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Triangular Factorizations of Special Polynomial Automorphisms

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In this paper we give explicit factorizations which demonstrate the stable tameness of all polynomial automorphisms arising from a recent construction of Hubbers and van den Essen. This is accomplished by two different factorizations of such an automorphism by triangular automorphisms, one which is concise but requires a large number of additional dimensions, and one which is more intricate but requires fewer dimensions. The stable tameness of automorphisms in this special class aligns with other existing evidence to suggest that all polynomial automorphisms may be stably tame. © 2001 Academic Press

INTRODUCTION

The most obvious types of polynomial automorphisms are linear automorphisms and elementary automorphisms (see 1.6 and 1.3). Automorphisms which are compositions of these types are called tame. A classical theorem of Jung and van der Kulk tells us that, over a field, all polynomial automorphisms in dimension 2 are tame [8, 9]. A major unsolved question asks whether this holds in higher dimensions. There are some partial results; for example, it follows from the main theorem of [13] that three-dimensional "cubic homogeneous" automorphisms over a field are tame.

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Nagata [10] established that not all automorphisms in dimension 2 are tame over an integral domain which is not a field, and by taking the domain to be the polynomial ring in one variable over a field and "releasing" the variable from the base ring (see 1.9), he produced an automorphism in dimension 3 which he conjectured to be non-tame. The truth of this conjecture remains unresolved. However, Smith [11] produced a factorization (using a key lemma which is also used in this paper—Lemma 3.14) showing that Nagata's example becomes tame when "extended by the identity" (see 1.8) to dimension 4. This raised the question of whether all automorphisms (perhaps over any base ring) are stably tame, i.e., become tame in some higher dimension. This question has appeared in numerous places in the literature ([1, 7], for example).

There are very few known methods of constructing automorphisms which are not obviously tame. One is Nagata's method in dimension 2, which makes use of a non-unit in the base ring. All known specific examples of this type have been shown to be stably tame. Another method takes an invertible matrix over a polynomial ring over a field in n variables, with $n \geq 2$, and builds a polynomial automorphism of dimension $n + 2$ by releasing the n variables (see 1.3 and 1.9). If the matrix is not the product of elementary and diagonal matrices, the resulting polynomial map does not appear to be tame. However, it becomes tame with the adjunction of one new variable, by a theorem of Suslin [12].

In [3] van den Essen and Hubbers produced an intriguing new way to construct automorphisms. They established that automorphisms produced by their new construction include recently discovered counterexamples to some well-known conjectures, such as the discrete Markus−Yamabe problem (see [4]), as well as some of the classes which categorize four-dimensional "cubic homogeneous" automorphisms, for which a complete classification was given in [6]. The automorphisms in this new class are not obviously tame. Van den Essen and Hubbers established in [5] that certain two-dimensional examples are in fact not tame. Other examples appear doubtful. They also showed in [5] that all automorphisms in this new class are stably tame. However, their proof does not provide an explicit factorization and it does not answer the question of how many additional dimensions are needed to achieve tameness.

This paper resolves these issues, giving algorithms for factoring automorphisms in this new class as products of triangular automorphisms, after extending the dimension. The first method, given in Section 2, factors such an n-dimensional automorphism after extending the dimension by $n(n-1)/2$. The second method, given in Section 3, uses some properties developed in [5] to accomplish a triangular factorization with only $n - 1$ added dimensions. Section 5 gives algorithms which arise from the proofs in Sections 2 and 3. The importance of these results lies in the methods of the factorizations, as they may provide some hint as to how one might factor automorphisms of a more general type using stabilization.

1. NOTATION AND TERMINOLOGY

1.1. *Polynomial Maps*

Throughout this paper A will denote an arbitrary commutative ring and X will denote a system of variables (indeterminates) $X_1, \ldots, X_n$. Hence $A[X]$ denotes the polynomial ring $A[X_1, \ldots, X_n]$ with A being the *base ring*. By an n-dimensional *polynomial map* we mean an A-endomorphism of the affine scheme $\mathbb{A}^n_A = \operatorname{Spec} A[X_1, \ldots, X_n]$. Such a map F corresponds to an A-algebra endomorphism ϕ_F of $A[X]$, which is given by n polynomials $F_1, \ldots, F_n \in A[X]$ defined by $F_i = \phi_F(X_i)$, $i = 1, \ldots, n$. Thus the set of n-dimensional polynomial maps is identified with $A[X]^n$. It is convenient to denote F by the column vector

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix},$$

which, to save space, will usually be written $F = (F_1, \ldots, F_n)^t$, where t denotes transpose. In fact, for any ring A we view elements of A^n as column vectors in this discussion. The polynomials $F_1, \ldots, F_n$ are called the *coordinate polynomials* of F. Given another polynomial map $G = (G_1, \ldots, G_n)^t$, the sum $F + G$ simply denotes column vector addition, and the composition $F \circ G$, also denoted $F(G)$, is given by the column

$$\big(F_1(G_1, \ldots, G_n), \ldots, F_n(G_1, \ldots, G_n)\big)^t.$$

Since we will be dealing with additional variables, we sometimes write $F(X)$ in place of F to indicate that the coordinate polynomials of F are to be seen as polynomials in $X_1, \ldots, X_n$. Note that X, considered as a column vector, gives the identity polynomial map. We say a polynomial map F is *invertible*, or is an *automorphism*, if there exists a polynomial map G such that $F \circ G = X$, in which case it automatically holds that $G \circ F = X$ as well. Given a polynomial map F and a matrix T of any dimension with entries in $A[X]$, we write $T|_F$, or $T(F)$, for the matrix obtained by substituting $F_1, \ldots, F_n$ for $X_1, \ldots, X_n$ in the entries of T. Note that a composition of polynomial maps $F \circ G$ $(= F(G))$ can also be expressed as $F|_G$.

1.2. *Structure*

The set of such n-dimensional polynomial maps forms a monoid with respect to composition, and, following notation introduced in [2], this monoid will be denoted $MA_n(A)$. The invertible polynomial maps are precisely the units in $MA_n(A)$, and this group is denoted $GA_n(A)$. Since $MA_n(A)$ is identified with $A[X]^n$ as a set, it inherits the additive structure $+$ of the latter, with respect to which composition $\circ$ is "right-distributive," i.e., satisfies $(F + F')\circ G = F \circ G + F' \circ G$ (but distributivity on the other side does not hold in general). Note that $MA_n(A)$ is anti-isomorphic to the monoid of A-algebra endomorphisms of $A[X]$.

1.3. *Matrices as Linear Polynomial Maps*

The monoid of $n \times n$ matrices with coefficients in A is denoted $M_n(A)$, and the group of invertible matrices is denoted $GL_n(A)$. The monoid $M_n(A)$ is naturally contained in $MA_n(A)$ (with $GL_n(A)$ contained in $GA_n(A)$) as the "linear endomorphisms," i.e., those for which the coordinate polynomials are linear forms. Under this containment a matrix T is identified with the polynomial map $(L_1, \dots, L_n)^t$ given by the formula $(L_1, \dots, L_n) = (T \cdot X)$. This simply creates linear forms from the rows of T. We denote this map by TX. Note that a linear endomorphism TX satisfies the *left distributivity property* (that fails in general): $(TX)\circ(F + F') = (TX)\circ F + (TX)\circ F'$ for all $F, F' \in MA_n(A)$. We will refer to this left distributivity property as LDP.

1.4. *Operations*

In this paper we will consistently use "$\circ$" for the composition of polynomial maps and "$\cdot$" for matrix multiplication, the latter of which will sometimes be omitted. We will also use a non-standard matrix operation denotes by "$\vartriangle$," which is defined by

$$S \vartriangle T = S(TX) \cdot T$$

for $S, T \in M_n(A[X])$. One easily verifies that this operation is associative.

LEMMA 1.5. *The "$\vartriangle$" operator has the property*

$$(S_1 \vartriangle S_2 \vartriangle \cdots \vartriangle S_k)X = (S_1 X)\circ(S_2 X)\circ \cdots \circ(S_k X)$$

for all integers $k \geq 1$.

Proof. The proof is by induction on k. For $k = 1$ it is obvious. Note that $(S_1 \vartriangle S_2)X = (S_1(S_2 X) \cdot S_2) \cdot X = S_1(S_2 X) \cdot (S_2 X) = (S_1 X)|_{S_2 X} =$

$(S_1X)\circ(S_2X)$, which proves the case $k = 2$. For $k > 2$ we have

$$\begin{aligned}(S_1 \triangle S_2 \triangle \cdots \triangle S_k)X &= \big(S_1 \triangle (S_2 \triangle \cdots \triangle S_k)\big)X \\ &= (S_1X)\circ\big((S_2 \triangle \cdots \triangle S_k)X\big) \\ &= (S_1X)\circ(S_2X)\circ\cdots\circ(S_kX),\end{aligned}$$

which proves the lemma. ∎

1.6. *Tame Automorphisms*

A polynomial map is called *elementary* if it has the form

$$(X_1, \ldots, X_{i-1}, X_i + f, X_{i+1}, \ldots, X_n)^t,$$

where $f \in A[(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$. Such a map is clearly invertible. A polynomial map is called *tame* if it can be written as a composition of automorphisms, each of which is either linear or elementary. A map is called *triangular*, if, up to rearrangement of the variables, it has the form $(X_1 + g_1, \ldots, X_n + g_n)$, where $g_i \in A[X_{i+1}, \ldots, X_n]$, for $i = 1, \ldots, n$. It is easily seen that triangular automorphisms are tame.

1.7. *Translations*

Given $a = (a_1, \ldots, a_n)^t \in A^n$, we form the *translation* $\mathrm{Tr}_a = X + a = (X_1 + a_1, \ldots, X_n + a_n)^t$, which is obviously triangular. For any linear polynomial map TX, one easily verifies that

$$(TX)\circ\mathrm{Tr}_a = \mathrm{Tr}_b\circ(TX), \qquad \text{where } b = T\cdot a. \tag{1}$$

1.8. *Stabilization*

Given $F = (F_1, \ldots, F_n)^t \in MA_n(A)$ and $m \geq 1$ an integer, we write $F^{[m]}$ for the polynomial map $(F_1, \ldots, F_n, X_{n+1}, \ldots, X_{n+m})^t$ of $MA_{n+m}(A)$. This procedure of "extending by the identity" gives a natural embedding $MA_n(A) \subset MA_{n+m}(A)$. It is easily seen that F is an automorphism if and only if $F^{[m]}$ is an automorphism. The extension of an n-dimensional polynomial map to a map in a higher dimension in this way is called *stabilization*, a term that comes from K-theory. It is only by stabilizing that certain automorphisms can be seen to be tame. A polynomial map F for which $F^{[m]}$ is tame for some $m \geq 1$ is called *stably tame*.

1.9. *Releasing of Variables*

Often a polynomial map can be viewed as a map in a larger dimension over a smaller base ring. This is an important concept in the study of polynomial maps. We observe that $MA_i(A[X_{i+1},\dots,X_n])$ is naturally contained in $MA_n(A)$ as follows: An element $F' \in MA_i(A[X_{i+1},\dots,X_n])$, which is given by $(F'_1,\dots,F'_i)^t$ with $F'_i \in A[X_{i+1},\dots,X_n][X_1,\dots,X_i] = A[X_1,\dots,X_n]$, is identified with the polynomial map $(F'_1,\dots,F'_i, X_{i+1}, \dots, X_n)^t \in MA_n(A)$. If we view these maps (anti-isomorphically) as ring endomorphisms of $A[X_1,\dots,X_n]$, this merely considers the $A[X_{i+1},\dots, X_n]$-algebra endomorphism F' as an A-algebra homomorphism, thus restricting the ring of "constants" from $A[X_{i+1},\dots,X_n]$ to A, i.e., "releasing" the variables $X_{i+1},\dots,X_n$ from the coefficient ring. We therefore refer to this as *releasing of variables*. Note that this is different from stabilization. Observe also that a linear map in $MA_i(A[X_{i+1},\dots,X_n])$ is not (in general) linear when considered in $MA_n(A[X_1,\dots,X_n])$.

1.10. *The New Class of Automorphisms*

In [3] a new class of polynomial maps, denoted by $\mathscr{H}_n(A)$, was introduced. We recall the definition of $\mathscr{H}_n(A)$.

DEFINITION 1.11. The set of n-dimensional polynomial maps $\mathscr{H}_n(A)$ is defined inductively as follows:

- $n = 1$: $\mathscr{H}_1(A) = A$. (This makes sense, since an element of A gives a one-dimensional polynomial map.)
- $n \geq 2$: A polynomial map $H \in MA_n(A)$ is in $\mathscr{H}_n(A)$ if and only if there exists $T \in M_n(A)$, $c \in A^n$, and $H' \in \mathscr{H}_{n-1}(A[X_n])$ such that

$$H = (\operatorname{adj} T).\begin{pmatrix} H' \\ 0 \end{pmatrix}\Bigg|_{TX} + c. \tag{2}$$

If H' is given by $(H'_1,\dots,H'_{n-1})^t$, then $(H',0)^t$ denotes the column $(H'_1,\dots,H'_{n-1},0)^t$, and everything above makes sense.

One easily verifies that the column $\operatorname{adj}(T)\cdot(H',0)^t|_{TX}$ which appears in (2) can also be realized as the composition of polynomial maps $((\operatorname{adj} T)X)\circ(H',0)^t\circ(TX)$; hence (2) can be expressed as

$$H = ((\operatorname{adj} T)X)\circ\begin{pmatrix} H' \\ 0 \end{pmatrix}\circ(TX) + c. \tag{3}$$

It was shown in [3] that for each $H \in \mathscr{H}_n(A)$, the polynomial map $F = X + H$ is invertible, thus introducing a new class of polynomial automorphisms. It was later established, in [5], that these automorphisms, while not being tame in general, are stably tame, although no bound on the integer m for which $F^{[m]}$ becomes tame is given.

The remainder of this paper will demonstrate two explicit methods for factoring a polynomial automorphism of the form $F = X + H$, with $H \in$

$\mathscr{H}_n(A)$, as a composition of triangular automorphisms, thereby giving a more concrete proof of stable tameness and providing an explicit integer m for which $F^{[m]}$ becomes tame.

2. THE QUICK METHOD

For $F = X + H$ with $H \in \mathscr{H}_n(A)$, this section provides an explicit recipe to factor $F^{[n(n+1)/2]}$ by triangular automorphisms, thereby proving:

THEOREM 2.1. *Let $F = X + H$ with $H \in \mathscr{H}_n(A)$. Then $F^{[n(n-1)/2]}$ is a product of triangular automorphisms, and hence is tame.*

The proof will require the following lemma, which is an extension of [3, Lemma 2.1].

LEMMA 2.2. *Let $H \in \mathscr{H}_n(A)$, $d \in A$, and $a = (a_1, \dots, a_n)^t \in A^n$. Then $H(dX_1 + a_1, \dots, dX_n + a_n) \in \mathscr{H}_n(A)$.*

Proof. (induction on n). For $n = 1$ the map H is constant, so $H(dX_1 + a_1) = H \in \mathscr{H}_1(A)$. Assume $n \geq 2$. Note that $H(dX_1 + a_1, \dots, dX_n + a_n)$ can be factored in $MA_n(A)$ as $H \circ \mathrm{Tr}_a \circ dX$, where Tr_a is the translation defined in 1.7 and dX is the map $(dX_1, \dots, dX_n)^t$. From Definition 1.11 and (3) we have $H = ((\mathrm{adj}\, T)X) \circ (H', 0)^t \circ (TX) + c$ with $H' \in \mathscr{H}_{n-1}(A[X_n])$. Note that the "$+c$" can be accommodated by composing $((\mathrm{adj}\, T)X) \circ (H', 0)^t \circ (TX)$ on the left with the translation Tr_c. Hence we get

$$\begin{aligned}
H(dX_1 + a_1, \dots, dX_n + a_n) &\\
&= H \circ \mathrm{Tr}_a \circ (dX) \\
&= \mathrm{Tr}_c \circ ((\mathrm{adj}\, T)X) \circ (H', 0)^t \circ (TX) \circ \mathrm{Tr}_a \circ (dX) \\
&= \mathrm{Tr}_c \circ ((\mathrm{adj}\, T)X) \circ (H', 0)^t \circ \mathrm{Tr}_b \circ (TX) \circ (dX) \\
&\qquad\qquad \text{where } b = (T \cdot a) \qquad (\text{see } (1)) \\
&= \mathrm{Tr}_c \circ ((\mathrm{adj}\, T)X) \circ (H', 0)^t \circ \mathrm{Tr}_b \circ (dX) \circ (TX) \\
&= \mathrm{Tr}_c \circ ((\mathrm{adj}\, T)X) \circ (H'(dX_1 + b_1, \dots, dX_n + b_n), 0)^t \circ (TX) \\
&= ((\mathrm{adj}\, T)X) \circ (H'(dX_1 + b_1, \dots, dX_n + b_n), 0)^t \circ (TX) + c.
\end{aligned}$$

We are done if we can show that $H'(dX_1 + b_1, \dots, dX_n + b_n) \in \mathscr{H}_{n-1}(A[X_n])$. Note that $X_1, \dots, X_{n-1}$ are variables and X_n is a constant in $\mathscr{H}_{n-1}(A[X_n])$. By induction we have $H'(dX_1 + b_1, \dots, dX_{n-1} + b_{n-1}, X_n) \in \mathscr{H}_{n-1}(A[X_n])$. To this we apply a base change using the substitution homomorphism of A-algebras $A[X_n] \to A[X_n]$ defined by $X_n \mapsto dX_n + b_n$, which yields $H'(dX_1 + b_1, \ldots, dX_n + b_n)$. According to [3, Lemma 2.1], $\mathscr{H}_n(A)$ is stable under base change, so the lemma is proved. ∎

Here is the main tool for the proof of Theorem 2.1:

PROPOSITION 2.3. *Let $F = X + H$ with $H \in \mathscr{H}_n(A)$, and let $Y = (Y_1, \ldots, Y_{n-1})$ be a system of variables. There exist $(2n-1)$-dimensional automorphisms U and V which are products of triangular automorphisms such that $U \circ F^{[n-1]} \circ V$ is of the form $(X, Y+K)^t$, where $K \in \mathscr{H}_{n-1}(A[X_1, \ldots, X_n])$ (with respect to the variables $Y_1, \ldots, Y_{n-1}$).*

Proof. Write $H = ((\operatorname{adj} T)X) \circ (H', 0)^t \circ (TX) + c$ as in (3). Observe that H can be written as $((\operatorname{adj} T)X) \circ (H'(TX), 0)^t + c$. Define $(2n-1)$-dimensional polynomial maps P, Q, and R in the variables $(X, Y) = (X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1})$ by

$$\begin{aligned} P &= (X, Y + H'(TX))^t \\ Q &= (X - c, Y)^t \\ R &= \Big(X - \big[((\operatorname{adj} T)X) \circ (Y, 0)^t\big], Y\Big)^t. \end{aligned} \tag{4}$$

Note that P, Q, and R are triangular automorphisms, and the inverse for each respective automorphism is obtained by replacing "+" by "−," or vice versa. Letting $U = R \circ Q$ and $V = P \circ R^{-1}$, we claim that

$$U \circ F^{[n-1]} \circ V = \Big(X, Y + H'\big((TX) + (dY, 0)^t\big)\Big)^t, \tag{5}$$

where $d = \det T$. This is verified in the calculation below. The underbraces indicate the next composition to be calculated.

$$\begin{aligned} &U \circ F^{[n-1]} \circ V \\ &\quad = R \circ \underbrace{Q \circ \Big(X + \big[((\operatorname{adj} T)X) \circ (H'(TX), 0)^t + c\big], Y\Big)^t}\circ P \circ R^{-1} \\ &\quad = R \circ \underbrace{\Big(X + \big[((\operatorname{adj} T)X) \circ (H'(TX), 0)^t\big], Y\Big)^t \circ P} \circ R^{-1} \\ &\quad = \underbrace{R \circ \Big(X + \big[((\operatorname{adj} T)X) \circ (H'(TX), 0)^t\big], Y + H'(TX)\Big)^t} \circ R^{-1} \end{aligned}$$

$$= \left(X - (\operatorname{adj} T)\cdot (Y,0)^t, Y + H'(TX)\right)^t \circ R^{-1}$$

(the above steps uses LDP for adj T—see 1.3)

$$= \left(X, Y + H'\left(T\cdot\left[X + ((\operatorname{adj} T)X)\circ (Y,0)^t\right]\right)\right)^t$$

$$= \left(X, Y + H'\left((TX) + (dY,0)^t\right)\right)^t$$

(again using LDP, plus the fact that $(TX)\circ((\operatorname{adj} T)X) = dX$).

The claim is proved, and, setting $K = H'((TX) + (dY,0)^t)$, the theorem is proved if we can show $K \in \mathscr{H}_{n-1}(A[X_1,\dots,X_n])$, with respect to the variables $Y_1,\dots,Y_{n-1}$.

Writing $TX = (L_1,\dots,L_n)$, with L_i a linear form in $A[X]$, we have

$$K = H'(dY_1 + L_1,\dots, dY_{n-1} + L_{n-1}, L_n).$$

We know $H'(X_1,\dots,X_n) \in \mathscr{H}_{n-1}(A[X_n])$, where $X_1,\dots,X_{n-1}$ are viewed as variables and X_n as a constant. It is innocent to replace $X_1,\dots,X_{n-1}$ by $Y_1,\dots,Y_{n-1}$. Furthermore, by making the base change $A[X_n] \to A[X_1,\dots,X_n]$ defined by $X_n \mapsto L_n$, we see that $H'(Y_1,\dots,Y_{n-1},L_n) \in \mathscr{H}_{n-1}(A[X_1,\dots,X_n])$, with respect to the variables $Y_1,\dots,Y_{n-1}$ [3, Lemma 2.1]. Finally, Lemma 2.2 above tells us that $H'(dY_1 + L_1,\dots,dY_{n-1} + L_{n-1}, L_n) \in \mathscr{H}_{n-1}(A[X_1,\dots,X_n])$ with respect to the new variables $Y_1,\dots,Y_{n-1}$, concluding the proof of the theorem. ∎

Proof of Theorem 2.1. For $n = 1$ this is a triviality. Assume $n \geq 2$, and apply Theorem 2.2 to get $(2n-1)$-dimensional automorphisms U and V which are products of triangular automorphisms such that $G = U \circ F^{[n-1]} \circ V = (X, Y + K)$ with $K \in \mathscr{H}_{n-1}(A[X_1,\dots,X_n])$. By induction we have $G^{[(n-1)(n-2)/2]}$ is tame; hence $F^{[n-1+(n-1)(n-2)/2]}$ is tame, so the theorem results from the equality $\frac{n(n-1)}{2} = n - 1 + \frac{(n-1)(n-2)}{2}$. ∎

This proof is summarized in 4.2, which derives an algorithm for factoring $F = X + H$, where $H \in \mathscr{H}_n(A)$, using the $\mathscr{D}_n(A)$ data for H (see Definition 3.2).

3. THE STRONG METHOD

Here we obtain a sharper upper bound, if $n \geq 3$, for the number of variables needed to factor $F = X + H$, $H \in \mathscr{H}_n(A)$, as a product of triangular automorphisms.

THEOREM 3.1. *Let $F = X + H$ with $H \in \mathscr{H}_n(A)$. Then $F^{[n-1]}$ is a product of triangular automorphisms.*

The method in the previous section was based directly on the definition of $\mathscr{H}_n(A)$. The method of this section uses the notion of $\mathscr{D}_n(A)$, which was introduced in [5]. For $n \geq 2$, $\mathscr{D}_n(A)$ is a set whose elements provide explicit data consisting of a sequence of matrices and vectors which determine a polynomial map $H \in \mathscr{H}_N(A)$. Each $H \in \mathscr{H}_n(A)$ arises from such data. We now restate the definition.

DEFINITION 3.2. Let $n \geq 2$. Then $\mathscr{D}_n(A)$ is the set of $(2n-1)$-tuples having the form[1]

$$(T;c) = (T^{(2)}, \dots, T^{(n)}; c^{(1)}, \dots, c^{(n)}),$$

where

$$T^{(i)} \in M_i(A[X_{i+1}, \dots, X_n])$$

for $2 \leq i \leq n$ and

$$c^{(i)} \in A[X_{i+1}, \dots, X_n]^i$$

for $1 \leq i \leq n$. (For $i = n$, these are to be read as $T^{(n)} \in M_n(A)$, $c^{(n)} \in A^n$.)

We will briefly explain how an element $E_n(T;c) \in \mathscr{H}_n(A)$ is constructed from an element $(T;c) \in \mathscr{D}_n(A)$. The construction is given inductively. For $n = 2$ we define

$$E_2(T;c) = (\operatorname{adj} T^{(2)}) \cdot \begin{pmatrix} c^{(1)} \\ 0 \end{pmatrix}\bigg|_{T^{(2)}X} + c^{(2)},$$

which can also be expressed as $[((\operatorname{adj} T^{(2)})X) \circ (c^{(1)}, 0)^t \circ (T^{(2)}X)] + c^{(2)}$. For $n \geq 3$, note that the pair of truncated sequences

$$(T';c') = (T^{(2)}, \dots, T^{(n-1)}; c^{(1)}, \dots, c^{(n-1)}) \tag{6}$$

lies in $\mathscr{D}_{n-1}(A[X_n])$. Letting $H' = E_{n-1}(T';c')$,[2] we then set

$$\begin{aligned} E_n(T;c) &= (\operatorname{adj} T^{(n)}) \cdot \begin{pmatrix} H' \\ 0 \end{pmatrix}\bigg|_{T^{(n)}X} + c^{(n)} \\ &= \left[((\operatorname{adj} T^{(n)})X) \circ (H',0)^t \circ (T^{(n)}X)\right] + c^{(n)}. \end{aligned}$$

This defines a function $E_n\colon \mathscr{D}_n(A) \to \mathscr{H}_n(A)$, for $n \geq 2$.

[1] In [5] the data items are denoted with subscripts. We have chosen to use superscripts here to avoid confusion with coordinate polynomials.

[2] There is a slight ambiguity in the symbol E_n in that it does not make reference to the ring A. Here E_{n-1} must be seen as a map $\mathscr{D}_{n-1}(A[X_n]) \to \mathscr{H}_{n-1}(A[X_n])$. The same abuse of notation will be tolerated in the definition of $E_{n,p}$ (Definition 3.3).

In order to prove certain properties of the polynomial map $E_n(T;c)$ defined above, it is helpful to write it as a summation $E_n(T;c) = \sum_{p=0}^{n-2} E_{n,p}(T;c) + c^{(n)}$, where the summands are defined below.

DEFINITION 3.3. Let $n \geq 2$ and $0 \leq p \leq n-2$. Then $E_{n,p}\colon \mathscr{D}_n(A) \to MA_n(A)$ is defined inductively by the following. For $(T;c) \in \mathscr{D}_n(A)$,

- $p = 0$: We let

$$E_{n,0}(T;c) = \left(\operatorname{adj} T^{(n)}\right) \cdot \begin{pmatrix} c^{(n-1)} \\ 0 \end{pmatrix} \Bigg|_{T^{(n)}X} = \left[\left((\operatorname{adj} T^{(n)})X\right) \circ \left(c^{(n-1)}, 0\right)^t \circ \left(T^{(n)}X\right)\right].$$

- $p \geq 1$: Setting $H' = E_{n-1,p-1}(T';c')$, where $(T';c')$ is as in (6), we let

$$E_{n,p}(T;c) = \left(\operatorname{adj} T^{(n)}\right) \cdot \begin{pmatrix} H' \\ 0 \end{pmatrix} \Bigg|_{T^{(n)}X} = \left[\left((\operatorname{adj} T^{(n)})X\right) \circ \left(H', 0\right)^t \circ \left(T^{(n)}X\right)\right].$$

This definition is from [5]. An easy induction, left to the reader, shows that

$$E_n(T;c) = \sum_{p=0}^{n-2} E_{n,p}(T;c) + c^{(n)}. \tag{7}$$

In [5] this summation is presented as the means to (directly) construct an element of $\mathscr{H}_n(A)$ from $(T;c) \in \mathscr{D}_n(A)$, with no mention of $E_n(T;c)$.

3.4. *Extending the Data*

The matrices and columns appearing in $(T;c)$ have different dimensions. In some situations it will be necessary to extend all of the data to dimension n. We do this by "extending $T^{(i)}$ by the identity," i.e., replacing it by

$$\begin{pmatrix} T^{(i)} & 0 \\ 0 & I_{n-i} \end{pmatrix},$$

and "extending $c^{(i)}$ by zero," i.e., replacing it by $(c^{(i)}, 0)^t$, so that we have $n \times n$ matrices and n-dimensional columns. These extended objects will again be denoted by $T^{(i)}$ and $c^{(i)}$; the context will clarify the intended dimension.

The following proposition uses the extended data and the non-standard matrix operation defined in 1.4 to express $E_{n,p}(T;c)$ more explicitly.

PROPOSITION 3.5. *Let $n \geq 2$, $0 \leq p \leq n-2$, and $(T;c) \in \mathscr{D}_n(A)$. Then*

$$E_{n,p}(T;c) = \operatorname{adj}(T^{(n-p)} \vartriangle T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)}) \cdot \left(c^{(n-p-1)}\big|_{(T^{(n-p)} \vartriangle T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)})X}\right).$$

The reader is referred to [5, Proposition 1.5].

Let ∂_i denote the differential operator $\partial/\partial X_i$ on $A[X]$. Let ∂ represent the column vector $(\partial_1, \ldots, \partial_n)^t$. Given $F = (F_1, \ldots, F_n)^t \in MA_n(A)$, the matrix multiplication $F^t \cdot \partial$ makes sense and denotes the derivation $F_1\partial_1 + \cdots + F_n\partial_n$ on $A[X]$.

The following result [5, Corollary 3.4] expresses the automorphism $X + E_{n,p}(T;c)$ as the exponential of a locally nilpotent derivation.

PROPOSITION 3.6. *Let $n \geq 2$, $0 \leq p \leq n-2$, and $(T;c) \in \mathscr{D}_n(A)$. The derivation $D = E_{n,p}(T;c)^t \cdot \partial$ is locally nilpotent, its exponential* $\exp(D)$ *is the polynomial map $X + E_{n,p}(T;c)$, and its inverse is given by* $\exp(-D) = X - E_{n,p}(T;c)$.

We now state the main theorem of [5][3]:

THEOREM 3.7. *Let $F = X + H$, where $H = E_n(T;c) = \sum_{p=0}^{n-2} E_{n,p}(T;c) + c^{(n)}$ for some $(T;c) \in \mathscr{D}_n(A)$. Then*

$$F = (X + c^{(n)}) \circ (X + E_{n,0}(T;c)) \circ \cdots \circ (X + E_{n,n-2}(T;c)).$$

3.8. *Notation.* We first observe from the realization of $E_{n,p}(T;c)$ given in Proposition 3.5 that it depends only on the components $T^{(n-p)}, \ldots, T^{(n)}$ and $c^{(n-p-1)}$ of $(T;c)$. For this reason, we discard the other components by introducing the set $\mathscr{D}_{n,p}(A)$ consisting of all $(p+2)$-tuples $(T^{(n-p)}, \ldots, T^{(n)}; f)$, where $T^{(n-p)}, \ldots, T^{(n)}$ are exactly as in Definition 3.2 and $f = (f_1, \ldots, f_{n-p-1}) \in A[X_{n-p}, \ldots, X_n]^{n-p-1}$. Note that with this shift in notation, the inductive realization of $E_{n,p}(T;f)$, given in Definition 3.2, now reads

$$\begin{aligned} E_{n,p}(T;f) &= (\operatorname{adj} T^{(n)}) \cdot \begin{pmatrix} H' \\ 0 \end{pmatrix}\Bigg|_{T^{(n)}X} \\ &= \left[((\operatorname{adj} T^{(n)})X) \circ (H', 0)^t \circ (T^{(n)}X)\right], \end{aligned} \tag{8}$$

[3] The statement of this theorem in [5], which appears there as Theorem 4.1, expresses the maps $X + E_{n,p}(T;c)$ as exponentials.

for $p \geq 1$, where $H' = E_{n-1,p-1}(T'; f)$. This is due to the fact that, for c and c' as in Definition 3.2, the $(n-p-1)$th component of c coincides with the $((n-1)-(p-1)-1)$th $= (n-p-1)$th component of c'. Note also that for $(T; f) \in \mathscr{D}_{n,p}(A)$, $E_{n,p}(T; f)$ can be realized by the $\triangle$-formula of Proposition 3.5, with f replacing $c^{(n-p-1)}$.

An additional observation that will be needed is that for $d \in A$ and $(T; c) \in \mathscr{D}_n(A)$ we have

$$d \cdot E_{n,p}(T; f) = E_{n,p}(T; df). \tag{9}$$

This is proved easily using induction and (8).

In view of the factorization given in the Theorem 3.7 the main goal of this section, which is the proof of Theorem 3.1, follows from the proposition below.

PROPOSITION 3.9. *Let* $F = X + E_{n,p}(T; f)$, *where* $(T; f) \in \mathscr{D}_{n,p}(A)$ *and* $0 \leq p \leq n-2$. *Then* $F^{[p+1]}$ *is a product of triangular automorphisms.*

The proof will follow from several lemmas. The explicit factorization will arise from the proof.

THEOREM 3.10. *Let* $(T; f), (T; g) \in \mathscr{D}_{n,p}(A)$. *Then*

$$\big(X + E_{n,p}(T; f)\big) \circ \big(X + E_{n,p}(T; g)\big) = \big(X + E_{n,p}(T; f+g)\big).$$

Proof. It is easy to see that $E_{n,p}(T; f) + E_{n,p}(T; g) = E_{n,p}(T; f+g)$. From this it is clear that the equation in the lemma is equivalent to

$$E_{n,p}(T; f) \circ \big(X + E_{n,p}(T; g)\big) = E_{n,p}(T; f), \tag{10}$$

which will be proved by induction on p.

- $p = 0$: By definition,

$$E_{n,0}(T; f) = \big((\operatorname{adj} T^{(n)})X\big) \circ (f_1, \dots, f_{n-1}, 0)^t \circ T^{(n)}$$

with $f_1, \dots, f_{n-1} \in A[X_n]$, and similarly for $E_{n,0}(T; g)$. Writing $(f, 0)^t$ in place of $(f_1, \dots, f_{n-1}, 0)^t$ and letting $d = \det(T^{(n)}) \in A$, we have

$$\begin{aligned}
&E_{n,0}(T; f) \circ (X + E_{n,0}(T; g)) \\
&\quad = \big((\operatorname{adj} T^{(n)})X\big) \circ (f, 0)^t \circ (T^{(n)}X) \\
&\qquad \circ \Big(X + \Big[\big((\operatorname{adj} T^{(n)})X\big) \circ (g, 0)^t \circ (T^{(n)}X)\Big]\Big) \\
&\quad = \big((\operatorname{adj} T^{(n)})X\big) \circ (f, 0)^t \circ \Big((T^{(n)}X) + \Big[(dX) \circ (g, 0)^t \circ (T^{(n)}X)\Big]\Big) \\
&\hspace{20em} \text{by LDP} \qquad \text{(see 1.3)} \\
&\quad = \big((\operatorname{adj} T^{(n)})X\big) \circ (f, 0)^t \circ \Big((T^{(n)}X) + \Big[(dg, 0)^t \circ (T^{(n)}X)\Big]\Big).
\end{aligned} \tag{11}$$

We note that the nth coordinate function of $(dg,0)\circ(T^{(n)}X)$ is 0, and, since f involves only X_n, the composition $(f,0)^t\circ((T^{(n)}X)+[(dg,0)^t\circ(T^{(n)}X)])$ is equal to $(f,0)^t\circ(T^{(n)}X)$. Therefore the composition of (11) is equal to

$$\big((\operatorname{adj}T^{(n)})X\big)\circ(f,0)^t\circ(T^{(n)}X)=E_{n,0}(T;f)$$

as desired.

- $p\geq 1$: letting $d=\det T^{(n)}$ and appealing to (8) we have

$$\begin{aligned}
&E_{n,p}(T;f)\circ\big(X+E_{n,p}(T;g)\big)\\
&\quad=\big((\operatorname{adj}T^{(n)})X\big)\circ\big(E_{n-1,p-1}(T';f),0\big)^t\\
&\qquad\underbrace{\circ\,(T^{(n)}X)\circ\Big(X+\Big[\big((\operatorname{adj}T^{(n)})X\big)\circ\big(E_{n-1,p-1}(T';g),0\big)^t\circ(T^{(n)}X)\Big]\Big)}\\
&\quad=\big((\operatorname{adj}T^{(n)})X\big)\circ\big(E_{n-1,p-1}(T';f),0\big)^t\\
&\qquad\circ\Big[(T^{(n)}X)+\underbrace{\big(d\cdot E_{n-1,p-1}(T';g),0\big)^t}\circ(T^{(n)}X)\Big]\\
&\quad=\big((\operatorname{adj}T^{(n)})X\big)\circ\big(E_{n-1,p-1}(T';f),0\big)^t\\
&\qquad\underbrace{\circ\Big[(T^{(n)}X)+\big(E_{n-1,p-1}(T';dg),0\big)^t\circ(T^{(n)}X)\Big]}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\text{by }(9))\\
&\quad=\big((\operatorname{adj}T^{(n)})X\big)\\
&\qquad\underbrace{\circ\big(E_{n-1,p-1}(T';f),0\big)^t\circ\Big[X+\big(E_{n-1,p-1}(T';dg),0\big)^t\Big]}\circ(T^{(n)}X)\\
&\quad=\big((\operatorname{adj}T^{(n)})X\big)\circ\big(E_{n-1,p-1}(T';f),0\big)^t\circ(T^{(n)}X)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\text{by induction on }(10))\\
&\quad=E_{n,p}(T;f)\qquad(\text{by }(8)).
\end{aligned}$$

This establishes (10), and hence the lemma. ∎

A consequence of this lemma is:

COROLLARY 3.11. *For* $f=(f_1,\ldots,f_{n-p-1})^t\in A[X_{n-p},\ldots,X_n]^{n-p-1}$, *the automorphism* $X+E_{n,p}(T;f)$ *is the composition, in any order, of the automorphisms*

$$X+E_{n,p}\big(T;(0,\ldots,0,f_i,0,\ldots,0)^t\big),$$

for $i = 1, \ldots, n-p-1$. *Here* f_i *appears in the ith position of the* $(n-p-1)$*-tuple* $(0, \ldots, 0, f_i, 0, \ldots, 0)$.

The next technical tool needed is:

LEMMA 3.12. *Let* e_i *be the* ith *unit column vector, and let* $E = (E_1, \ldots, E_n)^t \in A[X]^n$ *be defined by*

$$E = \left(\operatorname{adj}(T^{(n-p)} \vartriangle T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)}) \cdot e_i\right). \tag{12}$$

Let D *be the derivation on* $A[X]$ *defined by*

$$D = E^t \cdot \partial = E_1 \partial_1 + \cdots + E_n \partial_n.$$

Furthermore, let $h \in A[X]$ *be defined by*

$$h = f_i(X_{n-p}, \ldots, X_n)\Big|_{(T^{(n-p)} \vartriangle T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)})X}. \tag{13}$$

Then D *is a locally nilpotent derivation* $A[X]$, h *is in the kernel of* D, *and*

$$\exp(hD) = X + E_{n,p}\left(T; (0, \ldots, 0, f_i, 0, \ldots, 0)^t\right).$$

Proof. According to Proposition 3.5, $E_{n,p}(T; (0, \ldots, 0, f_i, 0, \ldots, 0))$ can be written as

$$\left(\operatorname{adj}(T^{(n-p)} \vartriangle T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)})\right) \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_i(X_{n-p}, \ldots, X_n)\Big|_{(T^{(n-p)} \vartriangle T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)})X} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here we have extended $(0, \ldots, 0, f_i, 0, \ldots, 0)$ to an n-tuple by adding zeros.

From this it is immediate that $E_{n,p}(T; (0, \ldots, 0, f_i, 0, \ldots, 0)^t) = hE = (hE_1, \ldots, hE_n)^t$, from whence it follows that

$$hD = hE \cdot \partial = E_{n,p}\left(T; (0, \ldots, 0, f_i, 0, \ldots, 0)^t\right) \cdot \partial.$$

According to Proposition 3.6, $\exp(hD)$ is a locally nilpotent derivation and $\exp(hD) = X + E_{n,p}(T; (0, \ldots, 0, f_i, 0, \ldots, 0))$. So it remains only to show that $D(h) = 0$. (The local nilpotence of D follows from this and the local nilpotence of hD.)

Let $(H_1, \dots, H_n)^t$ be the map $(T^{(n-p)} \triangle \cdots \triangle T^{(n)})X$, which, according to Lemma 1.5, can be expressed as $(T^{(n-p)}X) \circ \cdots \circ (T^{(n)}X)$. Then $h = f_i(H_{n-p}, \dots, H_n)$ and it suffices to show that D kills each of $H_{n-p}, \dots, H_n$.

At this point we introduce a new set of matrices which allows us to express $T^{(n-p)} \triangle \cdots \triangle T^{(n)}$ as an ordinary product of matrices. Viewing each $T^{(j)}$ as an n-dimensional matrix as in 3.4, set

$$S^{(j)} = T^{(j)}|_{S^{(j+1)} \cdots S^{(n)}X}. \tag{14}$$

For $j = n$ this reads $S^{(n)} = T^{(n)}$; hence (14) defines $S^{(j)}$ by descending induction.

LEMMA 3.13. *With $S^{(j)}$ defined as above, we have $T^{(n-p)} \triangle \cdots \triangle T^{(n)} = S^{(n-p)} \cdots S^{(n)}$.*

Proof. The proof is by induction on p. The case $p = 0$ is clear. For $p > 0$ we have

$$\begin{aligned}
&T^{(n-p)} \triangle \cdots \triangle T^{(n)} \\
&\quad = T^{(n-p)} \triangle \left(T^{(n-p+1)} \triangle \cdots \triangle T^{(n)}\right) \\
&\quad = T^{(n-p)}|_{(T^{(n-p+1)} \triangle \cdots \triangle T^{(n)})X} \cdot \left(T^{(n-p+1)} \triangle \cdots \triangle T^{(n)}\right) \\
&\quad = T^{(n-p)}|_{S^{(n-p+1)} \cdots S^{(n)}X} \cdot \left(S^{(n-p+1)} \cdots S^{(n)}\right) \qquad \text{(by induction)} \\
&\quad = S^{(n-p)} S^{(n-p+1)} \cdots S^{(n)}.
\end{aligned}$$

∎

Now we proceed with the task at hand, which is to show $D(H_r) = 0$ for $n - p \le r \le n$. The proof is by induction on $n - r$.

- $D(H_n) = 0$: Since each of the polynomial maps $T^{(n-p)}X, T^{(n-p+1)}X, \dots, T^{(n-1)}X$ fixes X_n, it is clear that $H_n = a_{n,1}X_1 + \cdots + a_{n,n}X_n$, where $(a_{n,1}, \dots, a_{n,n})$ is the bottom row of the matrix $T^{(n)}$. Then

$$\begin{aligned}
D(H_n) &= E^t \cdot \partial \cdot H_n \\
&= \left[\operatorname{adj}(T^{(n-p)} \triangle \cdots \triangle T^{(n)}) \cdot e_i\right]^t \cdot \partial \cdot H_n \\
&= \left[\operatorname{adj}(S^{(n-p)} \cdots S^{(n)}) \cdot e_i\right]^t \cdot \partial \cdot H_n \\
&= \left[(\operatorname{adj} S^{(n)}) \cdot \operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)}) \cdot e_i\right]^t \cdot \partial \cdot H_n \\
&= [e_i]^t \cdot \left[\operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)})\right]^t \cdot \left[\operatorname{adj}(S^{(n)})\right]^t \cdot (a_{n,1} \cdots a_{n,n})^t \\
&= [e_i]^t \cdot \left[\operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)})\right]^t \cdot \left[(a_{n,1}, \dots, a_{n,n}) \cdot \operatorname{adj}(S^{(n)})\right]^t
\end{aligned}$$

$$= [e_i]^t \cdot [\operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)})]^t \cdot [(a_{n,1}, \dots, a_{n,n}) \cdot \operatorname{adj}(T^{(n)})]^t$$

$$= [e_i]^t \cdot [\operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)})]^t \cdot (0, \dots, 0, d)^t$$

$$\text{where } d = \det T^{(n)}$$

$$= [(0, \dots, 0, d) \cdot \operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)}) \cdot e_i]^t.$$

Note from (14) that for $j < n$, the $S^{(j)}$ is extended from the lower dimension j. It follows that $\operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)})$, and hence its transpose, has the form

$$\begin{pmatrix} * & \cdots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 \\ 0 & \cdots & 0 & x \end{pmatrix} \tag{15}$$

and therefore

$$\begin{aligned} D(H_n) &= [(0, \dots, 0, d) \cdot \operatorname{adj}(S^{(n-p)} \cdots S^{(n-1)}) \cdot e_i]^t \\ &= [(0, \dots, 0, dx) \cdot e_i]^t \\ &= 0 \end{aligned}$$

since $i \le n - p - 1 < n$.

• $D(H_r) = 0$ for $n - p \le r < n$: Let $(G_1, \dots, G_{n-1})^t$ be the coordinate functions of the map $(T^{(n-p)}X) \circ (T^{(n-p+1)}X) \circ \cdots \circ (T^{(n-1)}X)$, considered as an element of $MA_{n-1}(A[X_n])$. Let $L = (L_1, \dots, L_n)^t \in MA_n(A)$ denote the linear polynomial map $T^{(n)}$. Then

$$\begin{aligned} (H_1, \dots, H_n)^t &= (T^{(n-p)}X) \circ \cdots \circ (T^{(n-1)}X) \circ (T^{(n)}X) \\ &= (G_1, \dots, G_{n-1}, X_n)^t \circ L \\ &= (G_1(L), \dots, G_{n-1}(L), L_n)^t, \end{aligned}$$

so $H_r = G_r(L)$.

Let $S = S^{(n-p)} \cdot S^{(n-p+1)} \cdots S^{(n)}$. According to (12) and Lemma 3.13, the derivation D can be expressed as

$$D = [(\operatorname{adj} S) \cdot e_i]^t \cdot \partial. \tag{16}$$

Define the $(n-1) \times (n-1)$-dimensional matrices $S'^{(n-p)}, \dots, S'^{(n-1)}$ by the $(n-1) \times (n-1)$-dimensional matrices defined according to (14), replacing n by $n-1$, so that $S' = S'^{(n-p)} \cdots S'^{(n-1)} = T^{(n-p)} \vartriangle \cdots \vartriangle T^{(n-1)}$. Denote this matrix by S', and note that, upon extending it to an $n \times n$ matrix, we have

$$\begin{aligned} S &= T^{(n-p)} \vartriangle \cdots \vartriangle T^{(n)} \\ &= S' \vartriangle T^{(n)} \\ &= S'(L) \cdot T^{(n)}. \end{aligned} \tag{17}$$

Continuing to view S' as an $n \times n$ matrix, let $(s_{j,k}) = \operatorname{adj} S'$, and note that $\operatorname{adj} S'$ has the form (15), and in particular,

$$s_{n,k} = 0 \qquad \text{for } 0 \le k \le n-1. \tag{18}$$

Now we compute $D(H_r)$.

$$\begin{aligned} D(H_r) &= D(G_r(L)) \\ &= \left[(\operatorname{adj} S) \cdot e_i\right]^t \cdot \partial \cdot G_r(L) \qquad (\text{by } (16)) \\ &= \left[(\operatorname{adj} T^{(n)}) \cdot (\operatorname{adj} S'(L)) \cdot e_i\right]^t \cdot \partial \cdot G_r(L) \qquad (\text{by } (17)) \\ &= [e_i]^t \cdot \left[\operatorname{adj}(S'(L))\right]^t \cdot \left[\operatorname{adj}(T^{(n)})\right]^t \cdot (\partial_1 G_r(L), \ldots, \partial_n G_r(L))^t \\ &= (\partial_1 G_r(L), \ldots, \partial_n G_r(L)) \cdot \operatorname{adj}(T^{(n)}) \cdot \operatorname{adj}(S'(L)) \cdot e_i, \end{aligned}$$

the last step being due to the fact that a 1×1 matrix is its own transpose. Let $T^{(n)} = (a_{j,k})$, and let $\operatorname{adj}(T^{(n)}) = (b_{j,k})$. The above is then equal to

$$\begin{aligned} &\sum_{t=1}^{n} \sum_{u=1}^{n} \partial_t G_r(L) b_{t,u} s_{u,i}(L) \\ &\quad = \sum_{t=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} (\partial_v G_r)(L)(\partial_t L_v) b_{t,u} s_{u,i}(L) \qquad (\text{by the chain rule}) \\ &\quad = \sum_{t=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} (\partial_v G_r)(L) a_{v,t} b_{t,u} s_{u,i}(L) \\ &\qquad\qquad (L_v = a_{v,1} X_1 + \cdots + a_{v,n} X_n) \\ &\quad = \sum_{u=1}^{n} \sum_{v=1}^{n} (\partial_v G_r)(L)\, d\, \delta_{v,u} s_{u,i}(L) \\ &\qquad\qquad (\text{where } d = \det T_n,\ \delta_{v,u} = \text{Kronecker delta}) \end{aligned}$$

$$= d \sum_{u=1}^{n} (\partial_u G_r)(L) s_{u,i}(L)$$

$$= d \sum_{u=1}^{n-1} (\partial_u G_r)(L) s_{u,i}(L)$$

$$\text{(because } s_{n,i} = 0 \text{ by (18), since } i \le n - p - 1 < n)$$

$$= d\big(D'(G_r)\big)(L),$$

where $D' = [(\operatorname{adj} S') \cdot e'_i]^t \cdot \partial' = [(\operatorname{adj}(S'^{(n-p)} \cdots S'^{(n)})) \cdot e'_i]^t \cdot \partial'$, where S' is viewed as an $(n-1) \times (n-1)$ matrix, e'_i is the $(n-1)$-dimensional ith unit column vector, and $\partial' = (\partial_1, \ldots, \partial_{n-1})^t$. By induction on $n - r$, we know that $D'(G_r) = 0$; hence $(D'(G_r))(L) = 0$ as desired.

Therefore $D(h) = 0$. ∎

The final tool needed in the proof of Proposition 3.9 is the often used result of Smith [11], which is:

LEMMA 3.14 (Smith's lemma). *Let D be a locally nilpotent derivation of a commutative ring R. Let $a \in \ker(D)$. Letting Y be a single indeterminate, extend D to the polynomial ring $R[Y]$ by setting $D(Y) = 0$. Note that aD and YD are locally nilpotent derivations of $R[Y]$. Let ρ be the R-automorphism of $R[Y]$ defined by $\rho(Y) = Y + a$. Then*

$$\exp(aD) = \rho^{-1} \circ \exp(-YD) \circ \rho \circ \exp(YD).$$

We are now prepared for the final step.

Proof of Proposition 3.4. Our goal is to show that if $F = X + E_{n,p}(T : f)$, where $(T; f) \in \mathscr{D}_{n,p}(A)$, $0 \le p \le n - 2$, then $F^{[p+1]}$ is tame. Our proof will go by induction on p, treating the case $p = 0$ in the process. So we fix p, and, if $p > 0$, we assume Proposition 3.9 holds for lower values of p (and any commutative ring A). By Corollary 3.11, we may assume F has the form $X + E_{n,p}(T; (0, \ldots, 0, f_i, 0, \ldots, 0)^t)$, which, according to Lemma 3.12, can be written as $\exp(hD)$ where $h = f_i(X_{n-p}, \ldots, X_n)|_{T^{(n-p)} \vartriangle \cdots \vartriangle T^{(n)} X}$, and $h \in \ker(D)$. We now apply Lemma 3.14 with $A[X]$ and h in the roles of R and a, respectively. Noting that ρ is a triangular automorphism of $A[X, Y]$, we see that we can factor $\exp(hD)^{[p+1]}$ by triangular automorphisms if we can do so for $\exp(YD)^{[p]}$. Since Y is fixed by the latter, we can consider the base ring to be $A[Y]$.

We form the $(n - p - 1)$-tuple $(0, \ldots, 0, Y, 0, \ldots, 0)$ with Y appearing in the ith position. Lemma 3.12 states that, upon extending $(0, \ldots, 0, Y, 0, \ldots, 0)$ to an n-tuple by adding zeros, $X + E_{n,p}(T; (0, \ldots, 0, Y, 0, \ldots, 0)^t) = \exp h'D$, where $h' = Y|_{T^{(n-p)} \vartriangle \cdots \vartriangle T^{(n)} X}$. Since Y is in the base ring, the substitution of $T^{(n-p)} \vartriangle \cdots \vartriangle T^{(n)} X$ into Y has no effect, so $h' = Y$ and we have

$$\exp YD = X + E_{n,p}\big(T;(0,\dots,0,Y,0,\dots,0)^t\big).$$

By Proposition 3.5 we have

$$\begin{aligned}
&E_{n,p}(T;(0,\dots,0,Y,0,\dots,0))\\
&\quad = \operatorname{adj}(T^{(n-p)} \triangle \cdots \triangle T^{(n)}) \cdot \begin{pmatrix} 0\\ \vdots \\ 0 \\ Y \\ 0 \\ \vdots \\ 0 \end{pmatrix}\Bigg|_{(T^{(n-p)} \triangle \cdots \triangle T^{(n)})X}\\
&\quad = \operatorname{adj}(T^{(n-p)} \triangle \cdots \triangle T^{(n)}) \cdot \begin{pmatrix} 0\\ \vdots \\ 0 \\ Y \\ 0 \\ \vdots \\ 0 \end{pmatrix}\\
&\qquad\qquad \text{(since the substitution has no effect)}\\
&\quad \underset{\text{Lemma 3.13}}{=} \operatorname{adj}(S^{(n-p)} \cdots S^{(n)}) \cdot \begin{pmatrix} 0\\ \vdots \\ 0 \\ Y \\ 0 \\ \vdots \\ 0 \end{pmatrix}\\
&\quad = \operatorname{adj}(S^{(n-p+1)} \cdots S^{(n)}) \cdot \left[(\operatorname{adj} S^{(n-p)}) \begin{pmatrix} 0\\ \vdots \\ 0 \\ Y \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]
\end{aligned}$$

$$= \operatorname{adj}(S^{(n-p+1)} \cdots S^{(n)}) \cdot \left[\operatorname{adj} T^{(n-p)}\big|_{S^{(n-p+1)} \cdots S^{(n)} X} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Y \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]$$

(by (14))

$$= \operatorname{adj}(S^{(n-p+1)} \cdots S^{(n)}) \cdot \left[\left(\operatorname{adj} T^{(n-p)}\right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Y \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]\Bigg|_{S^{(n-p+1)} \cdots S^{(n)} X}$$

$$= \operatorname{adj}(T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)}) \cdot \left[\left(\operatorname{adj} T^{(n-p)}\right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Y \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]\Bigg|_{(T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)})X} . \tag{19}$$

The inner column $[(\operatorname{adj} T^{(n-p)}) \cdot (0, \ldots, 0, Y, 0, \ldots, 0)^t]$ can be written as

$$\left((\operatorname{adj} T^{(n-p)})_{1,i} Y, \ldots, (\operatorname{adj} T^{(n-p)})_{n-p,i} Y, 0, \ldots, 0 \right)^t . \tag{20}$$

Note that the coordinates lie in $A[Y][X_{n-p+1}, \ldots, X_n]$. Now let g be the column of the first $n - p$ entries. If $p = 0$, then g is an n-tuple over $A[Y]$ and $g = E_{n,0}(T; (0, \ldots, 0, Y, 0, \ldots, 0)^t)$ (see Definition 3.3). Therefore

$$\exp YD = X + E_{n,0}\left(T; (0, \ldots, 0, Y, 0, \ldots, 0)^t\right) = X + g,$$

which is a translation over $A[Y]$, and thereby triangular. Thus the original F has become a product of triangular automorphisms with the adjunction of one new variable Y. If $p > 0$, we extend g to the n-tuple (20); appealing to (19), we write

$$\begin{aligned} E_{n,p}&\big(T;(0,\dots,0,Y,0,\dots,0)^t\big) \\ &= \operatorname{adj}(T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)}) \cdot \big(g|_{(T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)}X}\big) \\ &= E_{n,p-1}(T;g) \qquad \text{(by Proposition 3.5)}. \end{aligned}$$

The second equality holds because g has coordinates in $A[Y][X_{n-p+1}, \dots, X_n]$, viewing $A[Y]$ as the base ring. This is crucial; the argument would not have worked had we not replaced $(0,\dots,0,f,0,\dots,0)$ by $(0,\dots,0,Y,0,\dots,0)$ using Smith's lemma.

According to our induction on p, $X + E_{n,p}(T;(0,\dots,0,Y,0,\dots,0)^t)$ becomes a product of triangular automorphisms over $A[Y]$, with the addition of p new variables. Therefore the original $F = X + E_{n,p}(T;(0, \dots,0,f_i,0,\dots,0)^t)$ becomes a product of triangular automorphisms with the adjunction of $p+1$ new variables. This concludes the proof of Proposition 3.4, and thereby proves Theorem 3.1. ∎

4. THE ALGORITHMS

We will now derive from the proofs of Sections 2 and 3 explicit algorithms, based on the data $(T;c) \in \mathcal{D}_n(A)$, which factor $F = X + H$, with $H = E_n(T;c)$, as a product of triangular automorphisms.

The "quick method" of Section 2 did not use $\mathcal{D}_n(A)$ at all. The basic reduction occurs in Proposition 2.3. Note that when H is given by data $(T;c)$, then the matrix T and the column c that appear in the proof of Proposition 2.3 are entries $T^{(n)}$ and $c^{(n)}$ in $(T;c)$, and the H' that appears is constructed from the truncated data $(T';c')$ (see (6)). The reduction replaces $X+H$ by $(X, Y+K)^t$, where Y is a system of $n-1$ variables, $K = H'(dY_1 + L_1, \dots, dY_{n-1} + L_{n-1}, L_n)$, $L_1, \dots, L_n$ are the coordinate polynomials of $T^{(n)}X$, and $d = \det T^{(n)}$. This was seen to lie in $\mathcal{H}_{n-1}(A[X])$ by replacing $X_1, \dots, X_{n-1}$ by $Y_1, \dots, Y_{n-1}$ in $H' \in \mathcal{H}_{n-1}(A[X_n])$, making the base change $X_n \mapsto L_n$ (mapping $A[X_n]$ to $A[X_1, \dots, X_n]$), and appealing to Lemma 3.2 which allowed $Y_1, \dots, Y_{n-1}$ to be replaced by $dY_1 + L_1, \dots, dY_{n-1} + L_{n-1}$. To execute this reduction, it was necessary to know the matrix T_n. If $n = 2$, we are clearly done. If $n \geq 3$, then to continue the reduction we need to know the last matrix in the $\mathcal{D}_n(A[X])$ data for K; moreover we need to carry along this modified data for use in subsequent steps. Since H' is given by the truncated data $(T';c') = (T_2, \dots, T_{n-1}; c_1, \dots, c_{n-1})$, it is clear that variable replacement and the base change simply replace $(T';c')$ by $(T^*;c^*) = (T^{*(2)}, \dots, T^{*(n-1)}; c^{*(1)}, \dots, c^{*(n-1)})$, where

$T^{*(j)} = T^{(j)}(Y_1, \ldots, Y_{n-1}, L_n)$, $c^{*(j)} = c^{(j)}(Y_1, \ldots, Y_{n-1}, L_n)$. Thus we can continue if we can produce the data for the element $H(dX_1 + a_1, \ldots, dX_n + a_n) \in \mathscr{H}_n(A)$ of Lemma 2.2, given that H has data $(T; c)$. This is accomplished below.

LEMMA 4.1. *Given* $(T; c) \in \mathscr{D}_n(A)$ *and* $H = E_n(T; c) \in \mathscr{H}_n(A)$, *let* $d \in A$ *and* $a = (a_1, \ldots, a_n)^t \in A^n$. *Set* $\hat{T}^{(n)} = T^{(n)}$ *and let* $\ell^{(n)} = (\ell_1^{(n)}, \ldots, \ell_n^{(n)})^t = \hat{T}^{(n)} a \in A^n$, *and let* $W_n = dX_n + \ell_n^{(n)} \in A[X_n]$. *Given* r *with* $1 \le r < n$, *assume inductively that* $\ell^{(j)} \in A[X_{j+1}, \ldots, X_n]^j$ *and* $W_j \in A[X_j, \ldots, X_n]$ *have been defined for* $r < j \le n$. *Define*

$$\hat{c}^{(r)} = c^{(r)}(W_{r+1}, \ldots, W_n) \in A[X_{r+1}, \ldots, X_n]^r.$$

If $r \ge 2$ *define* $\hat{T}^{(r)}$ *and* W_r *as*

$$\hat{T}^{(r)} = T^{(r)}(W_{r+1}, \ldots, W_n) \in M_r(A[X_{r+1}, \ldots, X_n])$$

and

$$W_r = dX_r + \ell_r^{(r)} \in A[X_r, \ldots, X_n],$$

where

$$\ell^{(r)} = \hat{T}^{(r)} \cdot \begin{pmatrix} \ell_1^{(r+1)} \\ \vdots \\ \ell_r^{(r+1)} \end{pmatrix} \in A[X_{r+1}, \ldots, X_n].$$

Then $(\hat{T}; \hat{c}) = (\hat{T}^{(2)}, \ldots, \hat{T}^{(n)}; \hat{c}^{(1)}, \ldots, \hat{c}^{(n)}) \in \mathscr{D}_n(A)$ *and* $H(dX_1 + a_1, \ldots, dX_n + a_n) = E_n(\hat{T}; \hat{c})$.

This is proved without difficulty by using induction and following the proof of Lemma 2.2. The algorithm for the stable factorization of F by the quick method is now complete, and is summarized as follows:

4.2. *Algorithm for the Quick Method*

Given $F = X + H$ with $H = E_n(T; c) \in \mathscr{H}_n(A)$, $n \ge 2$, we adjoin variables $Y_1, \ldots, Y_n$. Defining U and V as in the proof of Proposition 2.3, replacing T and c by $T^{(n)}$ and $c^{(n)}$, we have $F^{[n-1]} = U^{-1} \circ (X, Y + K)^t \circ V^{-1}$, from (5), where $K = H'(dY_1 + L_1, \ldots, dY_{n-1} + L_{n-1}, L_n)$, $d = \det T^{(n)}$, and $(L_1, \ldots, L_n)^t = T^{(n)} X$. If $n = 2$ we are done. If $n > 2$, then to get the $\mathscr{D}_{n-1}(A[X])$ data for K, we recall from above that the data for $H'(Y_1, \ldots, Y_{n-1}, L_n)$ is $(T^*; c^*)$ where $T^{*(j)} = T^{(j)}(Y_1, \ldots, Y_{n-1}, L_n)$ for $2 \le j \le n-1$, $c^{*(j)} = c^{(j)}(Y_1, \ldots, Y_{n-1}, L_n)$ for $1 \le j \le n-1$. Therefore

$K = E_{n-1}(\hat{T}^*; \hat{c}^*)$, where $(\hat{T}^*, \hat{c}^*)$ is the data defined in Lemma 4.1 starting with (T^*, c^*), setting $d = \det T^{(n)}$ and $a = (L_1, \ldots, L_n)^t$. If $n > 2$ we proceed to the next step, using $\hat{T}^{*(n-1)}K$ and $\hat{c}^{*(n-1)}$ to define a new U and V. This eventually factors F using $n(n-1)/2$ new variables.

Since the strong method uses $\mathscr{D}_n(A)$, its algorithm is much more apparent from the proof. It goes as follows:

4.3. *Algorithm for the Strong Method*

Given $F = X + H$ with $H = E_n(T; c) \in \mathscr{H}_n(A)$, $n \geq 2$, we factor F according to Theorem 3.7. Thus it suffices to give an algorithm for factoring $X + E_{n,p}(T; f)$, where $0 \leq p \leq n-2$ and $(T; f) \in \mathscr{D}_{n,p}(A)$, using $p + 1$ new variables. Writing $f = (f_1, \ldots, f_{n-p-1})$, we use Corollary 3.11 to factor $X + E_{n,p}(T; f)$ as the composition of the $X + E_{n,p}(T; (0, \ldots, f_i, 0, \ldots, 0)^t)$ and proceed to factor each of these components. We now follow the proof of Proposition 3.4, which begins by employing Smith's lemma to adjoin one variable Y, and write

$$\begin{aligned}\left(X + E_{n,p}\left(T; (0, \ldots, f_i, 0, \ldots, 0)^t\right)\right)^{[1]} &= \rho^{-1} \circ \left(X + E_{n,p}\left(T; -(0, \ldots, Y, 0, \ldots, 0)^t\right), Y\right)^t \\ &\circ \rho \circ \left(X + E_{n,p}\left(T; (0, \ldots, Y, 0, \ldots, 0)^t\right), Y\right)^t,\end{aligned}$$

where

$$\rho = \left(X_1, \ldots, X_n, Y + f_i(X_{n-p}, \ldots, X_n)\big|_{(T^{(n-p)} \vartriangle T^{(n-p+1)} \vartriangle \cdots \vartriangle T^{(n)})X}\right)^t.$$

We can complete our procedure considering $A[Y]$ as the base ring and factoring $X + E_{n,p}(T; (0, \ldots, Y, 0, \ldots, 0)^t)$ (and in similar fashion its inverse) using p new variables. Letting

$$g = \left((\operatorname{adj} T^{(n-p)})_{1,i} Y, \ldots, (\operatorname{adj} T^{(n-p)})_{n-p,i} Y\right)$$

we have, if $p = 0$, $X + E_{n,0}(T; (0, \ldots, Y, 0, \ldots, 0)^t) = X + g$, which is triangular, finishing the task in this case. If $p > 0$ we have $X + E_{n,p}(T; (0, \ldots, Y, 0, \ldots, 0)^t) = E_{n,p-1}(T, g)$, to which we apply the algorithm recursively.

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