On the structure of cube tilings and unextendible systems of cubes in low dimensions

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ABSTRACT

We give a structural description of cube tilings and unextendible cube packings of $\mathbb{R}^3$. We also prove that up to dimension 4 each cylinder of a cube tiling contains a column and demonstrate by an example that the latter result does not hold in dimension 5.

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1. Introduction

We define a cube in the $n$-dimensional Euclidean space $\mathbb{R}^n$ to be any translate of the unit cube $[0, 1]^n$. Let $T$ be a subset of $\mathbb{R}^n$. The family $[0, 1]^n \oplus T := \{(0, 1)^n + t : t \in T\}$ is said to be a cube tiling of $\mathbb{R}^n$ if for each pair of distinct vectors $s, t \in T$ the cubes $[0, 1]^n + s$ and $[0, 1]^n + t$ are disjoint and 

$$\mathbb{R}^n = [0, 1]^n + T = \{x + t : x \in [0, 1]^n, t \in T\}.$$

We refer to $T$ as a set that determines a cube tiling. It is rather easy to observe that each cube tiling of $\mathbb{R}^2$ has a layered structure with layers parallel to one of the coordinate axes. Our main objective is to show that cube tilings of $\mathbb{R}^3$ can be understood on the same level: if a cube tiling of $\mathbb{R}^3$ is not layered, then its structure is as described in our Theorem 4.

As usual, we denote by $\mathbb{Z}$ the set of all integers while the set of positive integers is denoted by $\mathbb{N}$. Let $n \in \mathbb{N}$. The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. The following theorem is due to Ott-Heinrich Keller [9].

**Theorem 1.** If $[0, 1]^n \oplus T$ is a cube tiling of $\mathbb{R}^n$, then for each pair of distinct elements $s, t \in T$ there is $i \in [n]$ such that $|s_i - t_i| \in \mathbb{N}$.

(The reader can consult [8] or [13] for the proof.)

Keller’s result motivates the following definition. A set $S \subset \mathbb{R}^n$ satisfies Keller’s condition if for each pair of distinct elements $s, t \in S$ there is an index $i \in [n]$ such that $|s_i - t_i| \in \mathbb{N}$.

Let $A$ be a subset of $\mathbb{R}^n$, and $i \in [n]$. In what follows, the set of all $i$-th coordinates of the elements of $A$ is denoted by $A_i$, that is, $A_i = \{a_i : a = (a_1, \ldots, a_n) \in A\}$.

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Let $W$ be a non-empty subset of $\mathbb{R}^n$. The family $\{0, 1\}^n \oplus W$ is layered (in direction of the $i$-th vector $e_i$ of the standard basis) if there is a number $\alpha \in \mathbb{R}$ such that $W_i \subseteq \alpha + \mathbb{Z}$. If $W$ is not layered in any direction, then it is called non-layered.

The family $\{0, 1\}^n \oplus W$ is a cylinder of a tiling $\{0, 1\}^n \oplus T$ ($W$ determines a cylinder) in direction of the $i$-th vector $e_i$ of the standard basis, if there is a number $\alpha \in \mathbb{R}$ such that $W = \{t = (t_1, \ldots, t_n) \in T : t_i \in \alpha + \mathbb{Z}\}$. In particular, each cylinder is layered.

A system of cubes $\{0, 1\}^n \oplus S$ is a column ($S$ determines a column) if for arbitrary $s \in S$ and some $i \in [n]$

$$S = s + Z e_i = \{s + k e_i : k \in \mathbb{Z}\}.$$ Extending Minkowski’s conjecture concerning lattice cube tilings, in 1930 Keller [9] raised the question whether each cube tiling of $\mathbb{R}^n$ contains a cylinder. In 1992, Jeffrey Lagarias and Peter Shor [12], using an approach of Keresztély Corrádi and Sándor Szabó [3], constructed a cube tiling in $\mathbb{R}^{10}$ which contains no pair of cubes sharing an $(n-1)$-dimensional face. Ten years later John Mackey [14] found a relevant example in dimension 8 (see also [16, 10] where simplified approaches are presented). Building on an idea of Oscar Perron [15], the present authors proved [13] that the answer to Keller’s question is positive up to dimension 6. Therefore, Keller’s question remains open only in dimension 7. Here we show that up to dimension 4 each cylinder of a tiling contains a column (Theorem 6). We demonstrate by an example that this theorem is no longer valid in dimension 5.

A set $U \subseteq \mathbb{R}^n$ is unextendible, or determines an unextendible system of cubes in $\mathbb{R}^n$, if $U$ satisfies Keller’s condition and is not a proper subset of any other set which satisfies Keller’s condition. Otherwise, we say that $U$ is extendible, or determines an extendible system of cubes. It can be easily proved that unextendible systems of cubes in $\mathbb{R}^2$ coincide with cube tilings. It was already known to Keller [9] that there are nontrivial unextendible systems of cubes in $\mathbb{R}^3$. An example is discussed in [11, Appendix A]. Such systems relate to the so-called unextendible product bases, which are extensively studied within the framework of quantum information theory (see e.g. [1, 2, 4, 5]). For this reason, we found it important to classify all nontrivial unextendible sets in $\mathbb{R}^3$ (Theorem 5).

2. Results

If $A \subseteq \mathbb{Z}$, then $A'$ denotes the complement $\mathbb{Z} \setminus A$ of $A$ in $\mathbb{Z}$. We denote by $* + \mathbb{Z}$ an unspecified member of the family $\alpha + \mathbb{Z}$, $\alpha \in \mathbb{R}$. The expression $X \times (\alpha + \mathbb{Z})$ means an unspecified member of the family of sets $\bigcup_{x \in X} \{x\} \times (\alpha(x) + \mathbb{Z})$, where $\alpha$ runs over all functions from $X$ to $\mathbb{R}$. Meanings of the expressions $(\alpha + \mathbb{Z}) \times Y$ and $X \times (\alpha + \mathbb{Z}) \times Y$ are defined analogously.

Our first lemma is rather obvious.

Lemma 1. Let $A$ and $B$ be proper subsets of $\mathbb{Z}$, and $\alpha$ and $\beta$ be arbitrary real numbers. If

$$[0, 1) + \alpha + A = [0, 1) + \beta + B,$$

then $\alpha + A = \beta + B$ and $\alpha \equiv \beta \pmod{\mathbb{Z}}$.

Theorem 2. Each cylinder of a cube tiling of $\mathbb{R}^3$ contains a column.

Proof. Suppose the theorem does not hold. Then there is a cube tiling $[0, 1)^3 \oplus T$ and a set $W$ determining a cylinder of this tiling in direction $e_3$, which does not contain a column. Let $W_i = \{w_i : (w_1, w_2, w_3) \in W\}$. By the definition of a cylinder, there is a number $\gamma \in \mathbb{R}$ such that $W_3 = \gamma + \mathbb{Z}$. For each integer $k$ let us set

$$T^k = \{p \in \mathbb{R}^2 : \text{there is } q \in \mathbb{R} \text{ such that } (p, q) \in T, \ 0 \leq \gamma + k - q < 1\},$$

$$W^k = \{p \in T^k : \text{there is } q \in \mathbb{R} \text{ such that } (p, q) \in W\}.$$ Observe that each of the sets $T^k, k \in \mathbb{Z}$, determines a cube tiling of $\mathbb{R}^2$. Moreover, the sets $[0, 1)^2 + W^k, k \in \mathbb{Z}$, are identical. Consequently, each of the sets

$$T^{k_1} \cup W^l, \quad (k, l) \in \mathbb{Z}^2,$$
determines a cube tiling of \( \mathbb{R}^2 \). Now, if any of the sets \( W^k \) had as a subset a set \( K \) determining a column, then the set \( K \times \{ y + k \} \) would determine a column contained in \( [0, 1)^3 \oplus W \). Therefore, none of the sets \( W^k, \, k \in \mathbb{Z} \), contains a set determining a column.

Now, we prove the following

**Claim.** For every integer \( k \) there are reals \( \alpha^k \) and \( \beta^k \) such that \( W^k_1 \subseteq \alpha^k + \mathbb{Z} \) and \( W^k_2 \subseteq \beta^k + \mathbb{Z} \).

Clearly, it suffices to show that the supposition \( W^k_2 \not\subseteq \beta^k + \mathbb{Z} \) for a certain \( k \) and every \( \beta^k \in \mathbb{R} \) leads to the conclusion that \( W \) contains a set determining a column. Since every cube tiling of \( \mathbb{R}^2 \) is layered, there is \( \alpha^k \in \mathbb{R} \) such that \( T^k = \alpha^k + \mathbb{Z} \). In particular, \( W^k_1 \subseteq \alpha^k + \mathbb{Z} \). Let \( p \in W_1^k \). Then there is \( \beta \in \mathbb{R} \) such that \( \{ p \} \times (\beta + \mathbb{Z}) \subseteq T^k \). Since the set \( \{ p \} \times (\beta + \mathbb{Z}) \) determines a column, it cannot be a subset of \( W^k \). Consequently, there is \( S \subseteq T^k \setminus W^k \) such that \( S_1 = W^k_1 \) and \( S_2 = W^k_2 \mod \mathbb{Z} \). Let us consider the sets \( T^k_1 \), \( l \in \mathbb{Z} \). As \( S_1 \subseteq \alpha^k + \mathbb{Z} \), we have \( T^k_1 \subseteq \alpha^k + \mathbb{Z} \). Thus

\[
W^k_1 \subseteq \alpha^k + \mathbb{Z}, \quad \text{for every } l \in \mathbb{Z}.
\]

Therefore, if we take into account that \( [0, 1)^2 + W^l = [0, 1)^2 + W^k \), then we get

\[
W^l_1 = W^k_1, \quad \text{for every } l \in \mathbb{Z}.
\]

Let \( \ell = \{(p, q) : q \in \mathbb{R}\} \). Since \( \ell \not\subseteq [0, 1)^2 + W^l, \, l \in \mathbb{Z} \), (as in the other case \( [0, 1)^2 \oplus W^l \) would contain a column), according to the equation

\[
\ell \cap ([0, 1)^2 + W^l) = \ell \cap ([0, 1)^2 + W^k)
\]

and Lemma 1, we have \( \ell \cap W^l = \ell \cap W^k \). Since \( p \) is arbitrary, we deduce that \( W^l = W^k \) for every \( l \in \mathbb{Z} \). This equality implies that \( x + \mathbb{Z}e_3 \) is a subset of \( W \) for every \( x \in W \). Thus \( [0, 1)^3 \oplus W \) contains a column.

Our claim can be rephrased as follows: for every \( k \), there are reals \( \alpha^k, \beta^k \) such that

\[
W^k \subseteq (\alpha^k, \beta^k) + \mathbb{Z}.
\]

Let

\[
V^l = \{ x \in W^l : \text{there is } (x, y) \in W^l \text{ such that } \ell \cap ((x, y) + [0, 1)^2) \neq \emptyset \}.
\]

By (1), we have

\[
V^k + [0, 1) = V^l + [0, 1).
\]

Since \( V^l \subseteq W^l \subseteq \alpha^l + \mathbb{Z} \), we deduce from Lemma 1 that \( \alpha^k = \alpha^l \) for every pair \( k, l \). By the same argument, \( \beta^k = \beta^l \) for every \( k, l \). Therefore, there are reals \( \alpha \) and \( \beta \) such that

\[
W^k \subseteq (\alpha, \beta) + \mathbb{Z}^2.
\]

As the sets \( W^k, \, k \in \mathbb{Z} \), do not contain subsets determining columns and the sets \( [0, 1)^2 + W^k, \, k \in \mathbb{Z} \), are all equal, all the sets \( W^k \) have to be equal. Let us fix \( s \in W^0 \). The set \( \{ s \} \times (\gamma + \mathbb{Z}) \) is contained in \( W \) and determines a column, which is a contradiction.

**Lemma 2.** Let \( [0, 1)^3 \oplus T \) be a non-layered, unextendible system of cubes, which contains a column in direction \( e_3 \) and let \( S \subseteq T \) be the union of all sets determining columns in direction \( e_3 \) contained in \( [0, 1)^3 \oplus T \). Then there are proper subsets \( A, B \subseteq \mathbb{Z} \) and reals \( \alpha, \beta \) such that

\[
S = (\alpha + A) \times (\beta + B) \times (* + \mathbb{Z}).
\]

**Proof.** By the existence of a column, there are \( \alpha, \beta \in \mathbb{R} \) such that \( \{ \alpha \} \times \{ \beta \} \times (* + \mathbb{Z}) \subseteq T \). Suppose that \( \{ \alpha' \} \times \{ \beta' \} \times (* + \mathbb{Z}) \subseteq T \). Then

\[
\alpha \equiv \alpha' \pmod{\mathbb{Z}} \quad \text{and} \quad \beta \equiv \beta' \pmod{\mathbb{Z}},
\]

otherwise we could assume \( \alpha \not\equiv \alpha' \pmod{\mathbb{Z}} \). Then Keller's condition related to the set determining both columns would imply \( \beta \equiv \beta' \pmod{\mathbb{Z}} \). Now, let \((t_1, t_2, t_3)\) be any element of \( T \). Keller's condition
enforces $t_2 \equiv \beta \pmod{\mathbb{Z}}$. Therefore, $T$ would be layered, which is impossible. Suppose now that

$$\alpha \neq \alpha' \quad \text{and} \quad \beta \neq \beta'.$$

Then there is $z$ such that $(\alpha, \beta', z) \in T$. As in the other case, for any $t \in T$, we would have $t_1 \equiv \alpha \pmod{\mathbb{Z}}$ or $t_2 \equiv \beta \pmod{\mathbb{Z}}$. Consequently, $T \cup \{(\alpha, \beta', 0)\}$ would satisfy Keller’s condition which would contradict unextendability of $T$. In such circumstances, not only $(\alpha, \beta', z) \in T$ but also $(\alpha, \beta', z + k) \in T$, whenever $k \in \mathbb{Z}$. Therefore, $\{(\alpha) \times \{\beta'\} \times (\ast + \mathbb{Z}) \} \subset T$. By analogy, the same holds for the pair $\alpha'$, $\beta$. As a result, $S$ is of the form described by (2). The sets $A$ and $B$ have to be proper subsets of $\mathbb{Z}$, otherwise $T$ would be layered. □

**Theorem 3.** If $[0, 1)^3 \ominus T$ is an unextendible system of cubes in $\mathbb{R}^3$ which contains a column, then it is a tiling of $\mathbb{R}^3$.

**Proof.** Since each family of cubes in $\mathbb{R}^3$ which is unextendible and layered has to be a tiling, we may restrict our consideration to the case when $[0, 1)^3 \ominus T$ is supposed to be non-layered. We may also assume that the column is in direction $e_3$. Let us consider all columns in this direction contained in $[0, 1)^3 \ominus T$. Let $S$ be as defined in Lemma 2. These columns can be translated along the third axis so that the set $S$ will take the form $S = (\alpha + A) \times (\beta + B) \times \mathbb{Z}$. Then $T$ will change but the union $[0, 1)^3 \ominus T$ will not be affected. Clearly, the resulting set will remain unextendible. Moreover, translating the whole system, we may assume that both $\alpha$ and $\beta$ are equal to 0. Then

$$S = A \times B \times \mathbb{Z}.$$ 

Now, $T$ is either layered or not. If it is, then there is nothing to prove. Suppose that it is not. Then define subsets $U^i$, $i = 1, 2$ of $T$ so that $t \in U^i$ if and only if $t \notin \mathbb{Z}$. Clearly, they are disjoint with $S$ and non-empty. If we compare subsequently $S$ with $U^1$, $S$ with $U^2$ and finally $U^1$ with $U^2$, then according to Keller’s condition we obtain the following statements.

(i) If $x \in U^1$, then $x_2 \equiv 0 \pmod{\mathbb{Z}}$.
(ii) If $x \in U^2$, then $x_1 \equiv 0 \pmod{\mathbb{Z}}$.
(iii) If $x \in U^1 \cup U^2$, then there is $\eta \in \mathbb{R}$ such that $x_3 \equiv \eta \pmod{\mathbb{Z}}$.

The third of them possibly requires explanation. Fix an element $y \in U^2$ and define $\eta = y_3$. Then by the definition of $U^i$, claims (i) and (ii) and Keller’s condition, we get $x_3 = \eta \pmod{\mathbb{Z}}$ for every $x \in U^1$. The result follows now by interchanging $U^1$ and $U^2$.

Let $x \in U^1$. It is not hard to see that $x + ke_1 = (x_1 + k, x_2, x_3)$ belongs to $T$ for every $k \in \mathbb{Z}$. Indeed, suppose that it is not the case for certain $k$. Then fix an element $t \in T$. If $t \neq x \pmod{\mathbb{Z}}$, then the pair $t, x + ke_1$ satisfies Keller’s condition, as the pair $t, x$ does. If $t \equiv x \pmod{\mathbb{Z}}$, then stating that the pair $t, x + ke_1$ does not satisfy Keller’s condition would imply $t_1 = x_1 + k$. In particular, $t \in U^1$. Since, by our supposition, $t \neq x + ke_1$, we have $(t_2, t_3) \neq (x_2, x_3)$. Now, by claims (i) and (iii) we would draw the conclusion that the pair of vectors $t, x + ke_1$ would satisfy Keller’s condition contrary to the assumption that it would not. Therefore, we have proved that $[0, 1)^3 \ominus U^1$ splits into disjoint columns extending along the first axis; that is, there is a subset $D \subset \mathbb{R}^2$ such that $U^1 = (\ast + \mathbb{Z}) \times D$. Again, we can modify $T$ by replacing $U^1$ with $\mathbb{Z} \times D$. As before, the union $[0, 1)^3 \ominus T$ will remain unchanged but the resulting family of cubes will be now layered and unextendible. Therefore, it is a tiling, which implies that the family we start with is a tiling. □

**Theorem 4.** Let $T$ determine a non-layered cube tiling of $\mathbb{R}^3$. Then there are reals $\alpha, \beta$, $\gamma$ and proper subsets $A, B, C$ of $\mathbb{Z}$ such that $T$ is a disjoint union of the following sets

$$\begin{align*}
(\alpha + A) \times (\beta + B) \times (\ast + \mathbb{Z}), \\
(\alpha + A') \times (\ast + \mathbb{Z}) \times (\gamma + C), \\
(\ast + \mathbb{Z}) \times (\beta + B') \times (\gamma + C'), \\
(\alpha + A') \times (\beta + B) \times (\gamma + C'), \\
(\alpha + A) \times (\beta + B') \times (\gamma + C).
\end{align*}$$

(Fig. 1).
Proof. By Theorem 2, the tiling determined by $T$ contains a column. We may assume that it is a column in direction $e_3$. By Lemma 2, the set $M^3 \subseteq T$ which is the union of all sets determining the columns in direction $e_3$ can be expressed as follows

$$M^3 = (\alpha + A) \times (\beta + B) \times (\ast + \mathbb{Z}),$$

where $\alpha$ and $\beta$ are certain reals while $A$ and $B$ are proper subsets of $\mathbb{Z}$. Since the tiling $[0, 1)^3 \oplus T$ is non-layered, there is a number $\beta' \notin \beta + \mathbb{Z}$ such that the set $\{t \in T : t_2 \in \beta' + \mathbb{Z}\}$ determines a cylinder of $[0, 1)^3 \oplus T$. Again by Theorem 2, this cylinder contains a column. This column cannot be in direction $e_3$, as in that case the sets $\beta + B$ and $\beta' + \mathbb{Z}$ would have an element in common, which is impossible. It cannot also be in direction $e_1$, as in that case the set determining the column would be of the form $(* + \mathbb{Z}) \times \{\beta' + k\} \times \{\rho\}$, where $k$ is an integer, and $\rho$ a real. Then we could pick one element from this set and another from $M^3$ so that they would not satisfy Keller’s condition, which is impossible again. As a result, there is a column in direction $e_2$ which is disjoint with all the columns in direction $e_3$. By a variant of Lemma 2, where direction $e_3$ is replaced by $e_2$, we deduce that the set $K^2$ which is equal to the union of all sets that determine columns of our tiling in direction $e_2$ has the following form

$$K^2 = (\alpha^0 + A^0) \times (\ast + \mathbb{Z}) \times (\gamma + C),$$

where $\alpha^0$, $\gamma$ are certain reals and $A^0$, $C$ are proper subsets of the integers.

Let $L^2$ be a subset of $K^2$ consisting of all those vectors each of which belongs to a set determining a certain column in direction $e_2$, that is disjoint with all the columns in direction $e_3$, $L^2$ is nonempty, and by Keller’s condition satisfies the following relation

$$L^2 \subseteq M^2 := (\alpha + A') \times (\ast + \mathbb{Z}) \times (\gamma + C).$$

Since the tiling $[0, 1)^n \oplus T$ is non-layered, there is a number $\alpha' \notin \alpha + \mathbb{Z}$ such that the set $\{t \in T : t_1 \in \alpha' + \mathbb{Z}\}$ determines a cylinder of this tiling. By Theorem 2 this cylinder contains a column. Keller’s condition related to the union of $L^2$, $M^3$ and the set determining this column implies that this column extends along the first axis. Moreover, the vectors that determine it are outside of the union of $L^2$ and $M^3$. Let $L^1$ be the set consisting of all these vectors each of which belongs to a set determining a certain column in direction $e_1$, that is disjoint with all the columns in direction $e_3$ and the columns in direction $e_2$ determined by the elements of $L^2$. By Keller’s condition, we have

$$L^1 \subseteq M^1 = (\ast + \mathbb{Z}) \times (\beta + B') \times (\gamma + C').$$
Now, let \( t \in T \setminus \bigcup_{i=1}^{3} M(i) \). Keller’s condition implies

\[
t \in (\alpha + A') \times (\beta + B) \times (\gamma + C') \cup (\alpha + A) \times (\beta + B') \times (\gamma + C).
\]

Consequently,

\[
T \subseteq (\alpha + A') \times (\beta + B) \times (\gamma + C') \cup (\alpha + A) \times (\beta + B') \times (\gamma + C) \cup \bigcup_{i=1}^{3} M_i.
\]

As the set on the right hand side determines a disjoint family of cubes, \( T \) cannot be its proper subset. Therefore, we have obtained the expected representation of \( T \).

**Theorem 5.** Let \([0, 1)^3 \oplus T\) be an unextendible system of cubes in \( \mathbb{R}^3 \), which is not a tiling. Then there are reals \( \alpha^1, \alpha^2, \beta^1, \beta^2, \gamma^1, \gamma^2 \) such that \( \alpha^1 \not\equiv \alpha^2 \pmod{\mathbb{Z}} \), \( \beta^1 \not\equiv \beta^2 \pmod{\mathbb{Z}} \), \( \gamma^1 \not\equiv \gamma^2 \pmod{\mathbb{Z}} \), and proper subsets \( A, B, C, D, E, F \) of \( \mathbb{Z} \) such that \( T \) is the union of the following sets

\[
\begin{align*}
T^1 &= (\alpha^1 + A) \times (\beta^1 + B) \times (\gamma^1 + C), \\
T^2 &= (\alpha^1 + A') \times (\beta^1 + D) \times (\gamma^2 + E), \\
T^3 &= (\alpha^2 + F) \times (\beta^1 + B') \times (\gamma^2 + E'), \\
T^4 &= (\alpha^2 + F') \times (\beta^2 + D') \times (\gamma^1 + C').
\end{align*}
\]

*(Fig. 2).*

**Proof.** The proof consists of several claims.

**Claim 1.** For each \( i \in \{1, 2, 3\} \), the set \( T_i = \{ s : s = (s_1, s_2, s_3) \in T \} \) is a union of at least two cosets of \( \mathbb{Z} \).

Fix \( t = (t_1, t_2, t_3) \in T \). Observe that if there would exist an integer \( k \) such that \( T \) would not contain any vector whose \( i \)-th coordinate would differ from \( t_i \) on \( k \), then we could extend \( T \) by attaching the vector \( s \) such that \( s_i = t_i + k \) and \( s_j = t_j \), whenever \( j \neq i \). The resulting set would satisfy Keller’s condition. On the other hand, this set would violate the unextendability of \( T \). Therefore, we have \( t_i + \mathbb{Z} \subset T_i \), \( i = 1, 2, 3 \). There are at least two cosets contained in each \( T_i \), otherwise \([0, 1)^3 \oplus T\) would be a layered cube tiling.

For \( v \in T \), let us define

\[
W_v = \{ t \in T : t_1 \equiv v_1 \pmod{\mathbb{Z}} \}.
\]

**Claim 2.** There does not exist \( v \in T \) such that \( W_v \subseteq v + \mathbb{Z}^3 \).
Conversely, suppose that such a \( v \) exists. Then take any \( k \in \mathbb{Z} \). If \( t \in T \setminus W_v \), then \( t_1 \notin v_1 + \mathbb{Z} \). By Keller’s condition, there is \( j \in \{2, 3\} \) such that \( t_1 \neq v_1 \) and \( t_j \in v_j + \mathbb{Z} \). The same relations hold with \( v \) replaced by \( v + ke_1 \). Moreover, the latter vector belongs to \( v + \mathbb{Z}^3 \). Therefore, \( T \cup \{v + ke_1\} \) is a set that satisfies Keller’s condition. The unextendability of \( T \) implies that \( v + ke_1 \in T \). This leads to the conclusion that \( v + \mathbb{Z}e_1 \subseteq T \). Equivalently, \([0, 1)^3 \oplus T\) contains a column. By Theorem 3, \( T \) determines a cube tiling of \( \mathbb{R}^3 \), which contradicts our assumption that it does not.

**Claim 3.** For any \( v \in T \), there are no two elements \( p, q \in W_v \) such that \( p_i \equiv q_i \pmod{\mathbb{Z}} \) for some \( i \in \{2, 3\} \) and \( p_j \equiv q_j \pmod{\mathbb{Z}} \) for the other \( j \in \{2, 3\} \).

Suppose that for some \( W_v \), such a pair \( p, q \) exists. We may assume that \( i = 2 \) and \( j = 3 \). Let \( t \in T \setminus W_v \). Then Keller’s condition related to \( p, q \) and \( t \) implies that \( t_3 \in p_3 + \mathbb{Z} \). Therefore, \( T \setminus W_v \) is layered in direction \( e_3 \). Since \( T_1 \) consists of at least two cosets of \( \mathbb{Z} \) (Claim 1), there is \( u \in T \) such that \( W_u \) and \( W_v \) are disjoint. A fortiori, \( W_u \) is layered in direction \( e_3 \). Thus, according to Claim 2, \( W_u \) is not layered in direction \( e_2 \). These facts imply that there are two elements \( p' \) and \( q' \) in \( W_u \) such that \( p'_2 \equiv q'_2 \pmod{\mathbb{Z}} \) and \( p'_3 \equiv q'_3 \pmod{\mathbb{Z}} \). This in turn implies that \( W_v \) is layered in direction \( e_2 \). Hence, \( T \) is layered, which is impossible.

Our next claim is a straightforward consequence of Claims 2 and 3.

**Claim 4.** For every \( v \in T \), there are \( p, q \in W_v \) such that \( p_i \equiv q_i \pmod{\mathbb{Z}} \) for \( i = 2, 3 \);

**Claim 5.** Each \( T \) is a union of exactly two cosets.

Fix \( v \in T \). Let \( p \) and \( q \) be elements of \( W_v \) guaranteed by Claim 4. Then, by Keller’s condition, for any \( t \in T \setminus W_v \), we have \( t_i \equiv p_i \pmod{\mathbb{Z}} \) for some \( i \in \{2, 3\} \) and \( t_j \equiv q_j \pmod{\mathbb{Z}} \) for the other \( j \in \{2, 3\} \). In particular, for every \( k \in \{2, 3\} \), the set \( (T \setminus W_v)_k \) is contained in a union of at most two cosets of \( \mathbb{Z} \). These cosets are \( p_k + \mathbb{Z} \) and \( q_k + \mathbb{Z} \). On the other hand, since \( T_1 \) is a union of at least two cosets, there is \( w \in T \) such that \( W_w \) and \( W_v \) are disjoint. By our Claim 2 and the modular equations satisfied by the elements \( t \in T \setminus W_v \), there are two elements \( y \) and \( z \) in \( W_v \) such that

\[
y_2 \equiv p_2 \pmod{\mathbb{Z}} \quad \text{and} \quad y_3 \equiv q_3 \pmod{\mathbb{Z}};
\]

\[
z_2 \equiv p_2 \pmod{\mathbb{Z}} \quad \text{and} \quad z_3 \equiv q_3 \pmod{\mathbb{Z}}.
\]

Then, by the same reasoning as above, for every \( k \in \{2, 3\} \), the set \( (T \setminus W_v)_k \) is covered by the same cosets \( p_k + \mathbb{Z} \) and \( q_k + \mathbb{Z} \) as is the set \( (T \setminus W_v)_k \). Therefore, for every \( k \in \{2, 3\} \) the set \( T_k \) is covered by two cosets. Suppose now that there is \( t \in T \setminus (W_v \cup W_w) \). Then its form would be limited only by \( p \) and \( q \) but also by \( y \) and \( z \). Namely, such a \( t \) should satisfy the following proposition:

\[
(t_2, t_3) \in ((p_2, p_3) + \mathbb{Z}^2) \cup ((q_2, q_3) + \mathbb{Z}^2) \quad \text{and} \quad (t_2, t_3) \in ((p_2, p_3) + \mathbb{Z}^2) \cup ((q_2, q_3) + \mathbb{Z}^2).
\]

It is clear that this proposition is not consistent with our assumptions on \( p \) and \( q \). Therefore, \( T = W_v \cup W_w \) which shows that \( T \) is covered by two cosets of \( \mathbb{Z} \).

Now, by Claim 5, there are reals \( \alpha^1, \alpha^2, \beta^1, \beta^2, \gamma^1, \gamma^2 \) such that \( \alpha^1 \neq \alpha^2 \pmod{\mathbb{Z}}, \beta^1 \neq \beta^2 \pmod{\mathbb{Z}}, \gamma^1 \neq \gamma^2 \pmod{\mathbb{Z}} \) and \( T \) is contained in the union of the sets:

\[
V^1 = (\alpha^1 + \mathbb{Z}) \times (\beta^1 + \mathbb{Z}) \times (\gamma^1 + \mathbb{Z}),
\]

\[
V^2 = (\alpha^1 + \mathbb{Z}) \times (\beta^2 + \mathbb{Z}) \times (\gamma^2 + \mathbb{Z}),
\]

\[
V^3 = (\alpha^2 + \mathbb{Z}) \times (\beta^1 + \mathbb{Z}) \times (\gamma^2 + \mathbb{Z}),
\]

\[
V^4 = (\alpha^2 + \mathbb{Z}) \times (\beta^2 + \mathbb{Z}) \times (\gamma^1 + \mathbb{Z}),
\]

\[
V^5 = (\alpha^2 + \mathbb{Z}) \times (\beta^2 + \mathbb{Z}) \times (\gamma^1 + \mathbb{Z}),
\]

\[
V^6 = (\alpha^2 + \mathbb{Z}) \times (\beta^1 + \mathbb{Z}) \times (\gamma^1 + \mathbb{Z}),
\]

\[
V^7 = (\alpha^1 + \mathbb{Z}) \times (\beta^2 + \mathbb{Z}) \times (\gamma^1 + \mathbb{Z}),
\]

\[
V^8 = (\alpha^1 + \mathbb{Z}) \times (\beta^1 + \mathbb{Z}) \times (\gamma^2 + \mathbb{Z}).
\]

**Claim 6.** \( T \) is contained in one of the unions: \( U^1 = \bigcup_{i=1}^{4} V^i, U^2 = \bigcup_{i=5}^{8} V^i \).
For \( v \in T \), let us define

\[
X_v = \{ t \in T : t_2 \equiv v_2 \pmod{2} \}, \quad Y_v = \{ t \in T : t_3 \equiv v_3 \pmod{3} \}.
\]

It is clear that there are results corresponding to our claims for \( X_v \) and \( Y_v \). We may assume that \( V^1 \) intersects \( T \). Let \( v \) be an element of their common part. **Claim 3** implies that \( V^2 \) and \( V^5 \) are disjoint with \( T \). The corresponding claim for \( X_v \) implies that \( V^6 \) is disjoint with \( T \). Let \( w \in V^3 \). Then the pair \( v, w \) does not satisfy Keller’s condition. This implies that also \( V^7 \) is disjoint with \( T \). Therefore, \( T \subset U^1 \).

Clearly, the same holds true if any of the \( V^i \), \( i \leq 4 \) intersects with \( T \), otherwise \( T \subset U^2 \).

We are now ready to complete the proof. Observe first that by **Claim 4**, its variant for \( X_v \), and **Claim 5** all the intersections \( V^i \cap T \) are non-empty. Let us define the sets \( A, B, C, D, E \) and \( F \) by the equations:

\[
\begin{align*}
\alpha^1 + A &= (T \cap V^1)_1, \\
\beta^1 + B &= (T \cap V^1)_2, \\
\gamma^1 + C &= (T \cap V^1)_3, \\
\alpha^2 + D &= (T \cap V^2)_1, \\
\beta^2 + E &= (T \cap V^2)_2, \\
\gamma^2 + F &= (T \cap V^2)_3.
\end{align*}
\]

Now, appealing to Keller’s condition, one can easily observe that \( T^i \supseteq T \cap V^i \), for every \( i \leq 4 \). Therefore, by **Claim 6**, \( \bigcup_{i=1}^{4} T^i \supseteq T \). On the other hand, \( \bigcup_{i=1}^{4} T_i \) satisfies Keller’s condition. The unextendability of \( T \) gives \( T = \bigcup_{i=1}^{4} T_i \). \( \square \)

As a consequence of Theorems 3–5, we characterize those sets \( S \subset \mathbb{R}^3 \) satisfying Keller’s condition that cannot be extended to sets determining a cube tiling of \( \mathbb{R}^3 \) (compare with [11]).

**Corollary 1.** Let \( S \subset \mathbb{R}^3 \) satisfies Keller’s condition. There does not exist a set \( T \subset \mathbb{R}^3 \) determining a cube tiling and containing \( S \) if and only if there are vectors \( s^1, s^2, s^3, s^4 \) belonging to \( S \), reals \( \alpha^1, \alpha^2, \beta^1, \beta^2, \gamma^1, \gamma^2 \) such that \( \alpha^1 \neq \alpha^2 \pmod{2}, \beta^1 \neq \beta^2 \pmod{2}, \gamma^1 \neq \gamma^2 \pmod{2} \), and non-zero integers \( k^1, k^2, l^1, l^2, m^1, m^2 \) for which we have

\[
\begin{align*}
s^1 &= (\alpha^1, \beta^1, \gamma^1), \\
s^2 &= (\alpha^1 + k^1, \beta^1, \gamma^1), \\
s^3 &= (\alpha^2, \beta^1 + l^1, \gamma^2 + m^2), \\
s^4 &= (\alpha^2 + k^2, \beta^2 + l^2, \gamma^1 + m^1).
\end{align*}
\]

**Lemma 3.** Let \( [0, 1)^3 \oplus S \) and \( [0, 1)^3 \oplus T \) be cube tilings of \( \mathbb{R}^3 \). Let us suppose that there are pairwise disjoint subsets \( U, V \) and \( W \) of \( \mathbb{R}^3 \) such that \( S = U \cup W \) and \( T = V \cup W \) and \( U, V \) are non-empty. Then at least one of the sets \( [0, 1)^3 \oplus U \), \( [0, 1)^3 \oplus V \) contains a column.

**Proof.** By Theorem 2, there is nothing to prove if \( W \) is empty. Thus, we may assume that it is not. Suppose first that \( [0, 1)^3 \oplus S \) is layered. We may assume that there is \( \alpha \in \mathbb{R} \) such that \( S_3 = \alpha + \mathbb{Z} \). Since \( W \) is non-empty and \( W = S \cap T \), it follows that \( T_3 \supseteq \alpha + \mathbb{Z} \). Now, if the tiling determined by \( T \) would contain at least two cylinders in direction \( e_3 \), then there would exist a cylinder which would consist of boxes each of which would belong to \( [0, 1)^3 \oplus V \). By Theorem 2, this cylinder would contain a column we seek. Hence, we may assume that \( T_3 = \alpha + \mathbb{Z} \). For any set \( A \subset \mathbb{R}^3 \) and \( k \in \mathbb{Z} \), let us set \( A^k = \{ a \in A : a_3 = \alpha + k \} \). Since \( U \) is non-empty and \( U_3 \subseteq \alpha + \mathbb{Z} \), there is \( k_0 \) such that \( U^{k_0} \) is non-empty. We may assume that \( k_0 = 0 \). Since every cube tiling of \( \mathbb{R}^3 \) is layered, the layers \( [0, 1)^3 \oplus S^0 \) and \( [0, 1)^3 \oplus T^0 \) decompose into columns. Observe that each of these columns can be considered as containing a cube belonging to \( [0, 1)^3 \oplus W^0 \), otherwise we would have a column we seek. If the layer \( [0, 1)^3 \oplus S^0 \) decomposes into columns along the first axis, then the layer \( [0, 1)^3 \oplus T^0 \) decomposes along the second and conversely, as in the other case \( U^0 = V^0 \) would coincide which is impossible, as \( U \) and \( V \) are declared to be disjoint and \( U^0 \) is non-empty. As a result, there are numbers \( \beta, \gamma \in \mathbb{R} \) such that

\[
\begin{align*}
S^0 &= (\gamma + \mathbb{Z}) \times (\beta + \mathbb{Z}) \times \{ \alpha \}, \\
T^0 &= (\gamma + \mathbb{Z}) \times (\beta + \mathbb{Z}) \times \{ \alpha \}.
\end{align*}
\]

By Keller’s condition, we get

\[
W^0 \subseteq (\gamma + \mathbb{Z}) \times (\beta + \mathbb{Z}) \times \{ \alpha \}.
\]
Let us remind that each of the layers \([0,1)^3 \oplus S^0\) and \([0,1)^3 \oplus T^0\) decomposes into columns, each of which contains elements of \([0,1)^3 \oplus W^0\). Thus, both \(S^0\) and \(T^0\) are contained in \((\gamma + \mathbb{Z}) \times (\beta + \mathbb{Z}) \times \{\alpha\}\). Hence, these three sets coincide. As a consequence, \(U\) and \(V\) intersect, which is impossible.

Suppose now that neither \([0,1)^3 + S\) nor \([0,1)^3 + T\) are layered. By Theorem 4 there are reals \(\alpha, \beta, \gamma, \delta, \eta, \varphi\) all belonging to \([0,1)\), and proper subsets \(A, B, C, D, E, F\) of \(\mathbb{Z}\) such that \(S = \bigcup_{i=1}^5 S^i\) and \(T = \bigcup_{i=1}^5 T^i\) are the unions of the following sets:

\[
\begin{align*}
S^1 &= (\alpha + A) \times (\beta + B) \times (\gamma + \mathbb{Z}), \\
S^2 &= (\alpha + A') \times (\gamma + \mathbb{Z}) \times (\gamma + C), \\
S^3 &= (\gamma + \mathbb{Z}) \times (\beta + B') \times (\gamma + C'), \\
S^4 &= (\alpha + A') \times (\beta + B) \times (\gamma + C'), \\
S^5 &= (\alpha + A) \times (\beta + B') \times (\gamma + C), \\
T^1 &= (\delta + D) \times (\epsilon + E) \times (\gamma + \mathbb{Z}), \\
T^2 &= (\delta + D') \times (\gamma + \mathbb{Z}) \times (\varphi + F), \\
T^3 &= (\gamma + \mathbb{Z}) \times (\epsilon + E') \times (\varphi + F'), \\
T^4 &= (\delta + D') \times (\epsilon + E) \times (\varphi + F'), \\
T^5 &= (\delta + D) \times (\epsilon + E') \times (\varphi + F).
\end{align*}
\]

Let us take any element \(w \in S \cap T\). Clearly, such an element exists as \(S \cap T = W\) and \(W\) is assumed to be non-empty. Moreover, by the above block decompositions, there are \(i\) and \(j\) such that \(w \in S^i \cap T^j\). Therefore, at least one of the propositions, \(\alpha = \delta', \beta = \epsilon'\) and \(\gamma = \varphi\), is true. We may suppose that \(\gamma = \varphi\). Now, let us take any \(x \in S\) such that \(x_3 \neq \gamma (mod\ \mathbb{Z})\). Then \(x \in S^1\) and \(G = \{x_1 \times x_2 \times (x_3 + \mathbb{Z})\}\) is contained in \(S^1\). Hence, \(G\) determines a column. There is nothing to prove if \(G\) is contained in \(U\). Therefore, we may declare that there is an element \(w \in G \cap W\). Then \(w \in T\) and since \(w_3 \neq \varphi (mod\ \mathbb{Z})\), we deduce that \(w \in T^1\). Consequently, in \(x_i = w_i\), for \(i = 1, 2\), and \(x_3 = w_3\) (mod \(\mathbb{Z}\)), we have

\[
G = \{x_1\} \times \{x_2\} \times (x_3 + \mathbb{Z}) = \{w_1\} \times \{w_2\} \times (w_3 + \mathbb{Z}) \subseteq T.
\]

We have just shown that \(G \subseteq W\). Let us modify \(S\) and \(T\) replacing each such a \(G\) by \(H = \{x_1\} \times \{x_2\} \times (x_3 + \mathbb{Z})\). Clearly, \([0,1)^3 + G = [0,1)^3 + H\). Let us call the resulting sets \(S\) and \(T\). They both determine cube tilings layered in direction \(e_4\). Moreover, \(U = S \setminus T\) and \(V = S \setminus T\). Now, it follows from the previous case that at least one of the sets \(U\), \(V\) contains a set determining a column. \(\square\)

**Theorem 6.** Every cylinder of a cube tiling of \(\mathbb{R}^4\) contains a column.

**Proof.** Let \([0,1)^4 \oplus X\) be a cube tiling of \(\mathbb{R}^4\). It is clear that we may restrict our considerations to cylinders in direction \(e_4\). Therefore, let \(C\) determine such a cylinder, that is, there is \(\alpha \in \mathbb{R}\) such that \(C = \{x = (x_1, x_2, x_3, x_4) \in X : x_4 = \alpha + \mathbb{Z}\}\). Let \(C^i = \{x \in C : x_4 = \alpha + i\}\) for every \(i \in \mathbb{Z}\). For every \(A \subseteq \mathbb{R}^4\), let \(A^i\) be the image of \(A\) under the projection \((x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)\). If the cylinder \([0,1)^4 + C\) would not contain a column in direction \(e_4\), then it would contain a pair of layers \([0,1)^4 + C^k\) and \([0,1)^4 + C^l\) such that \(C^k \neq C^l + (k - l)e_4\). Equivalently,

\[
C^k \neq C^l + (k - l)e_4. \tag{4}
\]

Let \(S\) be the set consisting of all the images \(\pi(t)\), where \(t \in T\) and \(\alpha + k - 1 < t_4 \leq \alpha + k\). Clearly, \(S\) determines a cube tiling of \(\mathbb{R}^3\) and \(C^k \subseteq S\). Let \(T = (S \setminus C^k) \cup C^k\). \(T\) also determines a cube tiling of \(\mathbb{R}^3\). Let us set \(U = S \setminus T\), \(V = T \setminus S\). We have

\[
U = C^k \setminus C^l \quad \text{and} \quad V = C^l \setminus C^k. \tag{5}
\]

By (4) and the fact that \([0,1)^3 + C^k = [0,1)^3 + C^l\), we deduce that both \(U\) and \(V\) are non-empty. Now, it follows from Lemma 3 that at least one of these sets contains a set that determines a column. By their definitions, we deduce that at least one of the sets \([0,1)^4 + C^k, [0,1)^4 + C^l\) contains a column. Therefore, \([0,1)^4 + C\) contains a column. \(\square\)

3. Final remarks

(1) Lemma 3 and Theorem 6 cannot be extended to higher dimensions. It will be shown by an example which rests on the existence of some special tilings of \(\mathbb{R}^4\). These tilings were devised by Lagarias and Shor [12] in connection with their renowned example of a cube tiling in \(\mathbb{R}^{10}\) which
contains no pair of cubes sharing an \((n - 1)\)-dimensional face. Later on, Mackey [14] found a relevant example in \(\mathbb{R}^8\). It is rather interesting that he again employed the same special tilings. They can be described as follows. For numbers 0, 1, 2, 3, we define sets:

\[
A_0 = 2\mathbb{Z},
A_2 = 1 + 2\mathbb{Z},
A_1 = \frac{1}{3} + 2\mathbb{Z},
A_3 = \frac{3}{3} + 2\mathbb{Z}.
\]

For \(\varepsilon \in \{0, 1, 2, 3\}^4\), we set \(A_\varepsilon = A_{\varepsilon_1} \times A_{\varepsilon_2} \times A_{\varepsilon_3} \times A_{\varepsilon_4}\). Let \(P\), \(Q\) and \(R\) be subsets of \(\{0, 1, 2, 3\}^4\) described by the following table:

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<th>P</th>
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<th>R</th>
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<td>3132</td>
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</tbody>
</table>

Let us define \(U = \bigcup_{\varepsilon \in P} A_\varepsilon\), \(V = \bigcup_{\varepsilon \in Q} A_\varepsilon\) and \(W = \bigcup_{\varepsilon \in R} A_\varepsilon\). These sets are disjoint. It is easily seen that \(S = U \cup W\) and \(T = V \cup W\) determine cube tilings of \(\mathbb{R}^4\). Thus, \([0, 1)^4 + U = [0, 1)^4 + V\). On the other hand, neither \([0, 1)^4 \oplus U\) nor \([0, 1)^4 \oplus V\) contains a column. Consequently, the analogue of Lemma 3 does not hold in dimension 4. Now, let

\[
X = (U \times A_1) \cup (V \times A_3) \cup (W \times \mathbb{Z}).
\]

This set determines a cube tiling of \(\mathbb{R}^5\). The set \((U \times A_1) \cup (V \times A_3)\) determines a cylinder of this tiling, which does not contain a column. Therefore, the conclusion of Theorem 6 does not hold in dimension 5.

(2) It seems to be an important fact that a simple description of cube tilings in \(\mathbb{R}^4\) cannot be expected. Andrzej Kisielewicz (unpublished) found a tiling whose determining set cannot be decomposed into a finite number of product blocks similar to those considered in Theorem 4.

(3) The combinatorial structure of cube packings in low dimensions is investigated in [6,7] under the assumption that every set determining a packing in \(\mathbb{R}^n\) is \((2\mathbb{Z})^n\)-periodic and is contained in \(\left(\frac{1}{3}\mathbb{Z}\right)^n\). (In fact, all packings are rescaled by factor 2 in [7].) It is indicated in [7], among other things, that there is only one type of unextendible 3-dimensional cube packing.

Acknowledgment

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References