffices to check that $\widetilde{\upsilon}_j$ conducts f_s into $I_{\mathscr{V}} \subset \mathscr{O}$, for each j. But this is clear from the relations

$$\sum_{i=1}^{n} h_{ij} \frac{\partial f_s}{\partial t_i} + \sum_{t=1}^{m} h_{n+t,j} f_t = 0, \quad j = 1, \ldots, s,$$

which may be rewritten as

$$\widetilde{\upsilon}_j(f_s) = -\sum_{t=1}^{m} h_{n+t,j} f_t \in I_{\mathscr{V}},$$

as needed. □

Remarks 2.2. (i) If $\mathscr{V} = \{f = 0\}$ is already a hypersurface, then a set of generators for the $\mathscr{O}$-module of the ambient vector fields tangent to $\mathscr{V}$ may be derived from the syzygies of the (lifted) Jacobian ideal $(\mathscr{G}_f, f)$ of f, with no need of computing intersections.

(ii) The proof above remains exactly the same if one takes $\mathscr{V}$ to be a projective variety in some projective space $\mathbb{P}^n$ over k; that is, the method works regardless of any grading on the base polynomial ring. Of course, in full analogy with the affine setup considered up to now, we may naturally describe the set $\mathscr{D}_{\mathbb{P}^n} = \{\text{polynomial vector fields on } \mathbb{P}^n\}$ in terms of the usual $\mathbb{Z}$-graded derivation module of the standard graded polynomial ring $\mathscr{O} = \oplus_{s \geq 0} \mathscr{O}_s = k[t_1, \ldots, t_{n+1}]$. More precisely, given $s \geq -1$, its sth homogeneous piece may be thought of as

$$[\mathscr{D}_{\mathbb{P}^n}]_s = \bigoplus_{i=1}^{n+1} \mathscr{O}_{s+1} \frac{\partial}{\partial t_i}.$$

One puts $[\mathscr{D}_{\mathbb{P}^n}]_s = 0$ if $s \leq -2$. Given reduced projective varieties $\mathscr{V} \subset \mathscr{W} \subset \mathbb{P}^n$, one may also consider the $\mathscr{O}$-submodule $\mathfrak{T}_{\mathbb{P}^n/k}(\mathscr{V}, \mathscr{W}) \subset \mathscr{D}_{\mathbb{P}^n}$ (through the same definition as in the affine case), which is easily seen to be homogeneous as it inherits the grading above.

We illustrate our method determining $\mathfrak{T}_{\mathbb{P}^3/\mathbb{C}}(\mathscr{E})$, where $\mathscr{E}$ is the elliptic quartic curve in projective 3-space.

Example 2.3 ($k = \mathbb{C}$). The elliptic quartic curve $\mathscr{E} \subset \mathbb{P}^3$ may be viewed as the complete intersection $\mathscr{E} = \mathscr{V}_1 \cap \mathscr{V}_2$, where $\mathscr{V}_1$ and $\mathscr{V}_2$ are the hypersurfaces defined respectively by the polynomials

$$f_1 = t_1^2 + t_2^2 + t_3^2 + t_4^2, \qquad f_2 = t_1 t_3 + t_2 t_4$$

in $\mathscr{O} = \mathbb{C}[t_1, t_2, t_3, t_4]$. Let us obtain first the module $\mathbb{I}(\psi_1^{\mathscr{E}})$. Computing the syzygies of the ideal

$$\mathbb{J}_1^{\mathscr{E}} = \mathscr{G}_1 + I_{\mathscr{E}} = (2t_1, 2t_2, 2t_3, 2t_4, f_1, f_2)$$

and deleting the last two rows from its presentation matrix $\psi_1^{\mathscr{E}}$, one sees that $\mathbb{I}(\psi_1^{\mathscr{E}})$ is generated by the column-vectors of the matrix

$$\begin{pmatrix} -t_4 & 0 & 0 & -t_3 & 0 & -t_2 & t_1 & t_3 \\ 0 & -t_4 & 0 & 0 & -t_3 & t_1 & t_2 & t_4 \\ 0 & 0 & -t_4 & t_1 & t_2 & 0 & t_3 & 0 \\ t_1 & t_2 & t_3 & 0 & 0 & 0 & t_4 & 0 \end{pmatrix},$$

which, by the proof of Proposition 2.1, furnishes exactly the module $\mathcal{T}_{\mathbb{P}^3/\mathbb{C}}(\mathscr{E}, \mathscr{V}_1)$. In the same manner, from the syzygies of $\mathbb{J}_2^{\mathscr{E}} = (t_3, t_4, t_1, t_2, f_1, f_2)$, one obtains $\mathbb{I}(\psi_2^{\mathscr{E}}) = \mathcal{T}_{\mathbb{P}^3/\mathbb{C}}(\mathscr{E}, \mathscr{V}_2)$ as given by the column-vectors of

$$\begin{pmatrix} 0 & t_2 & 0 & t_1 & 0 & -t_4 & t_3 & 0 \\ 0 & 0 & t_2 & 0 & t_1 & t_3 & t_4 & 0 \\ -t_2 & 0 & 0 & -t_3 & -t_4 & 0 & t_1 & t_3 \\ t_1 & -t_3 & -t_4 & 0 & 0 & 0 & t_2 & t_4 \end{pmatrix}.$$

Finally, one computes the intersection $\mathbb{I}(\psi_1^{\mathscr{E}}) \cap \mathbb{I}(\psi_2^{\mathscr{E}})$ – with the aid, for instance, of *Macaulay* (cf. [1]) – and applies Proposition 2.1. It follows that a set of generators for the $\mathscr{O}$-module $\mathcal{T}_{\mathbb{P}^3/\mathbb{C}}(\mathscr{E})$ is as displayed in the matrix

$$\begin{pmatrix} -t_2 & -t_4 & t_3 & t_1 & t_3^2 & t_3t_4 & t_2t_3 & -t_2t_4 \\ t_1 & t_3 & t_4 & t_2 & t_3t_4 & -t_3^2 & t_2t_4 & t_2t_3 \\ -t_4 & -t_2 & t_1 & t_3 & -t_2t_4 & t_2t_3 & t_3t_4 & t_3^2 \\ t_3 & t_1 & t_2 & t_4 & t_2t_3 & t_2t_4 & -t_3^2 & t_3t_4 \end{pmatrix}.$$

This generating set is minimal, and notice that the standard Euler vector field ε appears in the fourth column. Now, denote by ϑ_1, ϑ_2 respectively the ambient vector fields associated to the first 2 columns, and consider

$$\mathcal{Z} = -t_4^2\, \vartheta_1 + t_2t_4\, \vartheta_2 + (t_3t_4 - t_1t_2)\, \varepsilon,$$

which then must be tangent to $\mathscr{E}$ – in other words, it leaves $\mathscr{E}$ invariant. Its expression in the open piece $\mathscr{U}_4$ where $t_4 \neq 0$, in affine coordinates $z_i = t_i/t_4$, $i = 1, 2, 3$, is $\mathcal{Z} = \mathcal{Z}|_{\mathscr{U}_4}$ given by

$$\mathcal{Z} = (-z_1^2z_2 + z_1z_3)\frac{\partial}{\partial z_1} + (-z_1z_2^2 + 2z_2z_3 - z_1)\frac{\partial}{\partial z_2} + (-z_1z_2z_3 - z_2^2 + z_3^2 + 1)\frac{\partial}{\partial z_3}.$$

This is precisely the vector field exhibited by Soares in [10, Example 2], while showing that one of his bounds (on degrees of smooth complete intersections invariant by foliations) is attained.

3. Linear type modules related to free divisors

Our goal in this section is to detect a family of linear type modules associated to suitable pairs of varieties $\mathscr{V} \subset \mathscr{F}$, with $\mathscr{F}$ a free divisor (in the sense of K. Saito). First, we briefly recall the necessary background.

The notion of linear type module depends viscerally on the definition of the Rees algebra of a module. Let A be a noetherian commutative ring with identity, and let E be a finitely generated A-module with (generic, constant, positive) rank. Let $\mathscr{S} = \mathrm{Sym}_A(E)$ be the symmetric algebra of E — which is, in some sense, the ancestor of the so-called *blowup algebras*. Usually, $\mathscr{S}$ is computed by means of some (in fact, any) free presentation of E over A. Let $\tau \subset \mathscr{S}$ be the A-torsion submodule of $\mathscr{S}$, which is seen to be an ideal of $\mathscr{S}$. Following [9], one defines the *Rees algebra* of E to be the residue class ring $\mathscr{R}_A(E) = \mathscr{S}/\tau$. We point out that an alternative was proposed in generality (for modules possibly with no rank) by Eisenbud et al. [5]. In spite of these comments, we shall not deal directly with any definition of Rees algebra; instead, for our purposes, it will suffice to establish a setting in order to apply a crucial auxiliary result (Lemma 3.1 below).

It may be difficult, in general, to predict about a presentation of the ring $\mathscr{R}_A(E)$. An indication of this fact is that its defining equations may have high degrees. Fortunately, there exists a distinguished class of modules for which this situation is under control. One says that E is *of linear type* if $\tau = (0)$, that is, the defining equations of the Rees algebra are precisely given by the *linear* forms defining the symmetric algebra $\mathscr{S}$. Concerning the subject, we quote a result that we shall use as an efficient tool. Recall that an ideal $\mathscr{B}$ of a noetherian ring A is said to be a *Bourbaki ideal* of a finitely generated A-module E with rank if there exists a short exact sequence

$$0 \longrightarrow G \longrightarrow E \longrightarrow \mathscr{B} \longrightarrow 0,$$

where G is a free A-module.

Lemma 3.1. *Let A be a noetherian ring and let E be a finitely generated A-module with rank. Let $\mathscr{B}$ be a Bourbaki ideal of E. If $\mathscr{B}$ is of linear type, then the A-module E is of linear type.*

A proof of this fact may be found in [9, Proposition 3.11(b)].

Now, we revisit the previous setting of modules of vector fields with a peculiar tangency feature. To a pair of varieties $\mathscr{V} \subset \mathscr{W} \subset \mathbb{A}^n$ we may associate the $\mathscr{O}$-module

$$\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{W}) = \{\vartheta \in \mathscr{D}_{\mathbb{A}^n} \mid \mathscr{V} \nsubseteq \mathrm{Sing}\,\vartheta,\ \vartheta_p \in T_p\mathscr{W},\ \forall\, p \in \mathscr{V} \setminus (\mathrm{Sing}\,\mathscr{W} \cup \mathrm{Sing}\,\vartheta)\}.$$

In virtue of the importance of discovering new families of linear type modules, we focus on the task of producing pairs of varieties $\mathscr{V} \subset \mathscr{W}$ for which the module $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{W})$ has this property, that is,

$$\mathscr{R}_{\mathscr{O}}\left(\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{W})\right) = \mathrm{Sym}_{\mathscr{O}}\left(\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{W})\right).$$

As it turned out, the investigation led us to K. Saito's well-known concept of *free divisor*, originally introduced in the local complex analytic setup, in his work [8]. Following Saito's idea, one says that a reduced algebraic hypersurface $\mathscr{F} \subset \mathbb{A}^n$ (as well as any $f \in \mathscr{O}$ defining $\mathscr{F}$) is *free* if its module of logarithmic vector fields $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{F})$ — typically denoted $\mathrm{Der}(\log \mathscr{F})$, also $\mathrm{Der}(-\log \mathscr{F})$ — is free, necessarily of rank n (the Krull dimension of $\mathscr{O}$).

In the same paper, Saito established an efficient way to check whether a given divisor is free, provided one has n vector fields tangent to it. More precisely, consider a reduced, non-constant $f \in \mathbb{C}[t_1, \ldots, t_n]$ and ambient vector fields $\vartheta_j = \sum_{i=1}^n g_{ij} \frac{\partial}{\partial t_i}$, $j = 1 \ldots, n$, each of them tangent to $\mathscr{F} = \{f = 0\}$. Saito's criterion guarantees that $\mathscr{F}$ is a free divisor if the matrix (g_{ij}) has determinant αf, for some non-zero $\alpha \in \mathbb{C}$ (we shall apply this later into Example 3.4). The subject has been target of intensive research, within the detection of new classes of free divisors and their connections to complex singularity theory (cf., *e.g.*, [4,11,12]).

Recall that a polynomial function $f \in \mathscr{O} = k[t_1, \ldots, t_n]$ is *quasi-homogeneous* if f becomes homogeneous after certain degrees are assigned to the t_i's. Hence, in this case, Euler's identity yields $f \in \mathscr{G}_f$ (in fact, a classical result of K. Saito states that the property $f \in \mathscr{G}_f$ characterizes quasi-homogeneity in the analytic setting; cf. [7]). The hypersurface defined by a quasi-homogeneous reduced polynomial is also said to be quasi-homogeneous.

We now prove the result of this section, which is stated for suitable pairs of varieties in an affine space, but it is immediately seen to hold in the projective setup as well.

Proposition 3.2. *Let $\mathscr{F} = \{f = 0\} \subset \mathbb{A}^n$ be a quasi-homogeneous free hypersurface and let $\mathscr{V} \subset \mathscr{F}$ be a reduced subvariety such that the ideal $\mathscr{G}_f \cap I_{\mathscr{V}} \subset \mathscr{O}$ is of linear type. Then, the $\mathscr{O}$-module $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F})$ is of linear type.*

Proof. First, Proposition 1.1 yields that $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F})$ may be realized as the $\mathscr{O}$-module formed with the k-derivations of $\mathscr{O}$ sending f into $I_{\mathscr{V}}$ (note that this module is torsion-free and has rank n).

Now, let $\upsilon \in \mathrm{Hom}_{\mathscr{O}}(\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F}), \mathscr{O})$ be given by $\upsilon(\vartheta) = \vartheta(f)$, for $\vartheta \in \mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F})$. Clearly, its kernel may be identified with the module

$$\mathcal{Z}(\mathscr{G}_f) = \left\{ (h_1, \ldots, h_n) \in \mathscr{O}^n \mid \sum_{i=1}^n h_i \frac{\partial f}{\partial t_i} = 0 \right\}$$

of first-order syzygies of the gradient ideal of f. As to the image of υ, it is easily seen to coincide with the ideal $\mathscr{G}_f \cap I_{\mathscr{V}} \subset \mathscr{O}$. It follows a short exact sequence of $\mathscr{O}$-modules

$$0 \longrightarrow \mathcal{Z}(\mathscr{G}_f) \longrightarrow \mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F}) \longrightarrow \mathscr{G}_f \cap I_{\mathscr{V}} \longrightarrow 0.$$

Setting $G = \mathcal{Z}(\mathscr{G}_f)$, we claim that G is free if f is a free divisor. In fact, put $F = \mathcal{T}_{\mathbb{A}^n/k}(\mathscr{F})$, which is free by hypothesis. By Proposition 1.1, it may be realized as the module formed with the $\chi \in \mathscr{D}_{\mathbb{A}^n}$ such that $\chi(f) \in (f)$. Consider the $\mathscr{O}$-linear map $\sigma: F \to \mathscr{O}$ given by $\sigma(\chi) = h_\chi$, for $\chi \in F$, where $h_\chi \in \mathscr{O}$ is the unique polynomial satisfying

$$\chi(f) = h_\chi f.$$

Of course, $\ker \sigma \simeq G$. Moreover, σ is surjective, as the quasi-homogeneity of $\mathscr{F}$ implies that there exist positive integers $s_1, \ldots, s_n$ satisfying $f = \sum_{i=1}^n \frac{1}{s_i} t_i \frac{\partial f}{\partial t_i}$, so that

$$\sum_{i=1}^n \frac{1}{s_i} t_i \frac{\partial}{\partial t_i} \in F, \qquad \sigma\left(\sum_{i=1}^n \frac{1}{s_i} t_i \frac{\partial}{\partial t_i}\right) = 1.$$

This shows that

$$F \simeq G \oplus \mathscr{O}$$

and hence that G is projective if f is a free divisor. Since $\mathscr{O}$ is a polynomial ring over a field, G must be free, as claimed. It follows, from the short exact sequence above, that the linear type ideal $\mathscr{G}_f \cap I_{\mathscr{V}}$ is a Bourbaki ideal of the module $\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F})$, which then, by Lemma 3.1, must be of linear type. □

Remarks 3.3. (i) A simple homological observation: keeping notation and hypotheses as in Proposition 3.2, one may dualize the short exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F}) \longrightarrow \mathscr{G}_f \cap I_{\mathscr{V}} \longrightarrow 0,$$

where G is the free module of syzygies of $\mathscr{G}_f$, in order to compare Ext modules,

$$\mathrm{Ext}^i_{\mathscr{O}}\left(\mathcal{T}_{\mathbb{A}^n/k}(\mathscr{V}, \mathscr{F}), \mathscr{O}\right) \simeq \mathrm{Ext}^{i+1}_{\mathscr{O}}\left(\frac{\mathscr{O}}{\mathscr{G}_f \cap I_{\mathscr{V}}}, \mathscr{O}\right), \quad \forall\, i \geq 1.$$

(ii) Once one has a quasi-homogeneous free divisor $\mathscr{F} = \{f = 0\}$ and a reduced subvariety $\mathscr{V} \subset \mathscr{F}$, the proposition above requires checking whether the ideal $\mathscr{G}_f \cap I_{\mathscr{V}}$ is of linear type. In general, one may resort to well-known effective criteria in order to test such property. A very useful one (in a suitable setup) is given, loosely speaking, in terms of heights of Fitting ideals, which in turn is equivalent to bounding number of generators locally at certain primes. Moreover, several classes of ideals are known to be of linear type. Central instances are complete intersections, ideals generated by d-sequences (see the comment below), and certain determinantal ideals; for an account and details, we refer to Vasconcelos' book [13] and its suggested references on the theme.

We invoke a concept due to C. Huneke. Let $\{g_1, \ldots, g_t\} \subset \mathscr{O}$ be a minimal generating set of an ideal $J \subset \mathscr{O}$. It is called a *d-sequence* if

$$(g_1, \ldots, g_i) \colon g_{i+1}\, g_s = (g_1, \ldots, g_i) \colon g_s, \quad i = 0, \ldots, t-1,\ s \geq i+1.$$

In this case, it is known that J is of linear type (see Huneke's paper [6], also [13, Theorem 2.3.2]). This will be used in the example below.

Example 3.4 ($k = \mathbb{C}$)**.** Applying Saito's freeness criterion to the non-smooth quartic hypersurface $\mathscr{F} = \{f = 0\} \subset \mathbb{P}^3$ defined by

$$f = \frac{1}{2} \det \begin{pmatrix} t_1 & 0 & 3t_3 & 3t_1 \\ t_2 & t_3 & 3t_4 & -t_2 \\ t_3 & t_1 & 4t_2 & t_3 \\ t_4 & t_2 & 0 & -3t_4 \end{pmatrix} = 8t_1t_2^3 + 9t_1^2t_4^2 - 18t_1t_2t_3t_4 - 3t_2^2t_3^2 + 6t_3^3t_4,$$

one sees that $\mathscr{F}$ is a free divisor (which probably belongs to a well-structured class of free divisors). Consider the planes $\Pi_i = \{t_i = 0\} \subset \mathbb{P}^3$, $i = 1, 3$, and set

$$\mathscr{L} = \Pi_1 \cap \Pi_3.$$

Then, $\mathscr{L}$ is a line contained in $\mathscr{F}$, and the ideal $I_{\mathscr{L}} \cap \mathscr{G}_f \subset \mathbb{C}[t_1, t_2, t_3, t_4]$ is minimally generated by the cubics

$$g_1 = 4t_1t_2^2 - t_2t_3^2 - 3t_1t_3t_4, \qquad g_2 = t_2^2t_3 + 3t_1t_2t_4 - 3t_3^2t_4, \qquad g_3 = 3t_1t_2t_3 - t_3^3 - 3t_1^2t_4.$$

With the aid of *Macaulay*, one verifies that $\{g_1, g_2, g_3\}$ is a d-sequence; by Huneke's result quoted above, $I_{\mathscr{L}} \cap \mathscr{G}_f$ must be of linear type, and hence Proposition 3.2 guarantees that so is the module $\mathcal{T}_{\mathbb{P}^3/\mathbb{C}}(\mathscr{L}, \mathscr{F})$ (which is non-free; see the comment after Corollary 4.4).

4. Freeness criteria

We now adopt the projective setup for convenience, that is, any variety will be defined by a (saturated) homogeneous ideal in the standard graded polynomial ring $\mathscr{O} = k[t_1, \ldots, t_{n+1}]$, seen as the homogeneous coordinate ring of $\mathbb{P}^n$ over k. See also Remark 2.2(ii).

Since clearly free modules are of linear type, one is interested – in virtue of the preceding section – only in pairs $\mathscr{V} \subset \mathscr{H} \subset \mathbb{P}^n$, with $\mathscr{H}$ a hypersurface, such that the $\mathscr{O}$-module $\mathcal{T}_{\mathbb{P}^n/k}(\mathscr{V}, \mathscr{H})$ is non-free. We shall derive criteria in order to characterize such pairs, with $\mathscr{H}$ non-smooth and not necessarily free. This will follow from the depth-formula observed below (depth taken with respect to the irrelevant ideal). We denote the coordinate ring of a variety $\mathscr{W}$ by $\mathscr{O}_{\mathscr{W}}$.

Proposition 4.1. *Let $\mathscr{V} \subset \mathscr{H} \subset \mathbb{P}^n$ be reduced varieties, with $\mathscr{H}$ a non-smooth hypersurface and $\mathscr{V} \nsubseteq \operatorname{Sing} \mathscr{H}$. Then,*

$$\operatorname{depth} \mathcal{T}_{\mathbb{P}^n/k}(\mathscr{V}, \mathscr{H}) = \operatorname{depth} \mathscr{O}_{\mathscr{V} \cap \operatorname{Sing} \mathscr{H}} + 2.$$

Proof. The idea relies on describing the module $\mathcal{T} = \mathcal{T}_{\mathbb{P}^n/k}(\mathscr{V}, \mathscr{H})$ as (isomorphic to) the module of syzygies, over $\mathscr{O}$, of an explicit ideal in a convenient factor ring of $\mathscr{O}$. If $\mathscr{H}$ is defined by a reduced homogeneous equation $h = 0$, let $\mathscr{G}_h$ be its gradient ideal. Consider the ideal

$$\mathscr{I} = \frac{(I_{\mathscr{V}}, \mathscr{G}_h)}{I_{\mathscr{V}}} \subset \mathscr{O}_{\mathscr{V}},$$

which is non-zero since $\mathscr{V} \nsubseteq \operatorname{Sing} \mathscr{H}$. Set $\mathscr{W} = \mathscr{V} \cap \operatorname{Sing} \mathscr{H}$. One has $\operatorname{depth} \mathscr{I} = \operatorname{depth} \mathscr{O}_{\mathscr{W}} + 1$, in virtue of the structural exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_{\mathscr{V}} \longrightarrow \mathscr{O}_{\mathscr{W}} \longrightarrow 0.$$

Now, we claim that

$$\mathscr{I} \simeq \operatorname{coker} (\mathcal{T} \hookrightarrow \mathscr{O}^{n+1}).$$

In fact, the rule

$$(g_1, \ldots, g_{n+1}) \longmapsto \sum_{i=1}^{n+1} g_i \frac{\partial h}{\partial t_i} \,(\operatorname{mod} I_{\mathscr{V}}),$$

for $g_i \in \mathscr{O}$, $i = 1, \ldots, n+1$, defines an $\mathscr{O}$-linear surjection $\mathscr{O}^{n+1} \to \mathscr{I}$ with kernel naturally isomorphic to $\mathcal{T}$ (this is easily seen to hold true if $\mathcal{T}$ is regarded as the module of the k-derivations conducting h into $I_{\mathscr{V}}$). It follows that depth $\mathcal{T} =$ depth $\mathscr{I} + 1$, and hence depth $\mathcal{T} =$ depth $\mathscr{O}_{\mathscr{W}} + 2$. □

It will be useful to notice that codim Sing $\mathscr{H} \geq 2$ whenever the (non-smooth) hypersurface $\mathscr{H}$ is reduced. The codimension $n - \dim \mathscr{W}$ of a variety $\mathscr{W}$ in the fixed ambient space $\mathbb{P}^n$ is denoted codim $\mathscr{W}$. Recall that $\mathscr{W}$ is said to be *arithmetically Cohen–Macaulay* if $\mathscr{O}_{\mathscr{W}}$ is a Cohen–Macaulay ring.

Corollary 4.2. *Let $\mathscr{V} \subset \mathscr{H} \subset \mathbb{P}^n$ be reduced varieties, with $\mathscr{H}$ a non-smooth hypersurface and $\mathscr{V} \nsubseteq$ Sing $\mathscr{H}$. The following are equivalent:*

(i) $\mathcal{T}_{\mathbb{P}^n/k}(\mathscr{V}, \mathscr{H})$ *is a free $\mathscr{O}$-module;*
(ii) depth $\mathscr{O}_{\mathscr{V} \cap \text{Sing}\, \mathscr{H}} = n - 1$*;*
(iii) $\mathscr{V} \cap$ Sing $\mathscr{H}$ *is an arithmetically Cohen–Macaulay variety of codimension* 2*.*

Proof. Since $\mathscr{O}$ has depth $n+1$, the equivalence between (i) and (ii) follows from Proposition 4.1 together with the Auslander–Buchsbaum formula in the graded case. Set $\mathscr{W} = \mathscr{V} \cap$ Sing $\mathscr{H}$. If (ii) holds, then $\dim \mathscr{O}_{\mathscr{W}} \geq n-1$, that is, $\dim \mathscr{W} \geq n-2$ and hence codim $\mathscr{W} \leq 2$, which must be an equality since, on the other hand,

$$\text{codim}\, \mathscr{W} \geq \text{codim Sing}\, \mathscr{H} \geq 2.$$

Thus, $\dim \mathscr{W} = n-2$, yielding $\dim \mathscr{O}_{\mathscr{W}} = n - 1 =$ depth $\mathscr{O}_{\mathscr{W}}$, as required. Conversely, if (iii) holds, then $\dim \mathscr{W} = n-2$ and hence $\dim \mathscr{O}_{\mathscr{W}} = n-1$, which by Cohen–Macaulayness means that depth $\mathscr{O}_{\mathscr{W}} = n-1$. □

At first sight, one might think that the equivalent conditions dealt with in the corollary above could never be satisfied in the case $\mathscr{V} \neq \mathscr{H}$, since for instance condition (iii) seems to imply that $\mathscr{V} \subseteq$ Sing $\mathscr{H}$, a situation that is ruled out by hypothesis. But this is not true, as the next example shows (which illustrates also that the $\mathscr{O}$-module $\mathcal{T}_{\mathbb{P}^3/k}(\mathscr{V}, \mathscr{H})$ may be free even if $\mathscr{H}$ is *not* a free divisor).

Example 4.3 ($k = \mathbb{C}$)**.** Consider the hypersurface $\mathscr{H} = \{h = 0\} \subset \mathbb{P}^3$, where $h = t_1t_2t_3t_4(t_1+t_2+t_3+t_4)$, and let $\mathscr{V} \subset \mathbb{P}^3$ be the reduced surface defined by the ideal $I_{\mathscr{V}} = (t_1t_2t_3, t_1t_2t_4) \subset \mathscr{O} = \mathbb{C}[t_1, t_2, t_3, t_4]$. Note that $\mathscr{V} \subset \mathscr{H}$, Sing $\mathscr{H} \nsubseteq \mathscr{V}$ and $\mathscr{V} \nsubseteq$ Sing $\mathscr{H}$. Further, the graded ring $\mathscr{O}_{\text{Sing}\, \mathscr{H}} = \mathscr{O}/\mathscr{G}_h$ has depth zero, so that $\mathscr{H}$ is not a free divisor by Corollary 4.4(ii) below. As to $\mathscr{W} = \mathscr{V} \cap$ Sing $\mathscr{H}$, its defining ideal may be written as

$$I_{\mathscr{W}} = I_{\mathscr{V}} + (t_1^2t_3t_4 + t_1t_3^2t_4 + t_1t_3t_4^2,\ t_2^2t_3t_4 + t_2t_3^2t_4 + t_2t_3t_4^2) \subset \mathscr{O}.$$

After a calculation with *Macaulay*, one gets depth $\mathscr{O}_{\mathscr{W}} = 2$, and thus Corollary 4.2 yields that the $\mathscr{O}$-module $\mathcal{T}_{\mathbb{P}^3/k}(\mathscr{V}, \mathscr{H})$ is free.

As a byproduct of Corollary 4.2, we recover well-known criteria for a non-smooth divisor to be free.

Corollary 4.4. *Let $\mathscr{F} \subset \mathbb{P}^n$ be a non-smooth reduced hypersurface. The following are equivalent:*

(i) $\mathscr{F}$ *is a free divisor;*
(ii) depth $\mathscr{O}_{\text{Sing}\, \mathscr{F}} = n-1$*;*
(iii) Sing $\mathscr{F}$ *is an arithmetically Cohen–Macaulay variety of codimension* 2*;*
(iv) $\mathcal{T}_{\mathbb{P}^n/k}(\mathscr{V}, \mathscr{F})$ *is a free $\mathscr{O}$-module, for every* (*in fact, any*) *reduced variety $\mathscr{V} \subset \mathbb{P}^n$ that satisfies*

$$\text{Sing}\, \mathscr{F} \subsetneq \mathscr{V} \subset \mathscr{F}.$$

Proof. One obtains the equivalence of (i), (ii) and (iii) immediately by taking $\mathscr{V} = \mathscr{H} = \mathscr{F}$ in the previous corollary. Item (iv) also follows readily. □

Revisiting Example 3.4, a run with *Macaulay* [1] yields that the depth of the (Cohen–Macaulay) graded ring $\mathscr{O}_{\text{Sing}\, \mathscr{F}}$ is $2 = n - 1$, as it must be since $\mathscr{F}$ is free. Now, the reduced variety associated to $\mathscr{L} \cap$ Sing $\mathscr{F}$ is a single point, to wit,

$$(\mathscr{L} \cap \text{Sing}\, \mathscr{F})_{\text{red}} = \{(0:0:0:1)\}.$$

It thus has codimension 3 in $\mathbb{P}^3$, so that, by Corollary 4.2(iii), the (linear type) $\mathscr{O}$-module $\mathcal{T}_{\mathbb{P}^3/\mathbb{C}}(\mathscr{L}, \mathscr{F})$ is *not* free. Moreover, since codim Sing $\mathscr{F} = 2$, we must have Sing $\mathscr{F} \nsubseteq \mathscr{L}$, in accordance with Corollary 4.4(iv).

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